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ON THE BOUNDARY OF THE SET OF THE CLOSURE OF CONTRACTIVE POLYNOMIALS

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Abstract

For $(r_1, \dots, r_d)^T = \mathbf{r} \in \mathbb{R}^d$ and $\mathbf{a} = (a_1, \dots, a_d)^T \in \mathbb{Z}^d$, let $\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{r}^T \mathbf{a} \rfloor)^T$. Further, let $a_{d+k+1} = -\lfloor \mathbf{r}^T \tau_{\mathbf{r}}^k(\mathbf{a}) \rfloor$. In this paper we prove that if some roots of the polynomial $X^d + r_d X^{d-1} + \dots + r_2 X + r_1$ are t-th roots of unity and the others lie in the open unit disc, then $|a_{k+t} - a_k| < c_1$ with a constant c_1 which does not depend on k. Under some conditions this yields an algorithm to decide whether the sequence $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}$ is, for all $\mathbf{a}$, ultimately periodic, or becomes divergent for some $\mathbf{a}$. We study the boundary of the closure of $\mathcal{D}_3$ and show that large parts of it belong to $\mathcal{D}_3$, while others lie outside $\mathcal{D}_3$.

1. Introduction

Let[2] $(r_1, \dots, r_d)^T = \mathbf{r} \in \mathbb{R}^d$. Akiyama, Borbély, Brunotte, Thuswaldner and the author introduced [1] the nearly linear mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \mapsto \mathbb{Z}^d$ such that if $\mathbf{a} = (a_1, \dots, a_d)^T \in \mathbb{Z}^d$ then

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{r}^T \mathbf{a} \rfloor)^T. \tag{1}$$

For $k \geq 0$ let

$$\tau^k(\mathbf{a}) = \begin{cases} \mathbf{a} & \text{if } k = 0, \\ \tau\left(\tau^{k-1}(\mathbf{a})\right) & \text{if } k > 0, \end{cases}$$

and $a_{d+k+1} = -\lfloor \mathbf{r}^T \tau_{\mathbf{r}}^k(\mathbf{a}) \rfloor$. They also defined the sets

$$\mathcal{D}_d = \{\mathbf{r} \; : \; \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k=0}^{\infty} \quad \text{is bounded for all} \quad \mathbf{a} \in \mathbb{Z}^d\}$$

and $\mathcal{E}_d$, which is the set of real monic polynomials, whose roots lie in the closed unit disc. They proved in the same paper that if $\mathbf{r} \in \mathcal{D}_d$ then $R(X) = X^d + r_d X^{d-1} + \dots + r_2 X + r_1 \in \mathcal{E}_d$, and if $R(X)$ lies in the interior of $\mathcal{E}_d$ then $\mathbf{r} \in \mathcal{D}_d$.

It is natural to ask what happens if $R(X)$ belongs to the boundary of $\mathcal{E}_d$, i.e., some of its roots lies on the unit circle. The case $d = 2$ was studied by Akiyama *et al.*

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[2]In this note a vector is always a column vector and $\mathbf{v}^T$ means its transpose.

in [4], but they were not able to completely settle it. They proved that $\mathcal{D}_2$ is equal to the closed triangle with vertices $(-1,0),(1,-2),(1,2)$, but without the points $(1,-2),(1,2)$, the line segment $\{(x,-x-1) \ : \ 0<x<1\}$ and, possibly, some points of the line segment $\{(1,y) \ : \ -2<y<2\}$. In the last case write $y=2\cos\alpha$ and $\omega=\cos\alpha+i\sin\alpha$. It is easy to see, that if $y=0,\pm1$ (i.e., $\alpha=0,\pm\pi/2$) then $(1,y)$ belongs to $\mathcal{D}_2$; we conjectured in [4] that this is true for all points of the line segment. In [3] the conjecture was proved for the golden mean, i.e., for $y=\frac{1+\sqrt{5}}{2}$; in [2] the conjecture was proved for those ω which are quadratic algebraic numbers.

Kirschenhofer, Pethő and Thuswaldner [5] studied the sequences $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}$ for $\mathbf{r}=(1,\lambda^2,\lambda^2)$, where λ denotes the golden mean. They not only proved that $\mathbf{r}\notin\mathcal{D}_3$, but found some connection between the Zeckendorf expansion of the coordinates of the initial vector $\mathbf{a}$ and the periodicity of $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}$.

In the present notes we continue the above investigations about the boundary of $\mathcal{E}_d$ for $d\geq 3$ in a systematic way. Our most general result is

Theorem 1. *Assume that some t-th roots of unity $\beta_1,\dots,\beta_s$ are simple zeroes of $R(X)$ and the other zeroes of it have modulus less than one. Then there exist constants c_1 depending on $\beta_1,\dots,\beta_s$ and c_2 depending on $\beta_1,\dots,\beta_s$ and $a_1,\dots,a_d$ such that if $k>c_2$ then*

$$|a_{k+t}-a_k|<c_1.$$

Further, if t is even and $\beta_1,\dots,\beta_s$ are primitive t-th roots of unity, then

$$|a_{k+t/2}+a_k|<c_1$$

holds as well.

The importance of Theorem 1 is that c_1 does not depend on the initial vector $\mathbf{a}$; in other words, the sequence $\{\tau_{\mathbf{r}}^n(\mathbf{a})\}$ is the union of a finite set and finitely many sequences of at most linear growth.

