A NOTE ON A CONJECTURE OF ERDŐS-TURÁN

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Received: 1/23/08, Revised: 6/20/08, Accepted: 6/29/08, Published: 7/18/08

Abstract

Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of nonnegative integers. We prove that for $F(x) = \sum_{n=1}^{\infty} x^{a_n}$ and $F(x)^2 = \sum_{n=0}^{\infty} R(n)x^n$, the condition $\limsup_{n\to\infty} R(n) = A$ for some positive integer A implies that $\liminf_{n\to\infty} R(n) \le A - 2\sqrt{A} + 1$.

1. Introduction

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers. Let

$$F(x) = \sum_{n=1}^{\infty} x^{a_n}$$

and

$$F(x)^2 = \sum_{n=0}^{\infty} R(n)x^n.$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is called an *additive basis of order two* if $R(n) > 0$ for every nonnegative integer n and an *asymptotic additive basis of order two* if $R(n) > 0$ for every sufficiently large n. The Erdős-Turán conjecture says that for any additive basis of order two $\{a_n\}_{n=1}^{\infty}$ the sequence $\{R(n)\}_{n=0}^{\infty}$ is unbounded. This conjecture can be rephrased in number theoretic language: Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers. Denote by $R(n)$ the number of solution $n = a_i + a_j$, i.e.,

$$R(n) = \#\{(i,j) : n = a_i + a_j\}.$$

Using this representation function the Erdős-Turán conjecture can be stated as follows,

[1]Supported by Hungarian National Foundation for Scientific Research, Grant No T 49693 and 61908

Conjecture 1 (Erdős-Turán conjecture for bases of order two) *Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of nonnegative integers such that $R(n) > 0$ for every nonnegative integer n. Then the sequence $\{R(n)\}_{n=0}^{\infty}$ is unbounded.*

Grekos, Haddad, Helou and Pihko [3] proved that $\limsup_{n\to\infty} R(n) \geq 6$ for every basis $\{a_n\}$. Later Borwein, Choi and Chu [1] improved it to $\limsup_{n\to\infty} R(n) \geq 8$.

If for some strictly increasing sequence nonnegative integers $\{a_n\}_{n=1}^{\infty}$ the representation function $R(n) > 0$ for every $n \geq n_0$ (that is $\{a_n\}_{n=1}^{\infty}$ forms an asymptotic additive basis), then the sequence $\{0, 1, \dots, n_0 - 1\} \cup \{a_n\}_{n=1}^{\infty}$ forms a basis and if its representation function is denoted by $R'(n)$ then $R'(n) \leq R(n) + n_0$. Therefore, we get that the above conjecture is equivalent to

Conjecture 2 (Erdős-Turán conjecture for asymptotic bases of order 2) *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers such that $R(n) > 0$ for every $n \geq n_0$. Then the sequence $\{R(n)\}_{n=0}^{\infty}$ is unbounded.*

This second version can be formulated as:

Conjecture 3 (Erdős-Turán conjecture for bounded representation function) *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers and*

$$\limsup_{n\to\infty} R(n) = A$$

for some positive integer A. Then we have $\liminf_{n\to\infty} R(n) = 0$.

In this note we give a non-trivial upper bound for $\liminf_{n\to\infty} R(n)$ if the sequence $\{R(n)\}_{n=0}^{\infty}$ is bounded.

Theorem 1 *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers and* $\limsup_{n\to\infty} R(n) = A$ *for some positive integer A. Then we have*

$$\liminf_{n\to\infty} R(n) \leq A - 2\sqrt{A} + 1.$$

2. Proof

If $a_N > N^2$ for some N, then

$$\#\{n : \quad 1 \leq n \leq N^2, \quad R(n) > 0\} \leq \binom{N}{2},$$

and therefore

$$\#\{n : \quad 1 \le n \le N^2, \quad R(n) = 0\} \ge \binom{N+1}{2}.$$

Hence it follows that if $a_n > n^2$ for infinitely many integers n, then $R(n) = 0$ for infinitely many integers n. Then we have $\liminf_{n\to\infty} R(n) = 0 \le A - 2\sqrt{A} + 1$, which proves the theorem.

