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ON SUMS OF PRIMES FROM BEATTY SEQUENCES

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Abstract

Let $k \geq 2$ and $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$ be reals such that the α_i's are irrational and greater than 1. Suppose further that some ratio α_i/α_j is irrational. We study the representations of an integer n in the form $p_1 + p_2 + \cdots + p_k = n$, where p_i is a prime from the Beatty sequence $\mathcal{B}_i = \{n \in \mathbb{N} : n = [\alpha_i m + \beta_i] \text{ for some } m \in \mathbb{Z}\}$.

1. Introduction

Ever since the days of Euler and Goldbach, number-theorists have been fascinated by additive representations of the integers as sums of primes. The most famous result in this field is I.M. Vinogradov's three primes theorem [7], which states that every sufficiently large odd integer is the sum of three primes. Over the years, a number of authors have studied variants of the three primes theorem with prime numbers restricted to various sequences of arithmetic interest. For instance, a recent work by Banks, Güloğlu and Nevans [1] studies the question of representing integers as sums of primes from a Beatty sequence. Suppose that α and β are real numbers, with $\alpha > 1$ and irrational. The *Beatty sequence* $\mathcal{B}_{\alpha,\beta}$ is defined by

$$\mathcal{B}_{\alpha,\beta} = \{n \in \mathbb{N} : n = [\alpha m + \beta] \text{ for some } m \in \mathbb{Z}\}.$$

(Henceforth, $[\theta]$ represents the integer part of the real number θ.) Banks *et al.* proved that if $k \geq 3$, then every sufficiently large integer $n \equiv k \pmod 2$ can be expressed as the sum of k primes from the sequence $\mathcal{B}_{\alpha,\beta}$, provided that $\alpha < k$ and α "has a finite type" (see below). In their closing remarks, the authors of [1] note that their method can be used to extend the main results of [1] to representations of an integer n in the form

$$p_1 + p_2 + \cdots + p_k = n, \tag{1}$$

where $p_i \in \mathcal{B}_{\alpha,\beta_i}$. However, they remark that "for a sequence $\alpha_1, \dots, \alpha_k$ of irrational numbers greater than 1, it appears to be much more difficult to estimate the number of representations

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of $n \equiv k \pmod 2$ in the form (1), where p_i lies in the Beatty sequence $\mathcal{B}_{\alpha_i,\beta_i}$." The main purpose of the present note is to address the latter question in the case when at least one of the ratios α_i/α_j, $1 \le i, j \le k$, is irrational.

Let $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$, $k \ge 2$, be real numbers, and suppose that $\alpha_1, \dots, \alpha_k$ are irrational and greater than 1. For $i = 1, \dots, k$, we denote by $\mathcal{B}_i$ the Beatty sequence $\mathcal{B}_{\alpha_i,\beta_i}$. We write

$$R(n) = R(n; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\substack{p_1+\cdots+p_k=n \\ p_i \in \mathcal{B}_i}} (\log p_1) \cdots (\log p_k), \tag{2}$$

where the summation is over the solutions of (1) in prime numbers $p_1, \dots, p_k$ such that $p_i \in \mathcal{B}_i$. Similarly to [1], we shall use the Hardy–Littlewood circle method to obtain an asymptotic formula for $R(n)$. The circle method requires some quantitative measure of the irrationality of the α_i's in the form of hypotheses on the rational approximations to the α_i's. Let $\|\theta\|$ denote the distance from the real number θ to the nearest integer. We say that an s-tuple $\theta_1, \dots, \theta_s$ of real numbers *is of a finite type*, if there exists a real number η such that the inequality

$$\|q_1\theta_1 + \cdots + q_s\theta_s\| < \max(1, |q_1|, \dots, |q_s|)^{-\eta} \tag{3}$$

has only finitely many solutions in $q_1, \dots, q_s \in \mathbb{Z}$. In particular, the reals in an s-tuple of a finite type are irrational and linearly independent over $\mathbb{Q}$. Our main result can now be stated as follows.