Define the integral vectors $\mathbf{1}=(1,\dots,1)^T$, $\bar{\mathbf{1}}=(1,-1,\dots,(-1)^{d-1})^T$, $\mathbf{i}=(1,0,-1,0,\dots)$ and $\bar{\mathbf{i}}=(0,1,0,-1,\dots)$. As a consequence of Theorem 1 we prove

Theorem 2. *Assume that $1,-1$ or i is a simple zero of $R(X)$ and the other zeroes of it have modulus less than one. Then there exists a computable finite set $A\subset\mathbb{Z}^d$ with the following property: for all $\mathbf{a}\in\mathbb{Z}^d$ there exist an integer k depending on the zeroes of $R(X)$ and $\mathbf{a}$ and integers L,K such that $\tau_{\mathbf{r}}^k(\mathbf{a}-L\mathbf{1})\in A,\tau_{\mathbf{r}}^k(\mathbf{a}-L\bar{\mathbf{1}})\in A$ and $\tau_{\mathbf{r}}^k(\mathbf{a}-L\mathbf{i}-K\bar{\mathbf{i}})\in A$, respectively.*

Theorem 2 immediately implies an algorithm to test $\mathbf{r}\in\mathcal{D}_d$ provided that $1,-1$ or i is a simple root of $R(X)$. Of course we have to test for all $\mathbf{a}\in A$ whether the sequence $\{\tau_{\mathbf{r}}^n(\mathbf{a})\}$ is ultimately periodic or divergent. We show that for $d=3$ both cases occur.

By a recent result of Paul Surer [10] the boundary of $\mathcal{E}_3$ can be parametrized by the union of the sets $B_1 = \{(-s, s-(s+1)t, (s+1)t-1) \ : \ -1 \leq s, t \leq 1\}$, $B_2 = \{(s, s+(s+1)t, (s+1)t+1) \ : \ -1 \leq s, t \leq 1\}$ and $B_3 = \{(v, 1+2tv, 2t+v) \ : \ -1 \leq t, v \leq 1\}$. We prove that large portions of B_1 belong to $\mathcal{D}_3$ and others do not belong to $\mathcal{D}_3$. For example, if $0 \leq (s+1)(t+1) < 1$ and $a_0 = 0, a_1 = 1, a_2 = 2$ then $\tau_{\mathbf{r}}^k(\mathbf{a}) = (k, k+1, k+2)$ holds for all k, i.e., $\mathbf{r} \notin \mathcal{D}_3$. On the other hand, if $s \geq 0, s \leq (s+1)t \leq 1$ then $\{\tau_{\mathbf{r}}^n(\mathbf{a})\}$ is ultimately constant, i.e., $\mathbf{r} \in \mathcal{D}_3$. Experiments show that these examples are typical for elements both of B_1 and B_2. This means that if $\{\tau_{\mathbf{r}}^n(\mathbf{a})\}$ is ultimately periodic then its period length is short, usually the sequence is ultimately constant. On the other hand, if $\{\tau_{\mathbf{r}}^n(\mathbf{a})\}$ is divergent then it is ultimately an arithmetical progression.

Choosing the values $s = 1, t = \frac{\lambda}{2}, v = 1$ shows that the point $\mathbf{r} = (1, \lambda^2, \lambda^2)^T$ studied in [5] belongs to $B_2 \cap B_3$.

2. Preparatory Results

To prove Theorem 1 we need some preparation from linear algebra and from linear recurring sequences. We recapitulate here with minor changes Chapter 2 of [7], because we need the notations in the sequel. First of all, we analyze the mapping $\tau = \tau_{\mathbf{r}}$ defined by Equation (1). Let $\mathbf{P} = \mathbf{P}(\mathbf{r}) \in \mathbb{Z}^{d \times d}$ be the companion matrix of $R(X)$, i.e.,

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -r_1 & -r_2 & \dots & -r_d \end{pmatrix}.$$

With this definition we have the following assertion.

Lemma 3. *Let* $\mathbf{a} = (a_1, \dots, a_d)^T \in \mathbb{Z}^d$ *and* $1 \leq k \in \mathbb{Z}$. *Then there exist* $-1 < \delta_1, \dots, \delta_k \leq 0$ *such that*

$$\tau^k(\mathbf{a}) = \mathbf{P}^k \mathbf{a} + \sum_{j=1}^{k} \mathbf{P}^{k-j} \boldsymbol{\delta}_j$$

holds, where $\boldsymbol{\delta}_j = (0, \dots, 0, \delta_j)^T \in \mathbb{R}^d$.

Proof. See the simple proof of Lemma 2 of [7]. □

Let $\{G_n\}_{n=0}^{\infty}$ be the linear recurring sequence defined by the initial terms $G_0 = \cdots = G_{d-2} = 0$, $G_{d-1} = 1$ and by the difference equation

$$G_{n+d} = -r_d G_{n+d-1} - \cdots - r_1 G_n. \tag{2}$$

Further, let $\mathbf{G}_n = (G_n, \dots, G_{n+d-1})^T$ and for $n \geq 0$ denote by $\mathcal{G}_n$ the $d \times d$ matrix, whose columns are $\mathbf{G}_n, \dots, \mathbf{G}_{n-d+1}$. Then we obviously have

$$\mathcal{G}_n = \mathbf{P}\mathcal{G}_{n-1} \quad \text{for} \quad n = 1, 2, \dots .$$

This implies

$$\mathcal{G}_n = \mathbf{P}^n \; \mathcal{G}_0 \qquad \text{for} \quad n \geq 0. \tag{3}$$

As

$$\mathcal{G}_0 = \begin{pmatrix} G_{d-1} & G_{d-2} & \dots & G_0 \\ G_d & G_{d-1} & \dots & G_1 \\ \vdots & \vdots & \ddots & \vdots \\ G_{2d-1} & G_{2d-2} & \dots & G_{d-1} \end{pmatrix}$$

is a lower triangular matrix with entries 1 in the main diagonal, it is non-singular and its inverse is

$$\mathcal{G}_0^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ r_d & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_3 & r_4 & \dots & 1 & 0 \\ r_2 & r_3 & \dots & r_d & 1 \end{pmatrix}.$$

Thus we get

$$\mathbf{P}^n = \mathcal{G}_n \mathcal{G}_0^{-1}. \tag{4}$$

Denoting by $p_{ij}^{(n)}$, $1 \leq i,\ j \leq d$, $n \geq 0$ the entries of $\mathbf{P}^n$ and setting $r_{d+1} = 1$ we obtain

$$p_{1j}^{(n)} = \sum_{u=0}^{d-j} r_{j+u+1} G_{n+u},\ j = 1, \dots, d, \tag{5}$$

in particular $p_{1d}^{(n)} = G_n$.

