# ON THE FROBENIUS PROBLEM FOR GEOMETRIC SEQUENCES

**Amitabha Tripathi**
*Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi – 110016, India*
atripath@maths.iitd.ac.in

*Received: 8/12/08, Revised: 8/29/08, Accepted: 9/16/08, Published: 10/7/08*

## Abstract

Let $a, b, k$ be positive integers, with $\gcd(a, b) = 1$, and let $\mathcal{A}$ denote the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$. Let $\Gamma(\mathcal{A})$ denote the set of integers that are expressible as a linear combination of elements of $\mathcal{A}$ with non-negative integer coefficients. We determine $g(\mathcal{A})$ and $n(\mathcal{A})$ which denote the *largest* (respectively, the *number* of) positive integer(s) not in $\Gamma(\mathcal{A})$. We also determine the set $\mathcal{S}^\star(\mathcal{A})$ of positive integers not in $\Gamma(\mathcal{A})$ which satisfy $n + \Gamma^\star(\mathcal{A}) \subset \Gamma^\star(\mathcal{A})$, where $\Gamma^\star(\mathcal{A}) = \Gamma(\mathcal{A}) \setminus \{0\}$.

## 1. Introduction

For a sequence of relatively prime positive integers $A = a_1, a_2, \ldots, a_k$, let $\Gamma(A)$ denote the set of all integers of the form $\sum_{i=1}^{k} a_i x_i$ where each $x_i \geq 0$. It is well known and not difficult to show that $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is a *finite* set. The *Coin Exchange Problem* of Frobenius is to determine the *largest* integer in $\Gamma^c(A)$. This is denoted by $g(A)$, and called the Frobenius number of $A$. The Frobenius number is known in the case $k = 2$ to be $g(a_1, a_2) = a_1a_2 - a_1 - a_2$, but is generally otherwise unsolved except in some special cases. A related problem is the determination of the number of integers in $\Gamma^c(A)$, which is denoted by $n(A)$ and known in the case $k = 2$ to be given by $n(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. More complete information on the Frobenius problem may be found in [3].

Ong and Ponomarenko recently determined the Frobenius number for geometric sequences in [2]. If we denote the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$ by $\mathcal{A}_k(a, b)$, and the corresponding Frobenius number by $\mathrm{G}_k = g\big(\mathcal{A}_k(a, b)\big)$, Ong & Ponomarenko proved their claim by showing that the sequence $\{\mathrm{G}_k\}_{k\geq 1}$ satisfies a certain first order recurrence, and then using induction. The main purpose of this note is to show that both the Frobenius number $g(A)$ and $n(A)$ follow in the case of geometric sequences from an old reduction formula due to Johnson [1] and Rödseth [4]. We further determine the set $\mathcal{S}^\star$, introduced in [5], in the case of geometric sequences. This gives another proof of the result for the Frobenius number since $g(A)$ is the largest integer in $\mathcal{S}^\star\big(A\big)$.

## 2. Main Results

Throughout this section, for positive integers $a, b, k$ with $\gcd(a, b) = 1$, we denote by $\mathcal{A}_k(a, b)$ the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$. We derive the values of both $g\big(\mathcal{A}_k(a, b)\big) := \mathrm{G}_k$ and $n\big(\mathcal{A}_k(a, b)\big) := \mathrm{N}_k$ by two methods. We first use a well-known reduction formula to derive recurrence relations for the two sequences $\{\mathrm{G}_k\}_{k\geq 1}$ and $\{\mathrm{N}_k\}_{k\geq 1}$, and then use telescoping sums to solve each recurrence. The second method to derive $g\big(\mathcal{A}_k(a, b)\big)$ consists in showing that $\mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)$ has exactly one element, which must then be $g\big(\mathcal{A}_k(a, b)\big)$. Our second proof of the result for $n\big(\mathcal{A}_k(a, b)\big)$ is indirect; we show that $2\,n\big(\mathcal{A}_k(a, b)\big) - 1 = g\big(\mathcal{A}_k(a, b)\big)$. We first recall the reduction formula that is central to our first derivation.