Theorem 1. *Let $k \ge 3$ and let $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$ be real numbers, with $\alpha_1, \dots, \alpha_k > 1$. Suppose that each individual α_i is of a finite type and that at least one pair $\alpha_i^{-1}, \alpha_j^{-1}$ is also of a finite type. Then, for any fixed $A > 0$ and any sufficiently large integer n, one has*

$$R(n; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\mathfrak{S}_k(n) n^{k-1}}{\alpha_1 \cdots \alpha_k (k-1)!} + O\big(n^{k-1} (\log n)^{-A}\big), \tag{4}$$

where $R(n; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is the quantity defined in (2) and $\mathfrak{S}_k(n)$ is given by

$$\mathfrak{S}_k(n) = \prod_{p \mid n} \left(1 + \frac{(-1)^k}{(p-1)^{k-1}}\right) \prod_{p \nmid n} \left(1 + \frac{(-1)^{k+1}}{(p-1)^k}\right). \tag{5}$$

The implied constant in (4) depends at most on $A, \boldsymbol{\alpha}, \boldsymbol{\beta}$.

Since $1 \ll \mathfrak{S}_k(n) \ll 1$ when $n \equiv k \pmod 2$, Theorem 1 has the following direct consequence.

Corollary 1. *Let $k \ge 3$ and suppose that $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$ are real numbers subject to the hypotheses of Theorem 1. Then, every sufficiently large integer $n \equiv k \pmod 2$ can be represented in the form (1) with $p_i \in \mathcal{B}_i$, $1 \le i \le k$.*

After some standard adjustments, the techniques used in the proof of Theorem 1 yield also the following result on sums of two Beatty primes.

Theorem 2. *Suppose that $\alpha_1, \beta_1, \alpha_2, \beta_2$ are real numbers, with $\alpha_1, \alpha_2 > 1$. Suppose further that the pair $\alpha_1^{-1}, \alpha_2^{-1}$ is of a finite type. Then, for any fixed $A > 0$, and all but $O(x(\log x)^{-A})$ integers $n \le x$, one has*

$$R(n; \boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_1\alpha_2)^{-1}\mathfrak{S}_2(n)n + O\big(n(\log n)^{-A}\big),$$

where $R(n; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is the quantity defined in (2) and $\mathfrak{S}_2(n)$ is given by (5) with $k = 2$. The implied constants depend at most on $A, \boldsymbol{\alpha}, \boldsymbol{\beta}$.

Since for even n, $1 \ll \mathfrak{S}_2(n) \ll \log\log n$, we have the following corollary to Theorem 2.

Corollary 2. *Suppose that $\alpha_1, \beta_1, \alpha_2, \beta_2$ are real numbers subject to the hypotheses of Theorem 2. Then, for any fixed $A > 0$, all but $O(x(\log x)^{-A})$ even integers $n \le x$ can be represented as sums of a prime $p_1 \in \mathcal{B}_1$ and a prime $p_2 \in \mathcal{B}_2$.*

By making some adjustments in the proof of Theorem 2, we can call upon a celebrated theorem by Montgomery and Vaughan [5] to improve on Corollary 2.

Corollary 3. *Suppose that $\alpha_1, \beta_1, \alpha_2, \beta_2$ are real numbers subject to the hypotheses of Theorem 2. Then there exists an $\varepsilon = \varepsilon(\boldsymbol{\alpha}) > 0$ such that all but $O(x^{1-\varepsilon})$ even integers $n \le x$ can be represented as sums of a prime $p_1 \in \mathcal{B}_1$ and a prime $p_2 \in \mathcal{B}_2$.*

Comparing Theorems 1 and 2 with the main results in [1], one notes that our theorems include no hypotheses similar to the condition $\alpha < k$ required in [1]. The latter condition is necessary in the case $\alpha_1 = \cdots = \alpha_k = \alpha$, if all large integers $n \equiv k \pmod 2$ are to be represented. However, it can be dispensed with when some pair α_i, α_j is linearly independent over $\mathbb{Q}$.