As a_{k+1} is the first coordinate of $\tau^k(\mathbf{a})$, Lemma 3 and (5) imply

$$a_{k+1} = \sum_{j=1}^{d} p_{1j}^{(k)} a_j + \sum_{j=1}^{k} p_{1d}^{(k-j)} \delta_j = \sum_{j=1}^{d} p_{1j}^{(k)} a_j + \sum_{j=1}^{k} G_{k-j} \delta_j. \tag{6}$$

On the other hand, if $\beta_1, \dots, \beta_h$ denote the distinct zeroes of the polynomial $R(X) = X^d + r_d X^{d-1} + \dots + r_1$ with multiplicities $e_1, \dots, e_h \geq 1$, respectively, then

$$G_n = g_1(n)\beta_1^n + \dots + g_h(n)\beta_h^n \tag{7}$$

holds for any $n \geq 0$. Here $g_i(X)$, $1 \leq i \leq h$, are polynomials with coefficients of the field $\mathbb{Q}(\beta_1, \dots, \beta_h)$ of degrees at most $e_i - 1$. (See e.g. [8].)

Equations (5) and (7) imply that there exist polynomials $g_{ij\ell}(X)$ with coefficients of the field $\mathbb{Q}(\beta_1, \ldots, \beta_h)$ of degrees at most $e_\ell - 1$ such that

$$p_{ij}^{(n)} = \sum_{\ell=1}^{h} g_{ij\ell}(n)\beta_\ell^n. \tag{8}$$

Using this equality, by (7) and (6) we obtain

$$a_{k+1} = \sum_{j=1}^{d} a_j \sum_{\ell=1}^{h} g_{1j\ell}(k)\beta_\ell^k + \sum_{j=1}^{k} \delta_j \sum_{\ell=1}^{h} g_\ell(k-j)\beta_\ell^{k-j}. \tag{9}$$

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Our starting point is Equation (9). It was used in a simpler form in [1] to prove that if all roots of $R(X)$ have modulus less than one, then $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}$ is ultimately periodic. This is true, because both summands in (9) are bounded. However, if one of the roots of $R(X)$ lies on the unit circle, then we usually have no control on the second summand – it can be bounded or unbounded. A closer look at (9) makes it possible to prove our theorem.

Let $t \geq 1$. Then Equation (9) implies

$$\begin{aligned} a_{k+t+1} - a_{k+1} \quad = \quad & \sum_{\ell=1}^{h} \beta_\ell^k \sum_{j=1}^{d} a_j \left(\beta_\ell^t g_{1j\ell}(k+t) - g_{1j\ell}(k)\right) \\ & + \sum_{j=k+1}^{k+t} \delta_j \sum_{\ell=1}^{h} g_\ell(k+t-j)\beta_\ell^{k+t-j} \\ & + \sum_{j=1}^{k} \delta_j \sum_{\ell=1}^{h} \beta_\ell^{k-j} \left(g_\ell(k+t-j)\beta_\ell^t - g_\ell(k-j)\right). \end{aligned}$$

As $\beta_1, \ldots, \beta_s$ are t-th roots of unity, we have $\beta_i^t = 1, i = 1, \ldots, s$. Further, as they are simple zeroes of $R(X)$, the polynomials $g_{1j\ell}(X), j = 1, \ldots, d, \ell = 1, \ldots, s$, and $g_\ell(X), \ell = 1, \ldots, s$, are constants depending only on $\beta_1, \ldots, \beta_h$. Thus

$$\beta_\ell^t g_{1j\ell}(k+t) - g_{1j\ell}(k) = g_\ell(k+t-j)\beta_\ell^t - g_\ell(k-j) = 0 \tag{10}$$

for all $\ell = 1, \ldots, s, j = 1, \ldots, d$. Thus our expression for $a_{k+t+1} - a_{k+1}$ simplifies to

$$\begin{aligned}|a_{k+t+1}-a_{k+1}| \quad &\leq \quad \left|\sum_{\ell=s+1}^{h}\beta_\ell^k\sum_{j=1}^{d}a_j\left(\beta_\ell^t g_{1j\ell}(k+t)-g_{1j\ell}(k)\right)\right| \\ &\quad + \left|\sum_{j=k+1}^{k+t}\delta_j\sum_{\ell=1}^{h}g_\ell(k+t-j)\beta_\ell^{k+t-j}\right| \\ &\quad + \left|\sum_{j=1}^{k}\delta_j\sum_{\ell=s+1}^{h}\beta_\ell^{k-j}\left(g_\ell(k+t-j)\beta_\ell^t-g_\ell(k-j)\right)\right|.\end{aligned}$$

Changing j to $j+k$ we can estimate the second summand as follows:

$$\left|\sum_{j=1}^{t}\delta_{j+k}\sum_{\ell=1}^{h}g_\ell(t-j)\beta^{t-j}\right| \leq \sum_{j=0}^{t-1}\sum_{\ell=1}^{h}|g_\ell(j)|.$$

As $|\beta_\ell|<1$ for $\ell=s+1,\dots,h$ and $|\delta_j|<1$ for $j=1,\dots,k$, there exists a constant c_3 depending only on the roots of $R(X)$ and $\mathbf{a}$ such that if $k\geq c_3$, then

$$\left|\beta_\ell^k\sum_{j=1}^{d}a_j\left(g_{1j\ell}(k+t)\beta_\ell^t-g_{1j\ell}(k)\right)\right| < \frac{1}{2h}.$$

By the same reason there exists a constant c_4, depending only on the roots of $R(X)$, such that if $k\geq c_4$, then

$$\left|\sum_{\ell=s+1}^{h}\beta_\ell^k\left(g_\ell(k+t)\beta_\ell^t-g_\ell(k)\right)\right| < |\beta_\ell|^{k/2}.$$