**Lemma 1. ([1, 4])** *Let $a_1, a_2, \ldots, a_k$ be positive integers. If $\gcd(a_2, \ldots, a_k) = d$ and $a_j = da_j'$ for each $j > 1$, then*

(a) $g(a_1, a_2, \ldots, a_k) = d\,g(a_1, a_2', \ldots, a_k') + a_1(d - 1)$;

(b) $n(a_1, a_2, \ldots, a_k) = d\,n(a_1, a_2', \ldots, a_k') + \frac{1}{2}(a_1 - 1)(d - 1)$.

**Theorem 1.** *Let $a, b, k$ be positive integers, with $\gcd(a, b) = 1$. Let $\mathcal{A}_k(a, b)$ denote the sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$, and let $\sigma_k(a, b)$ denote the sum of the integers in $\mathcal{A}_k(a, b)$. Then*

(a) $g(\mathcal{A}_k(a, b)) = \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1})$;

(b) $n(\mathcal{A}_k(a, b)) = \frac{1}{2}\big\{\sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}) + 1\big\}$.

*Proof.*

(a) For $k \geq 1$, by Lemma 1, with $a_1 = a^k$ and $d = b$, we have

$$\begin{aligned} g\big(\mathcal{A}_k(a, b)\big) &= b\,g(a^k, a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + a^k(b - 1) \\ &= b\,g(a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + a^k(b - 1) \\ &= b\,g\big(\mathcal{A}_{k-1}(a, b)\big) + a^k(b - 1). \end{aligned}$$

If we write $g\big(\mathcal{A}_k(a, b)\big) := \mathrm{G}_k$, then the sequence $\{\mathrm{G}_n\}_{n\geq 1}$ satisfies the first order recurrence

$$\mathrm{G}_n = b\,\mathrm{G}_{n-1} + a^n(b - 1), \quad \mathrm{G}_1 = g(a, b) = ab - a - b.$$

Dividing both sides of the recurrence by $b^n$, summing from $n = 2$ to $n = k$ and simplifying, we get

$$\frac{\mathrm{G}_k}{b^k} = \frac{\mathrm{G}_1}{b} + a^2(b - 1)\frac{b^{k-1} - a^{k-1}}{b^k(b - a)},$$

so that

$$\begin{aligned} g\big(\mathcal{A}_k(a, b)\big) &= \mathrm{G}_k = a^2(b - 1)\frac{b^{k-1} - a^{k-1}}{b - a} + b^{k-1}(ab - a - b) \\ &= \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}). \end{aligned}$$

(b) This is similar to part (a). For $k \geq 1$, by Lemma 1, with $a_1 = a^k$ and $d = b$, we have

$$\begin{aligned} n\big(\mathcal{A}_k(a,b)\big) &= b\,n(a^k, a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + \frac{1}{2}(a^k-1)(b-1) \\ &= b\,n(a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + \frac{1}{2}(a^k-1)(b-1) \\ &= b\,n\big(\mathcal{A}_{k-1}(a,b)\big) + \frac{1}{2}(a^k-1)(b-1). \end{aligned}$$

If we write $n\big(\mathcal{A}_k(a,b)\big) := \mathrm{N}_k$, then the sequence $\{\mathrm{N}_n\}_{n\geq 1}$ satisfies the first order recurrence

$$\mathrm{N}_n = b\,\mathrm{N}_{n-1} + \frac{1}{2}(a^n-1)(b-1), \quad \mathrm{N}_1 = n(a,b) = \frac{1}{2}(a-1)(b-1).$$

Dividing both sides of the recurrence by $b^n$, summing from $n = 2$ to $n = k$ and simplifying, we get

$$\frac{\mathrm{N}_k}{b^k} = \frac{\mathrm{N}_1}{b} + \frac{1}{2}a^2(b-1)\frac{b^{k-1}-a^{k-1}}{b^k(b-a)} - \frac{1}{2}\frac{b^{k-1}-1}{b^k},$$

so that

$$\begin{aligned} n\big(\mathcal{A}_k(a,b)\big) &= \mathrm{N}_k = \frac{1}{2}a^2(b-1)\frac{b^{k-1}-a^{k-1}}{b-a} - \frac{1}{2}(b^{k-1}-1) + \frac{1}{2}b^{k-1}(a-1)(b-1) \\ &= \frac{1}{2}\big\{1 + g(\mathcal{A}_k(a,b))\big\} \\ &= \frac{1}{2}\big\{\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1}+b^{k+1}) + 1\big\}. \end{aligned}$$ □

