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ON THE NUMBER OF ZERO-SUM SUBSEQUENCES OF RESTRICTED SIZE

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Abstract

Let $n = 2^\lambda m \geq 526$ with $m \in \{2, 3, 5, 7, 11\}$, and let S be a sequence of elements in $C_n \oplus C_n$ with $|S| = n^2 + 2n - 2$. Let $\mathsf{N}_0^{|G|}(S)$ denote the number of the subsequences with length $n^2 (= |G|)$ and with sum zero. Among other results, we prove that either $\mathsf{N}_0^{|G|}(S) = 1$ or $\mathsf{N}_0^{|G|}(S) \geq n^2 + 1$.

1. Introduction and Main Results

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ denote the set of integers. For $a, b \in \mathbb{Z}$ with $a \leq b$, we define $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Let G be an additively written finite abelian group. We denote by $|G|$ the *order* of G, and denote by $\exp(G)$ the *exponent* of G. Let $\mathcal{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we call $|S| = l$ the *length* of S. For every $g \in G, k \in \mathbb{N}$, let $\mathsf{N}_g^k(S)$ denote the number of subsets $I \subseteq [1, l]$ such that $|I| = k$ and $\sum_{i \in I} g_i = g$. The famous Erdős–Ginzburg–Ziv theorem asserts that if $|S| \geq 2|G| - 1$ then $\mathsf{N}_0^{|G|}(S) \geq 1$ [5].

When $G = C_n$ is the cyclic group of n elements, $\mathsf{N}_g^n(S)$ has been studied since 1967 by many authors including H.B. Mann, A. Bialostocki and M. Lotspeich, Z. Füredi and D.J. Kleitman, the first author, D.J. Grynkiewicz, and M. Kisin. Let p be a prime and let $S \in \mathcal{F}(C_p)$ with $|S| = 2p - 1$. H.B. Mann [19] proved that if no element occurs more than p times in S then $\mathsf{N}_g^p(S) \geq 1$ for every $g \in C_p$. With the same assumption above, the first author [8] proved that $\mathsf{N}_g^p(S) \geq p$ for every $g \in C_p \setminus \{0\}$, and either $\mathsf{N}_0^p(S) = 1$ or $\mathsf{N}_0^p(S) \geq p + 1$. In 1999, the first author [9] showed that for every positive integer n, if $|S| = 2n - 1$ then for every $g \in C_n \setminus \{0\}$ we have $\mathsf{N}_g^n(S) = 0$ or $\mathsf{N}_g^n(S) \geq n$, and either $\mathsf{N}_0^n(S) = 1$ or $\mathsf{N}_0^n(S) \geq n + 1$. In 1992, Bialostocki and Lotspeich [2] formulated the following conjecture.

Conjecture 1 Let $n \ge 2$ be a positive integer, and let $S \in \mathcal{F}(C_n)$. Then

$$\mathsf{N}_0^n(S) \ge \binom{\lfloor |S|/2 \rfloor}{n} + \binom{\lceil |S|/2 \rceil}{n}.$$

Conjecture 1.1 has been confirmed if one of the following conditions holds:

(i) $n = p^a q^b$ where p, q are primes (M. Kisin, [18]);

(ii) $|S| \ge n^{6n}$ (Füredi and Kleitman, [6]);

(iii) $|S| \le 6.5n$ (Grynkiewicz, [16]).

However, there is almost no result on $\mathsf{N}_g^{|G|}(S)$ for non-cyclic group G. In this paper we shall obtain some sharp results on $\mathsf{N}_g^{|G|}(S)$ for $G = C_n \oplus C_n$ and $|S| = n^2 + 2n - 2$.

Before we can state our main results (see Corollary 1.4 and 1.6 below) more precisely, let us introduce some notation and terminology first. We write sequence $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathrm{v}_g(S)}$$

with $\mathrm{v}_g(S) \in \mathbb{N}_0$ for all $g \in G$.