Therefore we may assume that

$$a_n \le n^2 \qquad \text{for } n \ge n_1. \tag{1}$$

Let us suppose that there exists a strictly increasing sequence of nonnegative integers $\{a_n\}_{n=1}^{\infty}$ such that $\limsup_{n\to\infty} R(n) = A$ but $\liminf_{n\to\infty} R(n) > A - 2\sqrt{A} + 1$. Then there exist an integer n_2 and $0 < \epsilon < \sqrt{A}$ for which $A - 2\sqrt{A} + 1 + \epsilon \le R(n) \le A$ for $n \ge n_2$. Set $C = A - \sqrt{A} + \epsilon$. By elementary calculus we have $f(x) = \frac{(x-C)^2}{x} < 1$ for every $x \in [A - 2\sqrt{A} + 1 + \epsilon, A]$, and therefore there exists a $\delta > 0$ such that

$$(R(n) - C)^2 \le (1-\delta)^2 R(n) \qquad \text{for } n \ge n_2. \tag{2}$$

Let

$$F(z) = \sum_{n=1}^{\infty} z^{a_n}.$$

Then

$$F(z)^2 = \sum_{n=0}^{\infty} R(n) z^n.$$

Let

$$z = (1 - \frac{1}{N})e^{2\pi i\alpha} = re^{2\pi i\alpha},$$

where N is a large integer. We give an upper and a lower bound for the integral

$$\int_0^1 |F(z)^2 - \sum_{n=0}^{\infty} Cz^n| d\alpha \tag{3}$$

to reach a contradiction. We get an upper bound for (3) by Cauchy's inequality, Parseval's formula and (2):

$$\int_0^1 |F(z)^2 - \sum_{n=0}^{\infty} Cz^n| d\alpha = \int_0^1 |\sum_{n=0}^{\infty} (R(n) - C) z^n| d\alpha \le \left(\int_0^1 |\sum_{n=0}^{\infty} (R(n) - C) z^n|^2 d\alpha \right)^{1/2} =$$

$$\left(\sum_{n=0}^{\infty} (R(n) - C)^2 r^{2n} \right)^{1/2} \le \left(c_1 + (1-\delta)^2 (\sum_{n=0}^{\infty} R(n) r^{2n}) \right)^{1/2} \le c_2 + (1-\delta) F(r^2). \tag{4}$$

Now here is the lower bound for (3). Obviously,

$$\int_0^1 |F(z)^2 - \sum_{n=0}^{\infty} Cz^n| d\alpha \ge \int_0^1 |F(z)^2| d\alpha - \int_0^1 |\sum_{n=0}^{\infty} Cz^n| d\alpha, \tag{5}$$

where by Parseval's formula

$$\int_0^1 |F^2(z)| d\alpha = \sum_{n=1}^{\infty} r^{2a_n} = F(r^2). \tag{6}$$

Moreover

$$\int_0^1 |\sum_{n=0}^{\infty} Cz^n| d\alpha = C \int_0^1 \frac{1}{|1-z|} d\alpha = 2C \int_0^{1/2} \frac{1}{|1-z|} d\alpha.$$

Since

$$|1-z|^2 = (1-r\cos 2\pi\alpha)^2 + (r\sin 2\pi\alpha)^2 = (1-r)^2 + 2r(1-\cos 2\pi\alpha) = (1-r)^2 + 4r\sin^2 \pi\alpha,$$

we have $|1-z| \geq max\{\frac{1}{N}, \alpha\}$ for every $0 < \alpha < \frac{1}{2}$. Hence

$$\int_0^1 |\sum_{n=0}^{\infty} Cz^n| d\alpha \leq 2C(\int_0^{1/N} N d\alpha) + \int_{1/N}^{1/2} \frac{1}{\alpha} d\alpha \leq c_3 \log N \tag{7}$$

for some $c_3 > 0$. By (4), (6) and (7) we have

$$F(r^2) - c_3 \log N \leq \int_0^1 |F^2(z) - \sum_{n=0}^{\infty} Cz^n| d\alpha \leq (1-\delta)F(r^2) + c_2;$$

therefore

$$\delta F(r^2) < c_2 + c_3 \log N, \tag{8}$$

but in view of (1)

$$F(r^2) = \sum_{n=1}^{\infty} r^{2a_n} \geq \sum_{n=n_1}^{\sqrt{N}} (1 - \frac{1}{N})^{2a_n} > c_4\sqrt{N}$$

for some positive c_4, which is a contradiction to (8) if N is large enough. □

References

[1] P. Borwein, S. Choi and F. Chu, *An old conjecture of Erdős-Turán on additive bases,* Math. Comp., 75(2006), no. 253, 475–484

[2] P. Erdős and P. Turán, *On a problem of Sidon in additive number theory and some related problems,* J. London Math. Soc., 16:212–215, 1941.

[3] G. Grekos, L. Haddad, C. Helou and J. Pihko, *On the Erdős-Turán conjecture,* J. Number Theory, 102(2): 339–352, 2003.