The formulae for both $g\big(\mathcal{A}_k(a,b)\big)$ and $n\big(\mathcal{A}_k(a,b)\big)$ in Theorem 1 display a nice symmetry in the variables $a, b$. From Theorem 1 we have $n\big(\mathcal{A}_k(a,b)\big) = \frac{1}{2}\big\{1+g(\mathcal{A}_k(a,b))\big\}$. If $m, n$ are integers with sum $g(\mathcal{A}_k(a,b))$, then it is easy to see that *at most* one of $m, n$ can belong to $\Gamma(\mathcal{A}_k(a,b))$. On the other hand, if for some such pair $m, n$, neither belongs to $\Gamma(\mathcal{A}_k(a,b))$, there would be less than $\frac{1}{2}\big\{1+g(\mathcal{A}_k(a,b))\big\}$ integers in $\Gamma^c\big(\mathcal{A}_k(a,b)\big)$. Thus, for every pair of non-negative integers $m, n$ with sum $g(\mathcal{A}_k(a,b))$, *exactly* one of $m, n$ belong to $\Gamma^c(\mathcal{A}_k(a,b))$. We use this to derive $n(\mathcal{A}_k(a,b))$, giving a second proof of the assertion in the second part of Theorem 1.

**Theorem 2.** *Let $a, b, k$ be positive integers, with $\gcd(a,b) = 1$. Let $\mathcal{A}_k(a,b)$ denote the sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$, and let $\sigma_k(a,b)$ denote the sum of the integers in $\mathcal{A}_k(a,b)$. If $m+n = g(\mathcal{A}_k(a,b))$, then $m \in \Gamma\big(\mathcal{A}_k(a,b)\big)$ if and only if $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$.*

*Proof.* Let $m+n = g(\mathcal{A}_k(a,b))$. If $m \in \Gamma\big(\mathcal{A}_k(a,b)\big)$, then $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$, for otherwise $m+n = g(\mathcal{A}_k(a,b)) \in \Gamma\big(\mathcal{A}_k(a,b)\big)$, which is impossible.

Conversely, suppose $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$. If $n < 0$, then $m > g(\mathcal{A}_k(a,b))$ and so $m \in \Gamma\big(\mathcal{A}_k(a,b)\big)$. We may therefore assume that $1 \leq n \leq g(\mathcal{A}_k(a,b))$ since both 0 and any integer greater than

$g(\mathcal{A}_k(a,b))$ belong to $\Gamma\big(\mathcal{A}_k(a,b)\big)$. Since $n+\lambda b^k \in \Gamma\big(a^k, a^{k-1}b, \ldots, ab^{k-1}\big)$ for all sufficiently large integer $\lambda$ and $n \notin \Gamma\big(a^k, a^{k-1}b, \ldots, ab^{k-1}\big)$, we may write $n = \sum_{i=0}^{k-1} a^{k-i}b^i x_i - b^k x_k$, where $x_i \geq 0$ for $0 \leq i \leq k-1$ and $x_k \geq 1$. If $x_0 > b$ in this representation, by repeatedly using the identity $a^k(x_0-b)+a^{k-1}b(x_1+a) = a^k x_0 + a^{k-1}bx_1$ we may assume that $0 \leq x_0 < b$ while maintaining $x_1 \geq 0$. Assuming that $x_0, x_1, \ldots, x_{j-1}$ are all non-negative integers less than $b$ for some $j < k$, by repeatedly using the identity $a^{k-j}b^j(x_j-b)+a^{k-j-1}b^{j+1}(x_{j+1}+a) = a^{k-j}b^j x_j + a^{k-j-1}b^{j+1}x_{j+1}$, we may assume that $0 \leq x_j < b$ and still have $x_{j+1} \geq 0$. Thus we may write

