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ASYMPTOTIC ESTIMATES FOR PHI FUNCTIONS FOR SUBSETS OF $\{M+1, M+2, \ldots, N\}$

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Abstract

Let $f(m,n)$ denote the number of relatively prime subsets of $\{m+1, m+2, \ldots, n\}$, and let $\Phi(m,n)$ denote the number of subsets A of $\{m+1, m+2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n. Let $f_k(m,n)$ and $\Phi_k(m,n)$ be the analogous counting functions restricted to sets of cardinality k. Simple explicit formulas and asymptotic estimates are obtained for these four functions.

A nonempty set A of integers is called *relatively prime* if $\gcd(A) = 1$. Let $f(n)$ denote the number of nonempty relatively prime subsets of $\{1, 2, \ldots, n\}$ and, for $k \geq 1$, let $f_k(n)$ denote the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ of cardinality k.

Euler's phi function $\varphi(n)$ counts the number of positive integers a in the set $\{1, 2, \ldots, n\}$ such that a is relatively prime to n. The Phi function $\Phi(n)$ counts the number of nonempty subsets A of the set $\{1, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n or, equivalently, such that $A \cup \{n\}$ is relatively prime. For every positive integer k, the function $\Phi_k(n)$ counts the number of sets $A \subseteq \{1, \ldots, n\}$ such that $\text{card}(A) = k$ and $\gcd(A)$ is relatively prime to n.

Nathanson [2] introduced these four functions for subsets of $\{1, 2, \ldots, n\}$, and El Bachraoui [1] generalized them to subsets of the set $\{m+1, m+2, \ldots, n\}$ for arbitrary nonnegative integers $m < n$.[2] We shall obtain simple explicit formulas and asymptotic estimates for the four functions.

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[2]Actually, our function $f(m,n)$ is El Bachraoui's function $f(m+1,n)$, and similarly for the other three functions. This small change yields formulas that are more symmetric and pleasing esthetically.

For every real number x, we denote by $[x]$ the greatest integer not exceeding x. We often use the elementary inequality $[x] - [y] \leq [x-y] + 1$ for all $x, y \in \mathbf{R}$.

Theorem 1. *For nonnegative integers $m < n$, let $f(m,n)$ denote the number of relatively prime subsets of $\{m+1, m+2, \ldots, n\}$. Then*

$$f(m,n) = \sum_{d=1}^{n} \mu(d) \left(2^{[n/d]-[m/d]} - 1\right)$$

and $0 \leq 2^{n-m} - 2^{[n/2]-[m/2]} - f(m,n) \leq 2n2^{[(n-m)/3]}$.

Proof. El Bachraoui [1] proved that

$$f(m,n) = \sum_{d=1}^{n} \mu(d) \left(2^{[n/d]} - 1\right) - \sum_{i=1}^{m} \sum_{d|i} \mu(d) 2^{[n/d]-i/d}.$$

Rearranging this identity, we obtain

$$\begin{aligned} f(m,n) &= \sum_{d=1}^{n} \mu(d) \left(2^{[n/d]} - 1\right) - \sum_{d=1}^{m} \mu(d) 2^{[n/d]} \sum_{\substack{i=1 \\ d|i}}^{m} 2^{-i/d} \\ &= \sum_{d=1}^{n} \mu(d) \left(2^{[n/d]} - 1\right) - \sum_{d=1}^{m} \mu(d) 2^{[n/d]} \sum_{j=1}^{[m/d]} 2^{-j} \\ &= \sum_{d=1}^{n} \mu(d) 2^{[n/d]} \left(1 - \sum_{j=1}^{[m/d]} 2^{-j}\right) - \sum_{d=1}^{n} \mu(d) \\ &= \sum_{d=1}^{n} \mu(d) \left(2^{[n/d]-[m/d]} - 1\right). \end{aligned}$$