We call $\mathrm{v}_g(S)$ the *multiplicity* of g in S. We say that S *contains* g if $\mathrm{v}_g(S) > 0$. The unit element $1 \in \mathcal{F}(G)$ is called the *empty sequence*. A sequence S_1 is called a *subsequence* of S if $S_1 | S$ in $\mathcal{F}(G)$ (equivalently, $\mathrm{v}_g(S_1) \le \mathrm{v}_g(S)$ for all $g \in G$), and it is called a *proper subsequence* of S if it is a subsequence with $1 \ne S_1 \ne S$. Let $S_1, S_2 \in \mathcal{F}(G)$, we denote by $S_1 S_2$ the sequence

$$\prod_{g \in G} g^{\mathrm{v}_g(S_1) + \mathrm{v}_g(S_2)} \in \mathcal{F}(G).$$

If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. For $g_0 \in G$, we set $g_0 + S = (g_0 + g_1) \cdot \ldots \cdot (g_0 + g_l) \in \mathcal{F}(G)$.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathrm{v}_g(S)} \in \mathcal{F}(G),$$

we call

$|S| = l = \sum_{g \in G} \mathrm{v}_g(S) \in \mathbb{N}_0$ the *length* of S,

$\mathsf{h}(S) = \max\{\mathrm{v}_g(S) | g \in G\} \in [0, |S|]$ the *maximum of the multiplicities* of S,

$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathrm{v}_g(S) g \in G$ the *sum* of S,

$\sum(S) = \left\{ \sum_{i \in I} g_i | I \subseteq [1, l] \text{ with} 1 \le |I| \le l \right\}$ the *set of all subsums* of S.

The sequence S is called

- *zero-sumfree* if $0 \notin \sum(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if it is a non-empty zero-sum sequence and every proper subsequence is zero-sumfree,
- a *short zero-sum sequence* if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$.

We denote by $\mathrm{D}(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \geq l$ has a nonempty zero-sum subsequence. The invariant $\mathrm{D}(G)$ is called the *Davenport constant* of G.

Let $n \geq 2$ be a positive integer. We say that n has *Property B* if every minimal zero-sum sequence in $\mathcal{F}(C_n \oplus C_n)$ of length $2n-1$ contains some element with multiplicity $n-1$. It has been conjectured that

Conjecture 2 Every positive integer $n \geq 2$ has Property B (e.g., see [11], [12], [15]).

Conjecture 1.2 has been confirmed for $n = 2^\lambda m$ and $m \in \{2, 3, 5, 7, 11\}$ (see [11], [14]).

Write the elements in $C_n \oplus C_n$ in the form (a, b). Let $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = (0, 1)$. Then every $(a, b) \in C_n \oplus C_n$ can be expressed as $(a, b) = a\mathbf{e_1} + b\mathbf{e_2}$ uniquely. Let $\mathbf{0} = (0, 0)$.

Now we can state our main results precisely.

Theorem 3 *Let $G = C_n \oplus C_n$ with $n \geq 2$, and let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = |G| + \mathrm{D}(G) - 1 = n^2 + 2n - 2$. If n has Property B then*

$$\mathsf{N}_g^{|G|}(S) = 0 \text{ or } \mathsf{N}_g^{|G|}(S) \geq n$$

for every $g \in G \setminus \{\mathbf{0}\}$.

Corollary 4 *Let $n = 2^\lambda m$ with $m \in \{2, 3, 5, 7, 11\}$, and let $G = C_n \oplus C_n$. If $S \in \mathcal{F}(G)$ is a sequence of length $|S| = |G| + \mathrm{D}(G) - 1 = n^2 + 2n - 2$, then*

$$\mathsf{N}_g^{|G|}(S) = 0 \text{ or } \mathsf{N}_g^{|G|}(S) \geq n$$

for every $g \in G \setminus \{\mathbf{0}\}$.