Thus

$$\begin{aligned}|a_{k+t+1}-a_{k+1}| \quad &\leq \quad 1/2+\sum_{j=0}^{t-1}\sum_{\ell=1}^{h}|g_\ell(j)| \\ &\quad + \left|\sum_{j=1}^{k-c_4}\delta_j\sum_{\ell=s+1}^{h}\beta_\ell^{k-j}\left(g_\ell(k+t-j)\beta_\ell^t-g_\ell(k-j)\right)\right| \\ &\quad + \left|\sum_{j=k-c_4+1}^{k}\delta_j\sum_{\ell=s+1}^{h}\beta_\ell^{k-j}\left(g_\ell(k+t-j)\beta_\ell^t-g_\ell(k-j)\right)\right|.\end{aligned}$$

The third summand is bounded by

$$\sum_{j=0}^{\infty}|\beta_\ell^{j/2}| = \frac{1}{1-|\beta_\ell^{1/2}|},$$

while the fourth summand can be estimated as above and we get for it the upper bound

$$\sum_{j=0}^{c_4-1} \sum_{\ell=s+1}^{h} \left| g_\ell(t+j)\beta_\ell^t - g_\ell(j) \right|,$$

which is a constant depending only on the roots of $R(X)$. The sum of these bounds depends only on the roots of $R(X)$ and we can choose it as c_1. To finish the proof of the first statement put $c_2 = \max\{c_3, c_4\}$.

If t is even, we estimate $|a_{k+t/2+1} + a_{k+1}|$ as in the previous case. The only important difference is that we use

$$\beta_\ell^t g_{1j\ell}(k+t) + g_{1j\ell}(k) = g_\ell(k+t-j)\beta_\ell^t + g_\ell(k-j) = 0$$

instead of (10). This is true because $\beta_1 \dots, \beta_s$ are primitive t-th roots of unity, thus $\beta_j^{t/2} = -1, j = 1, \dots, s$. □

Proof of Theorem 2. If $R(1) = 0$ then $\mathbf{r}^T\mathbf{1} = r_1 + \cdots + r_d = -1$, thus $\tau_{\mathbf{r}}(\mathbf{1}) = \mathbf{1}$. Let n be an integer. Then $\mathbf{r}^T(n\mathbf{1}) = nr_1 + \cdots + nr_d = -n$, thus $\tau_{\mathbf{r}}(n\mathbf{1}) = n\mathbf{1}$, i.e., $(n\mathbf{1})$ is a fixed point of $\tau_{\mathbf{r}}$ for all integers n.

We apply Theorem 1 with $t = 1$. Let $\mathbf{a} \in \mathbb{Z}^d$. There exists a constant c_1 such that if k is large enough, then $|a_{k+1} - a_k| < c_1$. Fix such a k and consider d consecutive terms $a_{k+i}, i = 0, \dots, d-1$ of $\{a_n\}$. Put $L = \min\{a_{k+i}, i = 0, \dots, d-1\}$ and assume that $L = a_{k+j}$ for some $j \in [0, d-1]$. If $h \in [0, d-1]$ then $0 \le a_{k+h} - L \le (d-1)c_1$. Indeed, the lower bound holds by the choice of L. To prove the upper bound assume that $h > j$. Then

$$\begin{aligned} a_{k+h} - L &= a_{k+h} - a_{k+j} = a_{k+h} - a_{k+h-1} + \cdots + a_{k+j+1} - a_{k+j} \\ &\le |a_{k+h} - a_{k+h-1}| + \cdots + |a_{k+j+1} - a_{k+j}| \\ &\le (d-1)c_1. \end{aligned}$$

The case $h < j$ can be handled similarly.

Let $\mathbf{b} = \mathbf{a} - L\mathbf{1}$. Then we have

$$\tau_{\mathbf{r}}^u(\mathbf{b}) = \tau_{\mathbf{r}}^u(\mathbf{a}) - \tau_{\mathbf{r}}^u(L\mathbf{1}) = \tau_{\mathbf{r}}^u(\mathbf{a}) - L\mathbf{1}$$

for all $u \ge 0$. Putting $u = k-1$ we get $\tau_{\mathbf{r}}^{k-1}(\mathbf{a} - L\mathbf{1}) = \tau_{\mathbf{r}}^{k-1}(\mathbf{a}) - L\mathbf{1} = (a_k - L, \dots, a_{k+d-1} - L)$. Thus the set $A = \{0, \dots, (d-1)c_1\}^d$ satisfies the assertion.

If $R(-1) = 0$ then $\mathbf{r}^T\bar{\mathbf{1}} = r_1 + r_2(-1) + \cdots + r_d(-1)^{d-1} = (-1)^{d+1}$, thus $\tau_{\mathbf{r}}(\bar{\mathbf{1}}) = (-1)^d\bar{\mathbf{1}}$. This implies that if n is an integer, then $\mathbf{r}^T(n\bar{\mathbf{1}}) = (-1)^d n\bar{\mathbf{1}}$, i.e., $n\bar{\mathbf{1}}$ is a fixed point of $\tau_{\mathbf{r}}$ or $\tau_{\mathbf{r}}^2$ according as d is even or odd. Using that -1 is a primitive second root of unity beside $|a_{k+2}-a_k| < c_1$ we also have $|a_{k+1}+a_k| < c_1$.

The rest of the proof is analogous to the case $R(1) = 0$ and we conclude that $A = \{0, \dots, (2d-1)c_1\}^d$ satisfies the assertion of the theorem.