It seems that the natural hypotheses for the above theorems are that all α_i's and some ratio α_i/α_j be irrational, but such generality is beyond the reach of our method. The finite type conditions above approximate these natural hypotheses without being too restrictive. For example, by a classical theorem of Khinchin's [4], almost all (in the sense of Lebesgue measure) real numbers are of a finite type.

2. Preliminaries

2.1. Notation

For a real number θ, $[\theta]$, $\{\theta\}$ and $\|\theta\|$ denote, respectively, the integer part of θ, the fractional part of θ and the distance from θ to the nearest integer; also, $e(\theta) = e^{2\pi i\theta}$. For integers a and b, we write (a, b) and $[a, b]$ for the greatest common divisor and the least common multiple of a and b. The letter p, with or without indices, is reserved for prime numbers. Finally, if $\mathbf{x} = (x_1, \ldots, x_s)$, we write $|\mathbf{x}| = \max(|x_1|, \ldots, |x_s|)$.

2.2.

For $i = 1, \dots, k$, we set $\gamma_i = \alpha_i^{-1}$ and $\delta_i = \alpha_i^{-1}(1-\beta_i)$. It is not difficult to see that $m \in \mathcal{B}_i$ if and only if $0 < \{\gamma_i m + \delta_i\} < \gamma_i$. Thus, the characteristic function of the Beatty sequence $\mathcal{B}_i$ is $g_i(\gamma_i m + \delta_i)$, where g_i is the 1-periodic extension of the characteristic function of the interval $(0, \gamma_i)$.

Our analysis will require smooth approximations to g_i. Suppose that $1 \le i \le k$ and Δ is a real such that

$$0 < \Delta < \tfrac{1}{4}\min(\gamma_i, 1-\gamma_i).$$

Then there exist 1-periodic C^∞-functions $g_i^\pm$ such that:

i) $0 \le g_i^-(x) \le g_i(x) \le g_i^+(x) \le 1$ for all real x;

ii) $g_i^\pm(x) = g_i(x)$ when $\Delta \le x \le \gamma_i - \Delta$ or $\gamma_i + \Delta \le x \le 1 - \Delta$;

iii) $\left|\dfrac{d^r g_i^\pm(x)}{dx^r}\right| \ll_r \Delta^{-r}$ for $r = 1, 2, \dots$.

Furthermore, the Fourier coefficients $\hat{g}_i^\pm(m)$ of $g_i^\pm$ satisfy the bounds

$$\hat{g}_i^\pm(0) = \gamma_i + O(\Delta), \qquad |\hat{g}_i^\pm(m)| \ll_r \frac{\Delta^{1-r}}{(1+|m|)^r} \quad (r = 1, 2, \dots), \tag{6}$$

where the latter bound follows from ii) and iii) above via partial integration.

2.3.

The proofs of Theorems 1 and 2 use the following generalization of the classical bound for exponential sums over primes in Vaughan [6, Theorem 3.1].

Lemma 1. *Suppose that α is real and a, q are integers, with $(a, q) = 1$ and $q \le N$. Then*

$$\sum_{p \le N} (\log p) e(\alpha p) \ll \left(Nq^{-1/2} + N^{4/5} + (Nq)^{1/2}\right)\left(1 + q^2|\theta|\right)(\log 2N)^4,$$

where $\theta = \alpha - a/q$.

The proof of the above lemma is essentially the same as that of [6, Theorem 3.1], which is the case $|\theta| \le q^{-2}$. The only adjustment one needs to make in the argument in [6] is to replace [6, Lemma 2.2] by the following variant.

Lemma 2. *Suppose that α, X, Y are real with $X \ge 1$, $Y \ge 1$, and a, q are integers with $(a, q) = 1$. Then*

$$\sum_{x \le X} \min\left(XYx^{-1}, \|\alpha x\|^{-1}\right) \ll \left(XYq^{-1} + X + q\right)\left(1 + q^2|\theta|\right)(\log 2Xq),$$

where $\theta = \alpha - a/q$.

2.4.

In the next lemma, we use the finite type of an s-tuple $\theta_1, \dots, \theta_s$ to obtain rational approximations to linear combinations of $\theta_1, \dots, \theta_s$.