Theorem 5 *Let $G = C_n \oplus C_n$ with $n \geq 526$, and let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = |G| + \mathrm{D}(G) - 1 = n^2 + 2n - 2$. If n has Property B then*

$$\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1 \text{ or } \mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq n^2 + 1.$$

Corollary 6 *Let $n = 2^\lambda m \geq 526$ with $m \in \{2, 3, 5, 7, 11\}$, and let $G = C_n \oplus C_n$. If $S \in \mathcal{F}(G)$ is a sequence of length $|S| = |G| + \mathrm{D}(G) - 1 = n^2 + 2n - 2$, then*

$$\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1 \text{ or } \mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq n^2 + 1.$$

Now let us give some examples concerning the above results.

Example 7 If $G = C_n \oplus C_n$, $S = \mathbf{0}^{n^2+2n-2}$, then $\mathsf{N}_g^{|G|}(S) = 0$, for every $g \in G \setminus \{\mathbf{0}\}$.

Example 8 If $G = C_n \oplus C_n$, $S = \mathbf{0}^{n^2-1}\mathbf{e_1}^n\mathbf{e_2}^{n-1}$, then $\mathsf{N}_{\mathbf{e_1}}^{|G|}(S) = n$.

Example 9 If $G = C_n \oplus C_n$, $n \geq 3$, $S = \mathbf{0}^{n^2}\mathbf{e_1}^{n-1}\mathbf{e_2}^{n-1}$, then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1$.

Example 10 If $G = C_n \oplus C_n$, $n \geq 3$, $S = \mathbf{0}^{n^2+1}\mathbf{e_1}^{n-2}\mathbf{e_2}^{n-1}$, then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = n^2+1$.

Example 11 If $G = C_2 \oplus C_2$, $S = (\mathbf{e_1} + \mathbf{e_2})^2\mathbf{e_1}^2\mathbf{e_2}^2$, then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 3$.

Remark 12 Example 7 and Example 8 show that the bounds in Theorem 3 are sharp. Example 9 and Example 10 show that the inequalities in Theorem 5 cannot be improved. Example 11 shows that the conclusion of Theorem 5 is not true for $G = C_2 \oplus C_2$. Perhaps this is the only exceptional case (see Conjecture 24 in Section 5). We believe that the conclusion of Theorem 5 is true for all $n \geq 3$, and we have checked it for all $n \leq 10$. It would be interesting to prove Theorem 5 for all $n \in [11, 525]$.

2. Preliminaries

To prove Theorem 3 and Theorem 5 we need some preliminaries, beginning with the following well-known result due to Olson [22].

Lemma 13 $D(C_n \oplus C_n) = 2n - 1$.

Lemma 14 ([15], Theorem 5.8.3) *Every sequence S in $C_n \oplus C_n$ with $|S| = 3n - 2$ contains a short zero-sum subsequence.*

Lemma 15 ([15], Theorem 5.8.7) *Let $G = C_n \oplus C_n$ with $n \geq 2$, and let $S \in \mathcal{F}(G)$ be a zero-sumfree sequence of length $|S| = 2n - 2$. If n has Property B then there is an automorphism ϕ over G such that $\phi(S) = \mathbf{e_2}^{n-1}\prod_{i=1}^{n-1}(\mathbf{e_1} + a_i\mathbf{e_2})$, or $\phi(S) = \mathbf{e_2}^{n-2}\prod_{i=1}^{n}(\mathbf{e_1} + a_i\mathbf{e_2})$ with $\sum_{i=1}^{n} a_i \equiv 1 \pmod n$ and $\mathsf{h}(S) = n - 2$.*

Lemma 16 *Let $n \geq 3$ have Property B, and let $G = C_n \oplus C_n$. Let $S_1, S_2 \in \mathcal{F}(G)$ with $|S_1| = |S_2| = 2n-2$. If $\mathsf{h}(S_1) \leq 2n-3$ and $\mathsf{h}(S_2) \leq 2n-3$, then there exist $T_1|S_1$ and $T_2|S_2$ such that $\sigma(T_1) = \sigma(T_2)$ and $|T_1| = |T_2| \in [1, 2n-2]$.*