Finally, if i is a root of $R(X)$, then $R(X) = (X^2+1)(X^{d-2}+q_{d-3}X^{d-3}+\cdots+q_0)$ with $q_{d-3}, \dots, q_0 \in \mathbb{R}$. It is easy to check that if $n, m \in \mathbb{Z}$ and $\mathbf{v} = n\mathbf{i} + m\bar{\mathbf{i}}$, then $\tau_{\mathbf{r}}^4(\mathbf{v}) = \mathbf{v}$. Further, as i is a primitive fourth root of unity we have $|a_{k+4} - a_k| < c_1$ and $|a_{k+2} + a_k| < c_1$. The rest of the proof is analogous again to the case $R(1) = 0$. $\square$

4. The Case $d = 3$

In this section we specialize the results of Theorems 1 and 2 to the case $d = 3$. First we compute $p_{1j}^{(n)}$ using (5) and get $p_{11}^{(n)} = r_2G_n + r_3G_{n+1} + r_4G_{n+2} = -r_1G_{n-1}, p_{12}^{(n)} = r_3G_n + G_{n+1}$ and $p_{13}^{(n)} = G_n$. Inserting these values into (6) we obtain

$$a_{k+1} = -r_1G_{k-1}a_1 + (G_{k+1} + r_3G_k)a_2 + G_ka_3 + \sum_{j=1}^{k} G_{k-j}\delta_j. \tag{11}$$

In the sequel we need the following lemma of M. Ward [9].

Lemma 4. *Let the linear recurring sequence* $\{G_n\}_{n=0}^{\infty}$ *be defined by* (2)*. Assume that* $R(X)$ *is square-free and denote by* $\alpha_1, \dots, \alpha_d$ *its roots. Then*

$$G_n = \sum_{h=1}^{d} \frac{\alpha_h^n}{R'(\alpha_h)},$$

where $R'(X)$ *denotes the derivative of* $R(X)$*.*

By a recent result of Paul Surer [10] the boundary of $\mathcal{E}_3$ is the union of the sets $B_1 = \{(-s, s-(s+1)t, (s+1)t-1) \ : \ -1 \le s, t \le 1\}$, $B_2 = \{(s, s+(s+1)t, (s+1)t+1) \ : \ -1 \le s, t \le 1\}$ and $B_3 = \{(v, 1+2tv, 2t+v) \ : \ -1 \le t, v \le 1\}$.

4.1. The Set B_1

In this case $R(X) = X^3 + ((s+1)t-1)X^2 + (s-(s+1)t)X - s = (X-1)(X^2 + (s+1)tX + s) = (X-1)(X-\alpha)(X-\beta)$. We have

$$(1-\alpha)(1-\beta) = R'(1) = 3 + 2((s+1)t-1) + (s-(s+1)t) = (s+1)(t+1).$$

Using this and Lemma 4 we get

$$\begin{aligned} G_n &= \frac{1}{R'(1)} + \frac{\alpha^n}{R'(\alpha)} + \frac{\beta^n}{R'(\beta)} \\ &= \frac{1}{(s+1)(t+1)} + \frac{\alpha^n(\beta-1) - \beta^n(\alpha-1)}{(\alpha-\beta)(\alpha-1)(\beta-1)} \\ &= \frac{1}{(s+1)(t+1)}\left(1 + \frac{\alpha^n(\beta-1) - \beta^n(\alpha-1)}{\alpha-\beta}\right). \end{aligned}$$

Later we need the difference of two consecutive terms of the sequence $\{G_n\}$, which is

$$\begin{aligned} & G_n - G_{n-1} \\ &= \frac{1}{(s+1)(t+1)}\left(\frac{\alpha^n(\beta-1) - \alpha^{n-1}(\beta-1) - \beta^n(\alpha-1) + \beta^{n-1}(\alpha-1)}{\alpha-\beta}\right) \\ &= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha-\beta}. \end{aligned}$$

Using this expression and (11) for any $k \geq 2$ we obtain

$$\begin{aligned} a_{k+1} - a_k &= sa_1\frac{\alpha^{k-2} - \beta^{k-2}}{\alpha-\beta} + a_2\left(\frac{\alpha^k - \beta^k}{\alpha-\beta} + ((s+1)t-1)\frac{\alpha^{k-1} - \beta^{k-1}}{\alpha-\beta}\right) \\ &+ \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha-\beta}a_3 + \sum_{j=1}^{k-1}\delta_j\frac{\alpha^{k-j-1} - \beta^{k-j-1}}{\alpha-\beta}. \end{aligned}$$

Observe that the summand $G_0\delta_k$ is zero, and therefore it is omitted. We estimate the last summand as

$$\left|\sum_{j=1}^{k-1}\delta_j\frac{\alpha^{k-j-1} - \beta^{k-j-1}}{\alpha-\beta}\right| \leq \frac{1}{|\alpha-\beta|}\left(\frac{1}{1-|\alpha|} + \frac{1}{1-|\beta|}\right).$$

Since $|\alpha|, |\beta| < 1$, the absolute value of the first three summands can be made arbitrarily small by choosing k large enough. Thus we get

Theorem 5. *Assume that* $-1 < s, t < 1$, $\mathbf{r} = (-s, s-(s+1)t, (s+1)t-1)^T$. *Let* α, β *be the roots of* $R(X) = X^3 + ((s+1)t-1)X^2 + (s-(s+1)t)X - s$, *which have modulus less than* 1. *Let*

$$c_{11} = \left\lfloor \frac{1}{|\alpha-\beta|}\left(\frac{1}{1-|\alpha|} + \frac{1}{1-|\beta|}\right)\right\rfloor$$

and $A = A(c_{11}) = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \ : \ 0 \leq x_1 \leq c_{11}, x_1 - c_{11} \leq x_2 \leq x_1 + c_{11}, x_2 - c_{11} \leq x_3 \leq x_2 + c_{11}\}$. *For any* $(a_1, a_2, a_3) \in \mathbb{Z}^3$ *there exist integers* L, k *such that* $\tau_{\mathbf{r}}^k(a_1 - L, a_2 - L, a_3 - L) \in A$.

Later we present an application of Theorem 5. Before that we show that a large part of B_1 does not belong to $\mathcal{D}_3$.

Theorem 6. *Assume that* $-1 < s, t < 1$, $\mathbf{r} = (-s, s-(s+1)t, (s+1)t-1)^T$ *and put* $u = (s+1)t$.

(1) *If* $u < -s$ *and* $\mathbf{a} = (0,1,2)^T$, *then* $a_{n+1} = a_n + 1$ *holds for all* $n \geq 0$.

