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ON K-IMPERFECT NUMBERS

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Abstract

A positive integer n is called a k-imperfect number if $k\rho(n) = n$ for some integer $k \geqslant 2$, where ρ is a multiplicative arithmetic function defined by $\rho(p^a) = p^a - p^{a-1} + p^{a-2} - \cdots + (-1)^a$ for a prime power p^a. In this paper, we prove that every odd k-imperfect number greater than 1 must be divisible by a prime greater than 10^2, give all k-imperfect numbers less than $2^{32} = 4\,294\,967\,296$, and give several necessary conditions for the existence of an odd k-imperfect number.

1. Introduction

Let $\sigma(n)$ be the sum of the positive divisors of a natural number n. Then n is said to be perfect if and only if $\sigma(n) = 2n$. Iannucci [4] defines a multiplicative arithmetic function ρ by $\rho(1) = 1$ and

$$\rho(p^a) = p^a - p^{a-1} + p^{a-2} - \cdots + (-1)^a \tag{1}$$

for a prime p and integer $a \geqslant 1$; it is a variation of the σ function. It follows that $\rho(n) \leqslant n$ with equality only for $n = 1$. He says that n is imperfect if $2\rho(n) = n$, and says n is k-imperfect if $k\rho(n) = n$ for a natural number k. He considers the function H, defined for natural numbers n, by

$$H(n) = \frac{n}{\rho(n)}. \tag{2}$$

Therefore n is a k-imperfect number if $H(n) = k$.

In fact, Martin [2] introduced the function ρ at the 1999 Western Number Theory Conference. He actually used the symbol $\widetilde{\sigma}$ by which to refer to ρ, and raised three questions (see Guy [3], p.72):

(1) Are there k-imperfect numbers with $k \geqslant 4$?
(2) Are there infinitely many k-imperfect numbers?
(3) Are there any odd 3-imperfect numbers?

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Iannucci gives several necessary conditions for odd 3-imperfect numbers and lists all k-imperfect numbers up to 10^9; these k-imperfect numbers are all even. If we can find an odd k-imperfect number, then Question (1) can be answered. In fact, if n is an odd k-imperfect number, since $H(2) = 2$, then $H(2n) = 2k \geqslant 4$.

In this paper, we prove that every odd k-imperfect number greater than 1 must be divisible by a prime greater than 10^2 and give all k-imperfect numbers less than $2^{32} = 4\,294\,967\,296$. We also give several necessary conditions for the existence of odd k-imperfect numbers.

2. Lemmas

For the remainder of this paper, p, q, and b, with or without subscripts, shall represent odd primes. We shall use a, d, d', r, e, and m to represent positive integers.

If $p \nmid a$ we let $\mathrm{ord}_p a$ denote the order of $a \in (\mathbb{Z}/p\mathbb{Z})^*$. We write $p^a \parallel n$ if $p^a \mid n$ and $p^{a+1} \nmid n$. We denote the n^{th} cyclotomic polynomial, evaluated at x, by $\Phi_n(x)$. From (1), we have

$$\rho(p^{2a}) = \frac{p^{2a+1}+1}{p+1}, \quad \rho(p^{2a+1}) = \frac{p^{2(a+1)}-1}{p+1},$$

and from the cyclotomic identity [5, Proposition 13.2.2]

$$x^n - 1 = \prod_{d|n} \Phi_d(x), \tag{3}$$

we have

$$\rho(p^{2a}) = \prod_{\substack{d|2a+1 \\ d>1}} \Phi_{2d}(p), \quad \rho(p^{2a+1}) = \prod_{\substack{d|2(a+1) \\ d\neq 2}} \Phi_d(p). \tag{4}$$

From Theorems 94 and 95 in Nagell [7], we have the following lemma.

Lemma 1. *Let $h = \mathrm{ord}_q(a)$. Then $q \mid \Phi_m(a)$ if and only if $m = hq^r$. If $r > 0$ then $q \parallel \Phi_{hq^r}(a)$.*

From Lemma 1, we easily obtain the fact: if $q \mid \Phi_m(a)$, then $q \mid m$ or $q \equiv 1 (\mathrm{mod}\ m)$. In the former (resp. latter) case, we say that q is intrinsic (resp. primitive). These terms were used by Murata and Pomerance [6].

Assume $n = \prod_{i=1}^{t} p_i^{\alpha_i} \prod_{j=1}^{s-t} p_j^{\beta_j}$, where α_i is even and β_j is odd. From (2) and (3) we have

$$H(n) \cdot \left(\prod_{i=1}^{t} \prod_{\substack{d|\alpha_i+1 \\ d>1}} \Phi_{2d}(p_i) \right) \cdot \left(\prod_{j=1}^{s-t} \prod_{\substack{d'|\beta_j+1 \\ d'\neq 2}} \Phi_{d'}(p_j) \right) = \prod_{i=1}^{t} p_i^{\alpha_i} \cdot \prod_{j=1}^{s-t} p_j^{\beta_j}. \tag{5}$$

By a result of Bang [1], we have the following lemma.