$$n = \sum_{i=0}^{k-1} a^{k-i}b^i x_i - b^k x_k,$$

with $0 \leq x_i \leq b-1$ for $0 \leq i \leq k-1$, and since $n \notin \Gamma\big(\mathcal{A}_k(a,b)\big)$, also $x_k \geq 1$. Writing $g(\mathcal{A}_k(a,b)) = (b-1)\sum_{i=0}^{k-1} a^{k-i}b^i - b^k$, we have

$$m = g(\mathcal{A}_k(a,b)) - n = \sum_{i=0}^{k-1} (b-1-x_i)a^{k-i}b^i + (x_k-1)b^k \in \Gamma\big(\mathcal{A}_k(a,b)\big).$$

This completes the proof. □

**Corollary 1.** *Let $a, b, k$ be positive integers, with* $\gcd(a,b)=1$. *Then*

$$n\big(\mathcal{A}_k(a,b)\big) = \frac{1}{2}\left\{1 + g\big(\mathcal{A}_k(a,b)\big)\right\}.$$

*Proof.* Consider pairs $\{m,n\}$ of integers in the interval $[0, g\big(\mathcal{A}_k(a,b)\big)]$ with $m+n = g\big(\mathcal{A}_k(a,b)\big)$. By Theorem 2, *exactly* one integer from each such pair is in $\Gamma^c\big(\mathcal{A}_k(a,b)\big)$. This completes the proof since no integer greater than $g\big(\mathcal{A}_k(a,b)\big)$ is in $\Gamma^c\big(\mathcal{A}_k(a,b)\big)$. □

**Remark 1.** *Let $a, b, k$ be positive integers, with* $\gcd(a,b)=1$. *Then $g\big(\mathcal{A}_k(a,b)\big)$ is an odd integer.*

The evaluation of $g$ given in Theorem 1 can also be derived by explicitly determining the set $\mathcal{S}^\star$, introduced in [5], since $g(a_1, a_2, \ldots, a_k)$ is the largest element in $\mathcal{S}^\star(a_1, a_2, \ldots, a_k)$. For positive and coprime integers $a_1, a_2, \ldots, a_k$, let $\Gamma(a_1, a_2, \ldots, a_k)$ denote the non-negative integers in the set $\{a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_j \geq 0\}$, let $m_j$ denote the least positive integer in $\Gamma(a_1, a_2, \ldots, a_k)$ that is congruent to $j$ mod $a_1$ for $1 \leq j \leq a_1 - 1$, and let $\Gamma^\star(a_1, a_2, \ldots, a_k) = \Gamma(a_1, a_2, \ldots, a_k) \setminus \{0\}$. Then

$$\begin{aligned}\mathcal{S}^\star(a_1, a_2, \ldots, a_k) \; &:= \; \{n \notin \Gamma(a_1, \ldots, a_k) : n + \Gamma^\star(a_1, \ldots, a_k) \subset \Gamma^\star(a_1, \ldots, a_k)\} \\ &\subseteq \; \{m_j - a_1 : 1 \leq j \leq a_1 - 1\}.\end{aligned}$$

Moreover,

$$m_j - a_1 \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k) \Longleftrightarrow m_j + m_i > m_{j+i} \text{ for } 1 \leq i \leq a_1 - 1. \tag{1}$$

We refer to [5] for more notations and results. With the notations above, we show that $\mathcal{S}^\star\big(\mathcal{A}_k(a,b)\big) = \{\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1}+b^{k+1})\}$. Since $g(a_1, a_2, \ldots, a_k) \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k)$, this further verifies the first result of Theorem 1.