Let $d \in \{1, 2, \ldots, n\}$. Then $m+1 \leq a \leq n$ and d divides a if and only if $[m/d]+1 \leq a/d \leq [n/d]$. It follows that $A \subseteq \{m+1, \ldots, n\}$ and $\gcd(A) = d$ if and only if $A' = (1/d) * A \subseteq \{[m/d]+1, \ldots, [n/d]\}$ and $\gcd(A') = 1$. Therefore,

$$\begin{aligned} 2^{n-m} - 1 &= \sum_{d=1}^{n} f([m/d], [n/d]) \\ &\leq f(m,n) + 2^{[n/2]-[m/2]} - 1 + \sum_{d=3}^{n} 2^{[n/d]-[m/d]} \end{aligned}$$

and we obtain the lower bound $f(m,n) \geq 2^{n-m} - 2^{[n/2]-[m/2]} - 2n2^{[(n-m)/3]}$. For the upper bound, we observe that the number of subsets of even integers contained in $\{m+1, \ldots, n\}$ is exactly $2^{[n/2]-[m/2]}$ and so $f(m,n) \leq 2^{n-m} - 2^{[n/2]-[m/2]}$. This completes the proof. □

Theorem 2. *For nonnegative integers $m < n$ and for $k \geq 1$, let $f_k(m,n)$ denote the number of relatively prime subsets of $\{m+1, m+2, \ldots, n\}$ of cardinality k. Then*

$$f_k(m,n) = \sum_{d=1}^{n} \mu(d) \binom{[n/d]-[m/d]}{k}$$

and

$$0 \leq \binom{n-m}{k} - \binom{[n/2]-[m/2]}{k} - f_k(m,n) \leq n\binom{[(n-m)/3]+2}{k}.$$

Proof. El Bachraoui [1] proved that

$$f_k(m,n) = \sum_{d=1}^{n} \mu(d) \binom{[n/d]}{k} - \sum_{i=1}^{m} \sum_{d|i} \mu(d) \binom{[n/d]-i/d}{k-1}.$$

We recall the combinatorial fact that for $k \geq 1$ and $0 \leq M \leq N$, we have

$$\binom{N}{k} - \sum_{j=1}^{M} \binom{N-j}{k-1} = \binom{N-M}{k}.$$

Then

$$\begin{aligned} f_k(m,n) &= \sum_{d=1}^{n} \mu(d) \binom{[n/d]}{k} - \sum_{d=1}^{m} \mu(d) \sum_{\substack{i=1 \\ d|i}}^{m} \binom{[n/d]-i/d}{k-1} \\ &= \sum_{d=1}^{m} \mu(d) \left(\binom{[n/d]}{k} - \sum_{j=1}^{[m/d]} \binom{[n/d]-j}{k-1} \right) + \sum_{d=m+1}^{n} \mu(d) \binom{[n/d]}{k} \\ &= \sum_{d=1}^{m} \mu(d) \binom{[n/d]-[m/d]}{k} + \sum_{d=m+1}^{n} \mu(d) \binom{[n/d]}{k} \\ &= \sum_{d=1}^{n} \mu(d) \binom{[n/d]-[m/d]}{k}. \end{aligned}$$

We obtain an upper bound for $f_k(m,n)$ by deleting k-element sets of even integers:

$$f_k(m,n) \leq \binom{n-m}{k} - \binom{[n/2]-[m/2]}{k}$$

and we obtain a lower bound from the identity

$$\begin{aligned} \binom{n-m}{k} &= \sum_{d=1}^{n} f_k([m/d],[n/d]) \\ &\leq f_k(m,n) + \binom{[n/2]-[m/2]}{k} + \sum_{d=3}^{n} \binom{[n/d]-[m/d]}{k} \\ &\leq f_k(m,n) + \binom{[n/2]-[m/2]}{k} + n\binom{[(n-m)/3]}{k}. \end{aligned}$$

□

Theorem 3. *For nonnegative integers $m < n$, let $\Phi(m,n)$ denote the number of subsets of $[m+1,n]$ such that* $\gcd(A)$ *is relatively prime to n. Then*

$$\Phi(m,n) = \sum_{d|n} \mu(d) 2^{(n/d)-[m/d]}.$$

If p^* is the smallest prime divisor of n, then

$$0 \leq 2^{n-m} - 2^{(n/p^*)-[m/p^*]} - \Phi(m,n) \leq 2n2^{[(n-m)/(p^*+1)]}.$$

Proof. El Bachraoui [1] proved that

$$\Phi(m,n) = \sum_{d|n} \mu(d) 2^{n/d} - \sum_{i=1}^{m} \sum_{d|(i,n)} \mu(d) 2^{(n-i)/d}$$