(2) *If* $u \geq -s$ *and* $s < 0$ *and* $\mathbf{a} = (0,0,1)^T$, *then* $a_3 = 1$ *and* $a_{n+2} = a_n + 1$ *holds for all* $n \geq 0$.

(3) *If* $1 - 2s \leq u < -s/2$ *and* $s > 2/3$ *and* $\mathbf{a} = (0,1,3)^T$, *then* $a_3 = 4, a_4 = 3$ *and* $a_{n+2} = a_n + 2$ *holds for all* $n \geq 0$.

(4) *If* $\frac{s+2}{2} < u < \frac{2s+3}{3}$ *and* $s > 3/4$ *and* $\mathbf{a} = (0,1,2)^T$, *then* $a_3 = 0, a_4 = 3$ *and* $a_{n+5} = a_n + 1$ *holds for all* $n \geq 0$.

(5) *If* $\frac{3s+4}{4} < u < \frac{4s+5}{5}$ *and* $s > 10/11$ *and* $\mathbf{a} = (0,3,2)^T$, *then* $a_3 = 1, a_4 = 4, a_5 = 0, a_6 = 5$ *and* $a_{n+7} = a_n + 1$ *holds for all* $n \geq 0$.

In the above cases $\mathbf{r}$ *does not belong to* $\mathcal{D}_3$.

Proof. (1) We have $a_k = k$ for $k = 0,1,2$. Assume that this is true for $k < n+2$. Then

$$\begin{aligned} a_{n+2} &= -\lfloor -s(n-1) + (s-u)n + (u-1)(n+1) \rfloor \\ &= -\lfloor -n - 1 + s + u \rfloor = n+2 \end{aligned}$$

because $u < -s$.

(2) We have $a_3 = -\lfloor u-1 \rfloor = 1$. Assume that $a_{2n} = a_{2n+1} = n$ and $a_{2n+2} = n+1$. Then

$$\begin{aligned} a_{2n+3} &= -\lfloor -sn + (s-u)n + (u-1)(n+1) \rfloor \\ &= -\lfloor -n - 1 + u \rfloor = n+1 = a_{2n+1} + 1. \end{aligned}$$

Similar computation shows that if $a_{2n+1} = n$ and $a_{2n+1} = a_{2n+2} = n+1$, then $a_{2n+4} = n+2 = a_{2n+2} + 1$.

(3) As $\mathbf{a} = (0,1,2)$ we have $a_3 = -\lfloor s + u - 2 \rfloor$. Using the inequalities for u and s we get

$$\begin{aligned} s + u - 2 &\geq s + s/2 - 1 > 0 \\ &< 5s/3 - 1 < 1, \end{aligned}$$

whence $a_3 = 0$. Similarly, $a_4 = -\lfloor s - 2u \rfloor$ and as

$$\begin{aligned} s - 2u &\geq s - s - 2 = -2 \\ &< s - 4s/3 - 2 = -s/3 - 2 > -3 \end{aligned}$$

$a_4 = 3$; $a_5 = -\lfloor -2s + 3u - 3 \rfloor$ and as

$$\begin{array}{rcl} -2s + 3u - 3 & \geq & -2s + 3s/2 + 3 - 3 > -1 \\ & < & 2s - 2s + 3 - 3 = 0 \end{array}$$

$a_5 = 1$; $a_6 = -\lfloor 3s - 2u - 1 \rfloor$ and as

$$\begin{array}{rcl} 3s - 2u - 1 & \geq & 3s - 4s/3 - 3 > -2 \\ & < & 3s - s - 2 - 1 < -1 \end{array}$$

$a_6 = 2$; $a_7 = -\lfloor -2s + u - 2 \rfloor$ and as

$$\begin{array}{rcl} -2s + u - 2 & \geq & -2s + s/2 > -3 \\ & < & -2s + 2s/3 - 1 < -2 \end{array}$$

$a_7 = 3$. Since $(a_5, a_6, a_7) = (a_0, a_1, a_2) + \mathbf{1}$ and $\tau_{\mathbf{r}}^k(\mathbf{a} + \mathbf{1}) = \tau_{\mathbf{r}}^k(\mathbf{a}) + \mathbf{1}$ for $k \geq 0$, the assertion follows.

The proof of cases (4), (5) is similar, therefore it is omitted. □

In contrast with the last theorem, we prove now that large parts of B_1 belong to $\mathcal{D}_3$.

Theorem 7. *Assume that* $-1 < s, t < 1$, $\mathbf{r} = (-s, s - (s+1)t, (s+1)t - 1)^T$ *and* $\mathbf{a} \in \mathbb{Z}^3$. *If*

(1) $-s, s - (s+1)t, (s+1)t - 1 \leq 0$ *or*

(2) $s \in (0.334, 0.399)$ *and* $t = -\frac{s}{s+1}$,

then $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}$ *is ultimately constant, i.e.,* $\mathbf{r} \in \mathcal{D}_3$.

Proof. (1) As $\tau_{\mathbf{r}}^k(\mathbf{a} + L\mathbf{1}) = \tau_{\mathbf{r}}^k(\mathbf{a}) + L\mathbf{1}$, adding $L\mathbf{1}$ to $\mathbf{a}$ with a suitable integer L we have that all coordinates of $\mathbf{a} + L\mathbf{1}$ are non-negative. Thus we may assume that this is valid already for the initial vector $\mathbf{a}$. Now assume that $a_{n-1}, a_n, a_{n+1} \geq 0$ for some $n \geq 1$. Then

$$\begin{array}{rcl} -\max\{a_{n-1}, a_n, a_{n+1}\} & \leq & -sa_{n-1} + (s - (s+1)t)a_n + ((s+1)t - 1)a_{n+1} \\ & \leq & -\min\{a_{n-1}, a_n, a_{n+1}\} \end{array}$$

and equality holds if and only if $a_{n-1} = a_n = a_{n+1}$, in which case we are done. Otherwise, $\min\{a_{n-1}, a_n, a_{n+1}\} + 1 \leq a_{n+2} \leq \max\{a_{n-1}, a_n, a_{n+1}\}$, i.e., the minimum of three consecutive terms is increasing, but their maximum is not, thus the sequence becomes constant after some steps.