**Lemma 2.** *Let* $a_1, a_2, \ldots, a_k$ *be positive integers with* $\gcd(a_2, \ldots, a_k) = d$. *Define,* $a'_j = a_j/d$ *for* $2 \le j \le k$. *Let* $m_j$ *(respectively,* $m'_j$*) denote the least positive integer in* $\Gamma(a_1, a_2, \ldots, a_k)$ *(resp., in* $\Gamma(a_1, a'_2, \ldots, a'_k)$*) that is congruent to* $j$ *mod* $a_1$. *Then* $m_j - a_1 \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k)$ *if and only if* $m'_j - a_1 \in \mathcal{S}^\star(a_1, a'_2, \ldots, a'_k)$ *for* $1 \le j \le a_1 - 1$.

*Proof.* Let $A$ denote the sequence $a_1, a_2, \ldots, a_k$ and $A'$ the sequence $a_1, a'_2, \ldots, a'_k$. Since each $m_j$ and $m'_j$ must also be representable as a non-negative linear combination of $a_2, \ldots, a_k$ and $a'_2, \ldots, a'_k$ respectively, it follows that $\{m_j : 1 \le j \le a_1 - 1\} = \{dm'_j : 1 \le j \le a_1 - 1\}$. Therefore, by (1), $m_j - a_1 \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k)$ if and only if $m_j + m_i > m_{j+i}$ for $1 \le i \le a_1 - 1$ if and only if $m'_j + m'_i > m'_{j+i}$ for $1 \le i \le a_1 - 1$ if and only if $m'_j - a_1 \in \mathcal{S}^\star(a_1, a'_2, \ldots, a'_k)$. This completes the proof. □

**Theorem 3.** *Let* $a, b, k$ *be positive integers, with* $\gcd(a, b) = 1$. *Let* $\mathcal{A}_k(a, b)$ *denote the sequence* $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$, *and let* $\sigma_k(a, b)$ *denote the sum of the integers in* $\mathcal{A}_k(a, b)$. *Then* $\mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big) = \big\{\sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1})\big\}$ *for* $k \ge 1$.

*Proof.* We apply Lemma 2 with $A = \mathcal{A}_k(a, b)$ and $a_1 = a^k$. Then $d = b$ and $m_j - a^k \in \mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)$ if and only if $\frac{1}{b} m_j - a^k \in \mathcal{S}^\star\big(a^k, a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}\big) = \mathcal{S}^\star\big(\mathcal{A}_{k-1}(a, b)\big)$. Therefore, by Theorem 1 in [5], $\big|\mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)\big| = \big|\mathcal{S}^\star\big(\mathcal{A}_1(a, b)\big)\big| = 1$ for each $k > 1$. Since we have $g(\mathcal{A}_k(a, b)) \in \mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)$, there can be no other integer in this set. □

**Corollary 2.** *Let* $a, b, k$ *be positive integers, with* $\gcd(a, b) = 1$. *Then*

$$g\big(\mathcal{A}_k(a, b)\big) = \max \mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big) = \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}).$$

**Remark 2.** *The proof of Theorem 3 shows that the sequence of Frobenius numbers* $\{g(\mathcal{A}_k(a, b))\}_{k \ge 1}$ *satisfies the recurrence* $\mathrm{G}_k = b\,\mathrm{G}_{k-1} + a^k(b - 1)$ *since* $g(\mathcal{A}_k(a, b)) = m_j - a^k$ *is the only element in* $\mathcal{S}^\star\big(\mathcal{A}_k(a, b)\big)$. *This result coincides with the result in the first part of Theorem 1.*

**Acknowledgement.** The author is grateful to the referee for several comments that has resulted in a clearer exposition of this work.

## References

[1] S. M. Johnson, A Linear Diophantine Problem, *Canad. J. Math.* **12** (1960), 390–398.

[2] D. C. Ong and V. Ponomarenko, The Frobenius Number of Geometric Sequences, *Integers* **8**, no. A33 (2008), 1–3.

[3] J. L. Ramírez Alfonsín, The Frobenius Diophantine Problem, Oxford Lecture Series in Mathematics and its Applications, no. 30, Oxford University Press, 2005.

[4] Ö. J. Rödseth, On a linear Diophantine problem of Frobenius, *Crelle* **301** (1978), 171–178.

[5] A. Tripathi, On a variation of the Coin Exchange Problem for Arithmetic Progressions, *Integers* **3**, no. A01 (2003), 1–5.