Lemma 3. *Suppose that the s-tuple $\theta_1, \dots, \theta_s$ has a finite type and let $\eta > 1$ be such that (3) has finitely many solutions. Let $0 < \varepsilon < (2\eta)^{-1}$, let Q be sufficiently large, and let $\mathbf{m} = (m_1, \dots, m_s) \in \mathbb{Z}^s$, with $0 < |\mathbf{m}| \le Q^\varepsilon$. Then there exist integers a and q such that*

$$|q(m_1\theta_1 + \cdots + m_s\theta_s) - a| \le Q^{-1}, \quad Q^\varepsilon \le q \le Q, \quad (a, q) = 1.$$

Proof. Since the sum $m_1\theta_1 + \cdots + m_s\theta_s$ is irrational, it has an infinite continued fraction. Let q and q' be the denominators of two consecutive convergents to that continued fraction, such that $q \le Q < q'$. By the properties of continued fractions,

$$\|q(m_1\theta_1 + \cdots + m_s\theta_s)\| \le (q')^{-1} < Q^{-1}.$$

If $1 \le q \le Q^\varepsilon$, then $Q^{-1} \le (q|\mathbf{m}|)^{-1/(2\varepsilon)}$, and hence

$$\|q(m_1\theta_1 + \cdots + m_s\theta_s)\| < (q|\mathbf{m}|)^{-1/(2\varepsilon)}$$

which contradicts the choice of η and ε. This completes the proof. □

3. Proof of Theorem 1

Without loss of generality, we may assume that the pair of a finite type in the hypotheses of the theorem is $\alpha_1^{-1}, \alpha_2^{-1}$. We also note that if α_i has a finite type, then so does $\gamma_i = \alpha_i^{-1}$. Indeed, let q be a sufficiently large solution of $\|q\gamma_i\| < q^{-\eta}$ and let a be the nearest integer to $q\gamma_i$. Then $q \ll a \ll q$ and

$$|a\alpha_i - q| < \alpha_i q^{-\eta} \ll a^{-\eta} < a^{-\eta+\varepsilon}.$$

Thus, if $\|q\gamma_i\| < q^{-\eta}$ has an infinite number of solutions in positive integers q, then so does $\|q\alpha_i\| < q^{-\eta+\varepsilon}$.

Recall the functions g_i and $g_i^\pm$ described in §2.2. We shall use those functions with Δ given by

$$\Delta = (\log n)^{-A}. \tag{7}$$

We have

$$R(n) = \sum_{p_1+\cdots+p_k=n} (\log p_1)\cdots(\log p_k) g_1(\gamma_1 p_1 + \delta_1)\cdots g_k(\gamma_k p_k + \delta_k),$$

so by the construction of $g_i^{\pm}$,

$$R^-(n) \le R(n) \le R^+(n), \tag{8}$$

where

$$R^{\pm}(n) = \sum_{p_1+\cdots+p_k=n} (\log p_1)\cdots(\log p_k) g_1^{\pm}(\gamma_1 p_1 + \delta_1)\cdots g_k^{\pm}(\gamma_k p_k + \delta_k).$$

We now proceed to evaluate the sums $R^+(n)$ and $R^-(n)$. We shall focus on $R^+(n)$, the evaluation of $R^-(n)$ being similar.

Substituting the Fourier expansions of $g_1^+, \dots, g_k^+$ into the definition of $R^+(n)$, we obtain

$$R^+(n) = \sum_{\mathbf{m}\in\mathbb{Z}^k} \hat{g}_1^+(m_1)\cdots\hat{g}_k^+(m_k) e(\delta_1 m_1 + \cdots + \delta_k m_k) R(n,\mathbf{m}), \tag{9}$$

where $\mathbf{m} = (m_1, \dots, m_k)$ and

$$R(n,\mathbf{m}) = \sum_{p_1+\cdots+p_k=n} (\log p_1)\cdots(\log p_k) e(\gamma_1 m_1 p_1 + \cdots + \gamma_k m_k p_k).$$

We note for the record that when $\mathbf{m} = \mathbf{0}$, we have

$$R(n,\mathbf{0}) = \frac{\mathfrak{S}_k(n) n^{k-1}}{(k-1)!} + O\big(n^{k-1}(\log n)^{-A}\big). \tag{10}$$

When $k = 3$, this is due to Vinogradov [7] (see also Vaughan [6, Theorem 3.4]), and the result for $k \ge 4$ can be proved similarly (see Hua [3]).