Lemma 2. *$\Phi_d(a)$ has no primitive prime factors if and only if $d = 2, a = 2^e - 1$, or $d = 6, a = 2$.*

Lemma 3. *If n is a squarefree k-imperfect number, then $n = 2$ or 6.*

Proof. Let $n = \prod_{i=1}^{s} p_i$ $(p_1 < p_2 < \cdots < p_s)$. Then $\rho(n) = \prod_{i=1}^{s}(p_i - 1)$. If $p_1 > 2$, then n is odd and $\rho(n)$ is even; thus $\rho(n) \nmid n$, a contradiction. Thus $p_1 = 2$. If $s \geqslant 3$, then $4 \mid n$ and thus $2^2 \mid n$, a contradiction. If $s = 1$, then $n = p_1 = 2$. If $s = 2$, since $\rho(n) \mid n$, we have $p_2 - 1 \mid 2p_2$; thus $p_2 = 3$, so that $n = 6$. □

Lemma 4. *Let p be a prime and $\Phi_{2p}(x)$ denote the $2p^{\text{th}}$ cyclostomic polynomial. We have $\Phi_{2p}(x) = x^{p-1} - x^{p-2} + x^{p-3} - \cdots + 1$.*

Proof. By (3), we have
$$\Phi_p(x) = \frac{x^p - 1}{\Phi_1(x)}.$$
Then
$$\Phi_{2p}(x) = \frac{x^{2p} - 1}{\Phi_1(x)\Phi_2(x)\Phi_p(x)} = x^{p-1} - x^{p-2} + x^{p-3} - \cdots + 1.$$ □

By Lemma 4, trial division, and the help of a computer, we easily obtain the following lemma and Table 1.

Lemma 5. *Suppose that $3 \leqslant p \leqslant 97, 3 \leqslant q \leqslant 41, b \leqslant 100$ and $b^m \parallel \Phi_q(p)$. All such prime powers b^m are given by the table below. If $3 \leqslant p \leqslant 97$ and $q = 43, 47$, then $\Phi_{2q}(p)$ has no prime factors less than 100. Moreover, if*

$$(p, q) \in \Big\{(3,3), (3,5), (5,3), (7,3), (11,3), (17,3), (19,3), (23,3), (31,3), (37,3)\Big\},$$

then all prime factors of $\Phi_{2q}(p)$ are less than 100.

Lemma 6. *[4,* Theorem 6*] An odd 3-imperfect number contains at least 18 distinct prime divisors.*

3. Main Results and Proofs

Let n denote an odd k-imperfect number. For an odd prime p, it is clear from (1) that $\rho(p^a)$ is odd if and only if a is even. Therefore n is a square, and we may assume
$$n = p_1^{2\alpha_1} p_2^{2\alpha_2} \cdots p_s^{2\alpha_s}.$$

p	$q=3$	$q=5$	$q=7$	$q=11$	$q=13$	$q=17$	$q=19$	$q=23$	$q=29$	$q=31$	$q=37$	$q=41$
3	7	61		67								
5	$3,7$		29	$23,67$				47				83
7	43	11		23	53							
11	$3,37$			$23,89$	53			47	59			
13		11	$7,29$					47	59			83
17	$3,7,13$	$11,71$		23	$53,79$							
19	7^3	$5,11$		23				47				83
23	$3,13^2$	$31,41$	71					47	59			
29	3	$5,11,31$			53			47				
31	$7^2,19$	41						47	59			
37	$31,43$			23	53		19		59			
41	3	$11,61$	$7,71$		79			47				
43	13			$11,23,67$	53			47	59			83
47	$3,7$								59			83
53	3			$23,67$								83
59	$3,7$	5			53							
61	7	11		23	79				59	31		
67			29	23		17		47^2	59			83
71	3		29		79							83
73	7	11		89				47	59		37	83
79		$5,11$		23								83
83	3	11	7	23					59			
89	$3,7$	$5,31$		23					59			83
97	67		$7,71$	23					59			83

Table 1: All Prime Powers b^m of $\Phi_{2q}(p)$ with $b^m \parallel \Phi_{2q}(p)$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are positive integers. From (4) we have

$$H(n) \cdot \prod_{i=1}^{s} \prod_{\substack{d \mid 2\alpha_i+1 \\ d>1}} \Phi_{2d}(p_i) = \prod_{i=1}^{s} p_i^{2\alpha_i}. \tag{6}$$

Proposition 7. *Let* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ *be a* k*-imperfect number greater than* 6*, and* $p = \max(p_i)$*. Then* $\max(\alpha_i) \leqslant p-1$*. Moreover, if* n *is odd then* $\max(\alpha_i) \leqslant \frac{p-3}{2}$*.*