(2) In this case we apply Theorem 5. In the actual case the polynomial $R(X)$ has the form $R(X) = (X - 1)(X^2 - sX + s)$. Its roots α, β for $0 \leq s \leq 1$ are conjugate

complex numbers, hence $|\alpha| = |\beta| = \sqrt{s}$. Further, $|\alpha - \beta| = \sqrt{4s - s^2}$. Using these expressions Theorem 5 implies $c_{11} = \frac{2}{(1-\sqrt{s})\sqrt{4s-s^2}}$. It is easy to see that c_{11} as a function of s is always larger than 4 and is less than 5 provided $s \in (0.079, 0.478)$.

For the initial points $\mathbf{a} \in A(4)$ we tested the sequence $\{a_n\}$ for $s \in (0.334, 0.399)$. Of course it is impossible to do this directly, because there are uncountably many values in the interval. However, the convexity property of the mapping $\tau_{\mathbf{r}}$ (see [1] Theorem 4.6) allows us to test only the end points of the interval. We have done this by using the computer algebra system MAPLE 9 and found that $\tau_{\mathbf{r}(0.334)}(\mathbf{a}) = \tau_{\mathbf{r}(0.399)}(\mathbf{a})$ except when $\mathbf{a} = (0, 4, 0), (0, -4, 0)$. If $\mathbf{a} = (0, -4, 0)$ then $\{a_n\} = (0, -4, 0, 3, 3, 2, 2, 3, 4, 4, 4)$, if $0.334 \le s \le 0.375$ and $\{a_n\} = (0, -4, 0, 4, 4, 3, 3, 4, 5, 5, 5)$, if $0.375 < s \le 0.468$. Similarly if $\mathbf{a} = (0, 4, 0)$ then $\{a_n\} = (0, 4, 0, -2, -1, 1, 2, 2, 2)$, if $0.334 \le s < 0.375$ and $\{a_n\} = (0, 4, 0, -3, -2, 0, 1, 1, 1)$, if $0.375 \le s \le 0.468$. This completes the proof. □

Note that the examples of the last two theorems seem to be typical in the sense that if $\{a_n\}$ is bounded then it is ultimately constant.

4.2. The Set B_2

In this case $R(X) = X^3 + ((s+1)t + 1)X^2 + (s + (s+1)t)X + s = (X+1)(X^2 + (s+1)tX + s) = (X+1)(X-\alpha)(X-\beta)$. We show again that large parts of B_2 belong to $\mathcal{D}_3$ and others do not.

First we prove the analogue of Theorem 5 for the actual case.

Theorem 8. *Assume that* $-1 < s, t < 1$, $\mathbf{r} = (s, s+(s+1)t, (s+1)t+1)^T$. *Let* α, β *be the roots of* $R(X) = X^3 + ((s+1)t+1)X^2 + (s+(s+1)t)X + s$, *which have modulus less than* 1. *Let*

$$c_{12} = \left\lfloor \frac{1}{|\alpha - \beta|} \left(\frac{\max\{1, |\alpha + 1|\}}{1 - |\alpha|} + \frac{\max\{1, |\beta + 1|\}}{1 - |\beta|} \right) \right\rfloor$$

and $A = A(c_{12}) = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \;:\; 0 \le x_1 \le c_{12}, -x_1 - c_{12} \le x_2 \le -x_1 + c_{12}, x_2 - c_{12} \le x_3 \le x_2 + c_{12}, -x_3 - c_{12} \le x_4 \le -x_3 + c_{12}\}$. *For* $(a_1, a_2, a_3)^T \in \mathbb{Z}^3$ *and* $L, k \in \mathbb{Z}$ *define* $a_{d+k+1}^{(L)} = -\lfloor \mathbf{r}^T \tau_{\mathbf{r}}^k(a_1 + L, a_2 - L, a_3 + L)$. *Then for any* $(a_1, a_2, a_3) \in \mathbb{Z}^3$ *there exist integers* L, k *such that* $(a_k^{(L)}, a_{k+1}^{(L)}, a_{k+2}^{(L)}, a_{k+3}^{(L)}) \in A$.

Proof. The proof is analogous to the proof of Theorem 5, therefore we present only the important differences. We have

$$G_n = \frac{1}{(s+1)(1-t)} \left((-1)^n + \frac{\alpha^n(\beta+1) - \beta^n(\alpha+1)}{\alpha - \beta} \right).$$

Thus

$$G_{n+1} + G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad G_{n+2} - G_n = \frac{\alpha^n(\alpha+1) - \beta^n(\beta+1)}{\alpha - \beta},$$

which imply the inequalities

$$|a_{k+1}+a_k| \le \frac{1}{|\alpha-\beta|}\left(\frac{1}{1-|\alpha|}+\frac{1}{1-|\beta|}\right)$$

and

$$|a_{k+2}-a_k| \le \frac{1}{|\alpha-\beta|}\left(\frac{|\alpha+1|}{1-|\alpha|}+\frac{|\beta+1|}{1-|\beta|}\right).$$

Taking the maximum of the right hand sides and using that $a_{k+1}+a_k$ and $a_{k+2}-a_k$ are integers we get the assertion. □

In the next theorem we show that some part of B_2 belongs to $\mathcal{D}_3$, while some other part does not belong to $\mathcal{D}_3$.

Theorem 9. *Assume that* $-1 < s, t < 1$, $\mathbf{r} = (s, s+(s+1)t, (s+1)t+1)^T$ *and put* $u = (s+1)t$.

(1) *If* $-1 < s \le 0$ *and* $t > 0$, *but* $(s,t) \ne (-1,1)$ *and* $\mathbf{a} = (0,0,1)^T$, *then* $a_{2n+f} = (-1)^f n, n = 0,1,\dots, f = 0,1$.