We now set

$$M = \Delta^{-1}(\log n) = (\log n)^{A+1}. \tag{11}$$

Combining (6), (7), (9) and (10), we deduce that

$$R^+(n) = \frac{\gamma_1\cdots\gamma_k}{(k-1)!}\mathfrak{S}_k(n) n^{k-1} + O\big(n^{k-1}(\log n)^{-A}\big) + O(\Sigma_1 + \Sigma_2), \tag{12}$$

where

$$\Sigma_1 = \sum_{0<|\mathbf{m}|\le M} |\hat{g}_1^+(m_1)\cdots\hat{g}_k^+(m_k)||R(n,\mathbf{m})|,$$
$$\Sigma_2 = \sum_{|\mathbf{m}|>M} |\hat{g}_1^+(m_1)\cdots\hat{g}_k^+(m_k)||R(n,\mathbf{m})|.$$

We may use (6) to estimate Σ_2. It follows easily from the second bound in (6) that

$$\sum_{m\in\mathbb{Z}} |\hat{g}_i^+(m)| \ll \sum_{|m|\le M} \frac{1}{1+|m|} + \sum_{|m|>M} \frac{\Delta^{-1}}{(1+|m|)^2} \ll \log M. \tag{13}$$

Hence, by (10), (11) and (6) with $r = [A] + 3$,

$$\begin{aligned}\Sigma_2 &\ll R(n, \mathbf{0})(\log M)^{k-1} \sum_{1 \le i \le k} \sum_{|m| > M} |\hat{g}_i^+(m)| \\ &\ll n^{k-1}(\log n)(\Delta M)^{1-r} \ll n^{k-1}(\log n)^{-A}.\end{aligned} \tag{14}$$

Next, we use a variant of the circle method to bound $|R(n, \mathbf{m})|$ when $0 < |\mathbf{m}| \le M$. Define the exponential sum

$$S(\xi) = \sum_{p \le n} (\log p) e(\xi p).$$

By orthogonality,

$$R(n, \mathbf{m}) = \int_0^1 S(\xi + \gamma_1 m_1) \cdots S(\xi + \gamma_k m_k) e(-n\xi)\, d\xi. \tag{15}$$

Put

$$P = (\log n)^{2A+12}, \quad Q = nP^{-1}. \tag{16}$$

For $j = 1, \dots, k$, we write $\lambda_j = \lambda_j(\mathbf{m}) = \gamma_j m_j - \gamma_1 m_1$. Then

$$R(n, \mathbf{m}) = e(\gamma_1 m_1 n) \int_{1/Q}^{1+1/Q} S(\xi) S(\xi + \lambda_2) \cdots S(\xi + \lambda_k) e(-n\xi)\, d\xi.$$

We partition the interval $[1/Q, 1 + 1/Q)$ into Farey arcs of order Q and write $\mathfrak{M}(q, a)$ for the arc containing the Farey fraction a/q: if a'/q' and a''/q'' are, respectively, the left and right neighbors of a/q in the Farey sequence, then

$$\mathfrak{M}(q, a) = \left[\frac{a + a'}{q + q'}, \frac{a + a''}{q + q''} \right).$$

Thus,

$$|R(n, \mathbf{m})| \le \sum_{q \le Q} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{\mathfrak{M}(q,a)} |S(\xi) S(\xi + \lambda_2) \cdots S(\xi + \lambda_k)|\, d\xi. \tag{17}$$