Rearranging this identity, we obtain

$$\begin{aligned}
\Phi(m,n) &= \sum_{d|n} \mu(d) 2^{n/d} - \sum_{d|n} \mu(d) \sum_{\substack{i=1 \\ d|i}}^{m} 2^{(n-i)/d} \\
&= \sum_{d|n} \mu(d) 2^{n/d} - \sum_{d|n} \mu(d) \sum_{j=1}^{[m/d]} 2^{(n-jd)/d} \\
&= \sum_{d|n} \mu(d) 2^{n/d} \left[1 - \sum_{j=1}^{[m/d]} 2^{-j} \right] \\
&= \sum_{d|n} \mu(d) 2^{(n/d)-[m/d]}.
\end{aligned}$$

Let p^* be the smallest prime divisor of n. Deleting all subsets of $\{m+1, \ldots, n\}$ whose elements are all multiplies of p^*, we obtain the upper bound

$$\Phi(m,n) \leq 2^{n-m} - 2^{(n/p^*)-[m/p^*]}.$$

For the lower bound, we have

$$\begin{aligned}
\Phi(m,n) - \left(2^{n-m} - 2^{(n/p^*)-[m/p^*]}\right) &= \sum_{\substack{d|n \\ d>p^*}} \mu(d) 2^{(n/d)-[m/d]} \\
&\leq 2 \sum_{\substack{d|n \\ d>p^*}} 2^{[(n-m)/d]} \leq 2n2^{[(n-m)/(p^*+1)]}.
\end{aligned}$$

This completes the proof. □

Theorem 4. *For nonnegative integers $m < n$, let $\Phi_k(m,n)$ denote the number of subsets of cardinality k contained in the interval of integers $\{m+1, m+2, \cdots n\}$ such that* $\gcd(A)$ *is relatively prime to n. Then*

$$\Phi_k(m,n) = \sum_{d|n} \mu(d) \binom{n/d - [m/d]}{k}$$

and

$$0 \leq \binom{n-m}{k} - \binom{n/p^* - [m/p^*]}{k} - \Phi_k(m,n) \leq n\binom{[(n-m)/(p^*+1)]+1}{k}.$$

Proof. Let p^* be the smallest prime divisor of n. El Bachraoui [1] proved that

$$\Phi_k(m,n) = \sum_{d|n} \mu(d)\binom{n/d}{k} - \sum_{i=1}^{m} \sum_{d|\gcd(i,n)} \mu(d)\binom{(n-i)/d}{k-1}.$$

Rearranging this identity, we obtain

$$\begin{aligned}
\Phi_k(m,n) &= \sum_{d|n} \mu(d)\binom{n/d}{k} - \sum_{d|n} \mu(d) \sum_{\substack{i=1\\ d|i}}^{m} \binom{(n-i)/d}{k-1} \\
&= \sum_{d|n} \mu(d) \left(\binom{n/d}{k} - \sum_{j=1}^{[m/d]} \binom{n/d-j}{k-1} \right) \\
&= \sum_{d|n} \mu(d) \binom{n/d - [m/d]}{k} \\
&\geq \binom{n-m}{k} - \binom{n/p^* - [m/p^*]}{k} - \sum_{\substack{d|n\\ d>p^*}} \binom{n/d - [m/d]}{k} \\
&\geq \binom{n-m}{k} - \binom{n/p^* - [m/p^*]}{k} - \sum_{\substack{d|n\\ d>p^*}} \binom{[(n-m)/d]+1}{k} \\
&\geq \binom{n-m}{k} - \binom{n/p^* - [m/p^*]}{k} - n\binom{[(n-m)/(p^*+1)]+1}{k}.
\end{aligned}$$

Deleting k-element subsets of $\{m+1,\ldots,n\}$ whose elements are multiples of p^*, we get the upper bound

$$\Phi_k(m,n) \leq \binom{n-m}{k} - \binom{[n/p^*] - [m/p^*]}{k}.$$

This completes the proof. □

References

[1] M. El Bachraoui, *The number of relatively prime subsets and phi functions for* $\{m, m+1, \ldots, n\}$, Integers **7** (2007), A43, 8 pp. (electronic).

[2] M. B. Nathanson, *Affine invariants, relatively prime sets, and a phi function for subsets of* $\{1, 2, \ldots, n\}$, Integers **7** (2007), A1, 7 pp. (electronic).