(2) *If* $s \le 0$ *and* $1+2s < u < 1+\frac{s}{2}$ *and* $\mathbf{a} = (0,-1,3)^T$, *then* $a_3 = -4, a_4 = 6, a_5 = -7$ *and* $a_{n+6} = a_n + 9$ *holds for all* $n \ge 0$.

(3) *If* $s, u+1 \ge 0$ *and* $s+u < 0$, *then for all initial vectors the sequence* $\{a_n\}$ *is ultimately periodic with period* $L, -L$ *for some integer* L.

In cases (1) *and* (2), $\mathbf{r}$ *does not belong to* $\mathcal{D}_3$, *while in case* (3) *it does belong to* $\mathcal{D}_3$.

Proof. (1) For the initial vector the statement is true. Assume that it is true for $a_{2n}, a_{2n+1}, a_{2n+2}$. Then

$$\begin{aligned} a_{2n+3} &= -\lfloor sn-(s+u)n+(u+1)(n+1)\rfloor \\ &= -\lfloor n+1+u\rfloor = -(n+1) \end{aligned}$$

because $u = (s+1)t$ is positive and less than 1. The case $a_{2n+1}, a_{2n+2}, a_{2n+3}$ can be treated similarly.

(2) We have $a_3 = -\lfloor -(s+u)+3(u+1)\rfloor = -\lfloor -s+2u+3\rfloor = -4$, $a_4 = -\lfloor -s+3(s+u)-4(u+1)\rfloor = -\lfloor 2s-u-4\rfloor = 6$. The proofs of the remaining statements are similar.

(3) As $\tau_{\mathbf{r}}^k(\mathbf{a}+L(1,-1,1)^T) = \tau_{\mathbf{r}}^k(\mathbf{a})+(-1)^k L(1,-1,1)^T$ holds for all $\mathbf{a}\in\mathbb{Z}^3$ and $k \ge 0$, we may assume that $a_1, a_3 \ge 0$ and $a_2 \le 0$. Let $k = \min\{a_1, |a_2|, a_3\}$ and $K = \max\{a_1, |a_2|, a_3\}$, and assume that $k \ne K$, otherwise we are done. Then

$$sa_1+(s+u)a_2+(u+1)a_3 = sa_1-(s+u)|a_2|+(u+1)a_3.$$

Here all summands are non-negative, therefore the sum is greater than k and less than K and we get $-K+1 \leq a_4 \leq -k$. We have $a_2, a_4 \leq 0$ and $a_3 \geq 0$, which justify the equality

$$sa_2 + (s+u)a_3 + (u+1)a_4 = -(s|a_2| - (s+u)a_3 + (u+1)|a_4|).$$

As the summands in the bracket are non-negative we obtain $-K \leq sa_2 + (s+u)a_3 + (u+1)a_4 \leq -k$ and equality holds only if $a_2 = -a_3 = a_4$. If this is not true then $k+1 \leq a_5 \leq K$. This means that the lower bound for the absolute value of the terms $|a_n|$ is increasing, but the upper bound is not decreasing, thus $\{|a_n|\}$ must became ultimately constant. □

4.3. The Set B_3

By Surer's [10] characterization $R(X) = X^3 + (2t+v)X^2 + (2tv+1)X + v = (X+v)(X^2+2tX+1) = (X+v)(X-\alpha)(X-\bar{\alpha})$. We study only the case $t = 0, |v| \leq 1$ and prove

Theorem 10. *The points $\mathbf{r} = (v, 1, v)^T, |v| \leq 1$, belong to $\mathcal{D}_3 \setminus \mathcal{D}_3^0$, where $\mathcal{D}_3^0$ denotes the set of those $\mathbf{r} \in \mathbb{R}^3$ for which $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k=0}^{\infty}$ is for all $\mathbf{a} \in \mathbb{Z}^3$ the ultimately zero sequence.*

Proof. Let $\{a_n\}$ be a sequence of integers satisfying

$$0 \leq va_{n-1} + a_n + va_{n+1} + a_n < 1$$

for all $n \geq 1$. Putting $b_n = a_n + a_{n+2}, n \geq 0$ we rewrite the last inequality as

$$0 \leq vb_{n-1} + b_n < 1. \tag{12}$$

If $0 \leq v < 1$ then $v \in \mathcal{D}_1^0$ by Proposition 4.4 of [4], i.e., the sequence $\{b_n\}$ is ultimately zero. We prove that for the other values of v, i.e., $-1 \leq v < 0$ and $v = 1$, the sequence $\{b_n\}$ is ultimately constant. This is obviously true for $v = \pm 1$. If $b_0 = 0$ then $b_n = 0$ for all $n \geq 0$.

Assume that $-1 < v < 0$. If $b_0 > 0$ then $1 \leq b_n \leq b_{n-1}$ holds for all $n \geq 1$. Indeed, $b_n \geq -vb_{n-1} > 0$, which proves the left inequality. On the other hand, $b_n < 1 - vb_{n-1} < 1 + b_{n-1}$. As both b_n and b_{n-1} are integers we get the right-hand side inequality. We proved that $\{b_n\}$ is non-negative and monotonic decreasing, and thus it is ultimately constant. If $b_0 < 0$ then one can analogously prove that $\{b_n\}$ is non-positive and monotonically increasing, and thus it is ultimately constant too.

After this preparation we turn to the proof of the theorem. Without loss of generality we may assume that $b_n = b, n \geq 0$. Let $a_0, a_1 \in \mathbb{Z}$. Then $a_2 = b - a_0, a_3 = b - a_1$ and $a_{4k+j} = a_j$ for all $j = 0, 1, 2, 3; k = 0, 1, \ldots$. Thus $\{a_n\}$ is an ultimately periodic sequence, i.e., $\mathbf{r} \in \mathcal{D}_3$. As we may choose a_0, a_1 arbitrarily, e.g., such that $\{a_n\}$ is not the zero sequence, we have $\mathbf{r} \notin \mathcal{D}_3^0$. This completes the proof of the theorem. □

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