When $\xi \in \mathfrak{M}(q, a)$, with $P < q \le Q$, Lemma 1 yields

$$S(\xi) \ll nP^{-1/2}(\log n)^4 \ll n(\log n)^{-A-2}.$$

Inserting this bound into the right side of (17), we obtain

$$|R(n, \mathbf{m})| \le \sum_{q \le P} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{\mathfrak{M}(q,a)} |S(\xi) S(\xi + \lambda_2) \cdots S(\xi + \lambda_k)|\, d\xi + \Sigma_3, \tag{18}$$

where

$$\begin{aligned}\Sigma_3 &\ll n(\log n)^{-A-2} \sum_{P<q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} \int_{\mathfrak{M}(q,a)} |S(\xi+\lambda_2)\cdots S(\xi+\lambda_k)|\,d\xi \\ &\ll n^{k-2}(\log n)^{-A-2}\int_0^1 |S(\xi+\lambda_2)S(\xi+\lambda_3)|\,d\xi.\end{aligned} \tag{19}$$

By the Cauchy–Schwarz inequality and Parseval's identity,

$$\int_0^1 |S(\xi+\lambda_i)S(\xi+\lambda_j)|\,d\xi \ll n(\log n) \quad (1\le i,j\le k), \tag{20}$$

so we deduce from (19) that

$$\Sigma_3 \ll n^{k-1}(\log n)^{-A-1}. \tag{21}$$

To estimate the remaining sum on the right side of (18), we consider separately the cases $m_1 = 0$ and $m_1 \ne 0$.

Case 1: $m_1 = 0$. Since $\mathbf{m} \ne \mathbf{0}$, we have $m_i \ne 0$ for some $i = 2,\dots,k$. By Lemma 3 with $s = 1$ and $\theta_1 = \gamma_i$, there exist an $\varepsilon > 0$ and integers b and r such that

$$\left|r(m_i\gamma_i) - b\right| < Q^{-1/2}, \quad Q^{\varepsilon} \le r \le Q^{1/2}, \quad (b,r) = 1.$$

Suppose that $\xi \in \mathfrak{M}(q,a)$, where $1 \le q \le P$. It follows that

$$\left|\xi + \lambda_i - \frac{a}{q} - \frac{b}{r}\right| \le \frac{1}{qQ} + \frac{1}{rQ^{1/2}} \le \frac{2}{rQ^{1/2}}.$$

Let the integers a_1, q_1 be such that

$$\frac{a_1}{q_1} = \frac{a}{q} + \frac{b}{r}, \quad (a_1,q_1) = 1.$$

Then q_1 divides $[q,r]$ and is divisible by $[q,r]/(q,r)$, so

$$\left|\xi + \lambda_i - \frac{a_1}{q_1}\right| \le \frac{2P}{q_1Q^{1/2}}, \quad rP^{-1} \le q_1 \le rP, \quad (a_1,q_1) = 1.$$

Thus, Lemma 1 yields

$$\begin{aligned}S(\xi+\lambda_i) &\ll \left(nq_1^{-1/2} + n^{4/5} + (nq_1)^{1/2}\right)\left(1 + q_1PQ^{-1/2}\right)(\log n)^4 \\ &\ll \left(nq_1^{-1/2} + n^{4/5}P\right)(\log n)^4 \ll n^{1-\varepsilon/3}.\end{aligned} \tag{22}$$

Combining (22) and (20), we easily get

$$\sum_{q\le P} \sum_{\substack{1\le a\le q\\(a,q)=1}} \int_{\mathfrak{M}(q,a)} |S(\xi)S(\xi+\lambda_2)\cdots S(\xi+\lambda_k)|\,d\xi \ll n^{k-1-\varepsilon/4}. \tag{23}$$

Case 2: $m_1 \neq 0$. Then we apply Lemma 3 to the pair γ_1, γ_2. It follows that there exist an $\varepsilon > 0$ and integers b and r such that

$$\big|r(m_2\gamma_2 - m_1\gamma_1) - b\big| < Q^{-1/2}, \quad Q^{\varepsilon} \le r \le Q^{1/2}, \quad (b, r) = 1.$$