Proof. Let $\alpha = \max(\alpha_i)$. Lemma 3 implies $\alpha > 1$. Assume that $\alpha = 5$ and that the required inequality does not hold. Then it is necessary that $n = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}$ with $0 \leqslant \alpha_i \leqslant 5$ $(i = 1, 2, 3)$, $\max(\alpha_i) = 5$. Thus $n \leqslant 2^5 3^5 5^5 < 10^9$. But we find no such k-imperfect numbers n in [4, Table 1]. Hence we can assume that $\alpha \neq 1, 5$, so $\Phi_{\alpha+1}(p_j)$ or $\Phi_{2(\alpha+1)}(p_j)$ has a primitive prime divisor b from Lemma 2 ($\Phi_6(2) = 3$). Assume that $\alpha > p-1$; then $b \geqslant \alpha+2 > p$, a contradiction. For odd n, if $\alpha > \frac{p-3}{2}$, then $b \geqslant 2(\alpha+1)+1 > p$, a contradiction. □

Theorem 8. *Every odd* k*-imperfect number greater than* 1 *must be divisible by a prime greater than* 100*.*

Proof. Suppose that $n = \prod_{i=1}^{r} p_i^{e_i}$ is an odd k-imperfect number and $p_i < 100$ ($i = 1, 2, \ldots, r$). Then the left-hand side of (6) has no prime factors greater than 100 and e_i ($i = 1, 2, \ldots, r$) are even. From Proposition 1, we have $\max(e_i) \leqslant \frac{\max(p_i)-3}{2} \leqslant 47$. Therefore, it is necessary that the largest prime factor of $e_i + 1$ ($i = 1, 2, \ldots, r$) is at most 5 from Lemma 5.

By Lemma 5, we know that $p_i \in P = \{3, 5, 7, 11, 17, 19, 23, 31, 37\}$ for all $i \in \{1, 2, \ldots, r\}$. If $p_i = 3$, then by Lemma 5 and Table 1, we have $7 \mid n$, then $43 \mid n$. Thus n has a prime factor greater than 100 from Lemma 5, a contradiction. In this way, we have $p_i \notin P$ and thus $n = 1$, a contradiction. □

Theorem 9. *An odd k-imperfect number ($k \geqslant 3$) contains at least 18 distinct prime divisors.*

Proof. Suppose n is an odd k-imperfect number ($k \geqslant 3$). Then we may assume

$$n = q_1^{2\alpha_1} q_2^{2\alpha_2} \cdots q_s^{2\alpha_s}.$$

From (2), we have

$$H(p^{2a}) = \frac{p^{2a}(p+1)}{p^{2a+1}+1} = \frac{p+1}{p+\frac{1}{p^{2a}}} < \frac{p+1}{p},$$

and

$$\begin{aligned} H(p^{2a}) &= \frac{p^{2a}}{p^{2a} - p^{2a-1} + p^{2a-2} - \cdots + 1} = \frac{1}{1 - \frac{1}{p} + \frac{1}{p^2} - \cdots + \frac{1}{p^{2a}}} \\ &\geqslant \frac{1}{1 - \frac{1}{p} + \frac{1}{p^2}} = \frac{p^2}{p^2 - p + 1}. \end{aligned}$$

On the other hand, if $p < q$, then $\frac{q+1}{q} < \frac{p^2}{p^2-p+1}$, and so for any positive integers a, b, from (7), we have

$$H(q^{2b}) < H(p^{2a}). \tag{7}$$

Therefore

$$H(n) = \prod_{i=1}^{s} H(q_i^{2\alpha_i}) < \prod_{i=1}^{s} H(p_i^{2a}) < \prod_{i=1}^{s} \frac{p_i+1}{p_i},$$

where p_i is the $(i+1)^{\text{th}}$ prime. Since

$$\frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \frac{32}{31} \cdot \frac{38}{37} \cdot \frac{42}{41} \cdot \frac{44}{43} \cdot \frac{48}{47} \cdot \frac{54}{53} \cdot \frac{60}{59} \cdot \frac{62}{61} \cdot \frac{68}{67} < 5$$

and by Lemma 6, we have $s \geqslant 18$. □

Corollary 10. *If n is an odd k-imperfect number ($k \geqslant 3$), then $n > 3.4391411 \times 10^{49}$.*

Proof. From Theorems 1 and 2 and inequality (8), we have

$$\begin{aligned} n &\geqslant (3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61)^2 \cdot 10^4 \\ &> 3.4391411 \times 10^{49}. \end{aligned}$$