Suppose that $\xi \in \mathfrak{M}(q, a)$, where $1 \le q \le P$. Arguing similarly to Case 1, we find that there exist integers a_1, q_1 such that

$$\left|\xi + \lambda_2 - \frac{a_1}{q_1}\right| \le \frac{2P}{q_1 Q^{1/2}}, \quad rP^{-1} \le q_1 \le rP, \quad (a_1, q_1) = 1.$$

Using this rational approximation to $\xi + \lambda_2$, we can now apply Lemma 1 to show that

$$S(\xi + \lambda_2) \ll \big(nq_1^{-1/2} + n^{4/5}P\big)(\log n)^4 \ll n^{1-\varepsilon/3}.$$

We then derive (23) in a similar fashion to Case 1.

We conclude that (23) holds for all vectors $\mathbf{m}$ with $0 < |\mathbf{m}| \le M$. Together, (18), (21) and (23) yield

$$|R(n, \mathbf{m})| \ll n^{k-1}(\log n)^{-A-1}$$

for all $0 < |\mathbf{m}| \le M$, whence

$$\Sigma_1 \ll n^{k-1}(\log n)^{-A-1}(\log M)^k \ll n^{k-1}(\log n)^{-A}. \tag{24}$$

Finally, from (12), (14) and (24),

$$R^+(n) = \frac{\gamma_1 \cdots \gamma_k}{(k-1)!}\mathfrak{S}_k(n)n^{k-1} + O\big(n^{k-1}(\log n)^{-A}\big).$$

Since an analogous asymptotic formula holds for $R^-(n)$, the conclusion of the theorem follows from (8). □

4. Sketch of the Proof of Theorem 2

Let $R^+(n)$ and $R^-(n)$ be the quantities defined in §3 with $k = 2$. To prove Theorem 2 it suffices to establish the inequality

$$\sum_{n \le x} \big|R^{\pm}(n) - \gamma_1\gamma_2\mathfrak{S}_2(n)n\big|^2 \ll x^3(\log x)^{-3A}. \tag{25}$$

As in the proof of Theorem 1, we focus on the proof of the inequality for $R^+(n)$, the proof of the other inequality being similar.

We use the notation introduced in §3 with $k = 2$ and A replaced by $2A + 1$. When $k = 2$, the asymptotic formula (10) is not known, but we do have the upper bound (see [2])

$$R(n, \mathbf{0}) \ll n \prod_{p | n} \left(\frac{p}{p-1}\right) \ll n \log\log n.$$

This bound suffices to show similarly to (9)–(14) that

$$R^+(n) = \gamma_1\gamma_2 R(n, \mathbf{0}) + O\left(n(\log n)^{-2A} + \Sigma_1\right).$$

Furthemore, by [6, Theorem 3.7],

$$\sum_{n \le x} |R(n, \mathbf{0}) - \mathfrak{S}_2(n)n|^2 \ll x^3(\log x)^{-3A}.$$

Thus, (25) for $R^+(n)$ follows from the inequality

$$\sum_{n \le x} \Bigg| \sum_{0<|\mathbf{m}|\le M} |\hat{g}_1^+(m_1)\hat{g}_2^+(m_2)||R(n, \mathbf{m})| \Bigg|^2 \ll x^3(\log x)^{-3A}. \tag{26}$$

By (13) and Cauchy's inequality, the left side of (26) is

$$\ll (\log M)^4 \max_{0<|\mathbf{m}|\le M} \sum_{n\le x} |R(n, \mathbf{m})|^2,$$

so it suffices to show that

$$\sum_{n\le x} |R(n, \mathbf{m})|^2 \ll x(\log x)^{-3A-1} \tag{27}$$

for all $\mathbf{m}$, $0 < |\mathbf{m}| \le M$. By (15) and Bessel's inequality,

$$\sum_{n\le x} |R(n, \mathbf{m})|^2 \le \int_0^1 |S(\xi + \gamma_1 m_1)S(\xi + \gamma_2 m_2)|^2 \, d\xi, \tag{28}$$

where the definition of the exponential sum $S(\xi)$ has been altered to

$$S(\xi) = \sum_{p\le x} (\log p)e(\xi p).$$

We set

$$P = (\log x)^{3A+10}, \quad Q = xP^{-1}, \tag{29}$$

and obtain similarly to (17) that

$$\int_0^1 |S(\xi + \gamma_1 m_1)S(\xi + \gamma_2 m_2)|^2 \, d\xi \le \sum_{q\le Q} \sum_{\substack{1\le a\le q \\ (a,q)=1}} \int_{\mathfrak{M}(q,a)} |S(\xi)S(\xi + \lambda_2)|^2 \, d\xi. \tag{30}$$

As in §3,

$$\min\left(|S(\xi)|, |S(\xi + \lambda_2)|\right) \ll xP^{-1/2}(\log x)^4$$

for all ξ, so we deduce from (20), (29) and (30) that

$$\int_0^1 |S(\xi + \gamma_1 m_1)S(\xi + \gamma_2 m_2)|^2 \, d\xi \ll x^3P^{-1}(\log x)^9 \ll x^2(\log x)^{-3A-1}.$$

Inserting the last bound into (28), we obtain (27). □

5. Closing Remarks

In the proof of Theorem 1, we essentially showed that

$$R(n) = \gamma_1 \cdots \gamma_k R(n, \mathbf{0}) + \text{error terms}, \tag{31}$$

and then chose the parameters Δ, M, P, Q so that the error terms were $\ll n^{k-1}(\log n)^{-A}$. It is possible to alter the above choices so that the error terms in (31) are $\ll n^{k-1-\varepsilon}$ for some $\varepsilon > 0$ which depends only on the α_i's. For example, if $\eta > 1$ is such that each of the inequalities

$$\|q_1\alpha_1^{-1} + q_2\alpha_2^{-1}\| < \max(1, |q_1|, |q_2|)^{-\eta}, \qquad \|q\alpha_i\| < |q|^{-\eta} \quad (1 \le i \le k),$$

has finitely many solutions, then one may choose

$$\Delta = n^{\varepsilon}, \quad M = \Delta^{-1} n^{\varepsilon} = n^{2\varepsilon}, \quad P = n^{3\varepsilon}, \quad Q = nP^{-1},$$

with $0 < \varepsilon < (20\eta)^{-1}$. Thus, the quality of the error term in (4) is determined solely by the quality of the error term in the asymptotic formula (10) for the number of representations of an integer n as the sum of k primes. However, since no improvements on (10) are known, the improved bounds for the error terms in (31) have no effect on Theorem 1.

Similarly, when $k = 2$, a slight alteration of our choices in §4 yields the bound

$$R(n) = \gamma_1\gamma_2 R(n, \mathbf{0}) + O\big(n^{1-\varepsilon}\big) \tag{32}$$

for all but $O(x^{1-\varepsilon})$ values of $n \le x$. In this case, however, such a variation has a tangible effect: it yields Corollary 3. Indeed, by a well-known result of Montgomery and Vaughan [5], there is an absolute constant $\omega < 1$ such that the right side of (32) is positive for all but $O(x^{\omega})$ even integers $n \le x$.

Finally, a comment regarding our finite type hypotheses. We say that an s-tuple $\theta_1, \ldots, \theta_s$ of real numbers *is of subexponential type*, if for each fixed $\eta > 0$, the inequality

$$\|q_1\theta_1 + \cdots + q_s\theta_s\| < \exp(-|\mathbf{q}|^{\eta})$$

has only finitely many solutions $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{Z}^s$. Clearly, every s-tuple of a finite type is also of subexponential type, but not vice versa. It takes little effort to check that in the arguments in §3 and §4, it suffices to assume that each α_i and some pair $\alpha_i^{-1}, \alpha_j^{-1}$ are of subexponential type. Thus, our method reaches some Beatty sequences $\mathcal{B}_{\alpha,\beta}$ with α of an infinite type. On the other hand, under the weaker subexponential type hypotheses, we no longer have the improved remainder estimates in (31) and (32). In particular, we no longer have Corollary 3 (at least, not by the simple argument sketched above). This seems to be too steep a price to pay for such a modest gain in generality.

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