□

Clearly, if n is a k-imperfect number then, writing $n = 2^a m$ ($a > 0$), we have $\rho(2^a) \mid m$. From Corollary 1, if n is a k-imperfect number and $n < 3.4391411 \times 10^{49}$, then n must be even. Therefore if we want to find all k-imperfect numbers less than 3.4391411×10^{49}, we check only even numbers. A computer search produced all k-imperfect numbers less than $2^{32} = 4\,294\,967\,296$, there are in thirty-eight such numbers, including the thirty-three numbers less than 10^9 found in [4]; the five new numbers found by us are:

$$\begin{aligned} 1665709920 &= 2^5 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 43 \cdot 61, & H(n) &= 3; \\ 1881532800 &= 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17 \cdot 61, & H(n) &= 3; \\ 2082137400 &= 2^3 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 43 \cdot 61, & H(n) &= 3; \\ 2147450880 &= 2^{15} \cdot 3 \cdot 5 \cdot 17 \cdot 257, & H(n) &= 3; \\ 3094761600 &= 2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 17 \cdot 43, & H(n) &= 3. \end{aligned}$$

Proposition 11. *If n is a k-imperfect number, then $\omega(n) \geqslant k - 1$, where $\omega(n)$ denotes the number of distinct prime factors of n.*

Proof. Write

$$n = \prod_{\substack{i=1 \\ 2 \nmid \alpha_i}}^{t} p_i^{\alpha_i} \cdot \prod_{\substack{j=1 \\ 2 \mid \beta_j}}^{\omega(n)-t} p_j^{\beta_j},$$

where α_i, β_j are positive integers and p_i, p_j are primes. Since

$$\frac{p^\alpha (p+1)}{p^{\alpha+1}+1} < \frac{p^\alpha (p+1)}{p^{\alpha+1}-1} = \frac{p+1}{p - \frac{1}{p^\alpha}} \leqslant \frac{p+1}{p - \frac{1}{p}} = \frac{p}{p-1},$$

where α is a positive integer, we have

$$\begin{aligned} k &= H(n) = \prod_{\substack{i=1 \\ 2 \nmid \alpha_i}}^{t} \frac{p_i^{\alpha_i}(p_i+1)}{p_i^{\alpha_i+1}-1} \cdot \prod_{\substack{j=1 \\ 2 \mid \beta_j}}^{\omega(n)-t} \frac{p_j^{\beta_j}(p_j+1)}{p_j^{\beta_j+1}+1} \leqslant \prod_{r=1}^{\omega(n)} \frac{p_r}{p_r - 1} \\ &\leqslant \prod_{i=2}^{\omega(n)+1} \frac{i}{i-1} = \omega(n) + 1, \end{aligned}$$

and thus $\omega(n) \geqslant k - 1$. □

Theorem 12. *Suppose n is k_1-imperfect and $n \cdot q_1 q_2 \cdots q_t$ is k_2-imperfect, where $q_1 < q_2 < \cdots < q_t$ are primes not dividing n and $k_1, k_2 \geqslant 2$. Then $n \cdot q_1$ is k_3-imperfect with $k_3 \geqslant 2$, except when $t \geqslant 2$ and $q_1 q_2 = 6$, in which case $n \cdot q_1 q_2$ is $3k_1$-imperfect . Furthermore, if $n \cdot q_1$ is k-imperfect, then $q_1 \leqslant H(n) + 1$.*

Proof. We may assume $t \geqslant 2$. Suppose $q_1 \geqslant 3$. Since $n \cdot q_1 q_2 \cdots q_t$ is k_2-imperfect and H is multiplicative, we have

$$H(n \cdot q_1 q_2 \cdots q_t) = H(n) \prod_{i=1}^{t} H(q_i) = H(n) \prod_{i=1}^{t} \frac{q_i}{q_i - 1} = k_2.$$

Then

$$H(n) \prod_{i=1}^{t} q_i = k_2 \prod_{i=1}^{t} (q_i - 1).$$

Since $q_1 - 1 < q_2 - 1 < \cdots < q_t$, we have $q_t \mid k_2$, and then

$$H(n \cdot q_1 q_2 \cdots q_{t-1}) = H(n) \prod_{i=1}^{t-1} \frac{q_i}{q_i - 1} = \frac{k_2}{q_t} \cdot (q_t - 1).$$

Let $\frac{k_2}{q_t} \cdot (q_t - 1) = k_4$ ($k_4 \geqslant 2$). Applying the same argument to the k_4-imperfect number $n \cdot q_1 q_2 \cdots q_{t-1}$, and repeating it as necessary, leads to our result in this case. If $q_1 q_2 = 6$, then $H(n \cdot q_1 q_2) = 3k_1$. Thus $n \cdot q_1 q_2$ is $3k_1$-imperfect. If $n \cdot q_1$ is k-imperfect then $H(nq_1) = H(n)\frac{q_1}{q_1 - 1}$. Then we have $q_1 - 1 \mid H(n)$ and so $q_1 \leqslant H(n) + 1$. □

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