Proof. It is easy to check the lemma for $n = 3$. So, we assume that $n \geq 4$. Let

$$S_1 = \prod_{i=1}^{2n-2} (a_i\mathbf{e_1} + b_i\mathbf{e_2})$$

and

$$S_2 = \prod_{i=1}^{2n-2} (c_i\mathbf{e_1} + d_i\mathbf{e_2}).$$

Let P_{2n-2} denote the symmetric group on $[1, 2n-2]$. Clearly, it suffices to prove that $S_1 - \delta(S_2)$ is not zero-sumfree for some $\delta \in P_{2n-2}$, where $\delta(S_2) = \prod_{i=1}^{2n-2}(c_{\delta(i)}\mathbf{e_1} + d_{\delta(i)}\mathbf{e_2})$.

Assume to the contrary that $S_1 - \delta(S_2)$ is zero-sumfree for every $\delta \in P_{2n-2}$. By Lemma 15, $\mathsf{h}(S_1 - \delta(S_2)) = n-1$ or $n-2$ holds for every $\delta \in P_{2n-2}$.

`Case 1:` $\mathsf{h}(S_1 - \delta(S_2)) = n-2$ holds for every $\delta \in P_{2n-2}$.

Especially, $\mathsf{h}(S_1 - S_2) = n-2$. Again by Lemma 15, there exists an automorphism ϕ over G such that

$$\phi(S_1 - S_2) = \mathbf{e_2}^{n-2}\prod_{i=1}^{n}(\mathbf{e_1} + z_i\mathbf{e_2}).$$

Without loss of generality, we may assume that $\phi = \mathrm{id}$. Furthermore, by rearranging the subscripts, if necessary, we assume that

$$(a_1 - c_1)\mathbf{e_1} + (b_1 - d_1)\mathbf{e_2} = \cdots = (a_{n-2} - c_{n-2})\mathbf{e_1} + (b_{n-2} - d_{n-2})\mathbf{e_2} = \mathbf{e_2}$$

and

$$(a_j - c_j)\mathbf{e_1} + (b_j - d_j)\mathbf{e_2} = \mathbf{e_1} + z_{j-n+2}\mathbf{e_2}$$

for every $j \in [n-1, 2n-2]$.

Since $\mathsf{h}(S_1 - S_2) = n-2$, we may assume that

$$z_1 \neq z_2.$$

Claim 1. $a_i - c_j \in \{1, 2\}$ holds for any $i, j \in [n+1, 2n-2]$ with $i \neq j$.

Let $i, j \in [n+1, 2n-2]$ with $i \neq j$, and let τ be the transposition $(i, j) \in P_{2n-2}$. Then

$$S_1 - \tau(S_2) = \mathbf{e_2}^{n-2}\left((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\right)\left((a_j - c_i)\mathbf{e_1} + (b_j - d_i)\mathbf{e_2}\right) \times \prod_{k \neq i-n+2, j-n+2}(\mathbf{e_1} + z_k\mathbf{e_2}).$$

If $a_i - c_j = 0$ then $(a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2} = (b_i - d_j)\mathbf{e_2} \neq \mathbf{e_2}$ follows from $\mathsf{h}(S_1 - \tau(S_2)) = n - 2$. Therefore, $\mathbf{0} \in \sum\left(\mathbf{e_2}^{n-2}\left((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\right)\right) \subseteq \sum\left(S_1 - \tau(S_2)\right)$, a contradiction.

Now we assume that $a_i - c_j \in [3, n-1]$. Let $I \subseteq [1, n] \setminus \{1, 2, i-n-2, j-n-2\}$ be a subset with $|I| = n - (a_i - c_j) - 1 \in [0, n-4]$. Then $a_i - c_j + 1 + \sum_{k\in I} 1 = 0$. Therefore

$$\left\{\left(b_i - d_j + z_1 + \sum_{k\in I} z_k\right)\mathbf{e_2}, \left(b_i - d_j + z_2 + \sum_{k\in I} z_k\right)\mathbf{e_2}\right\}$$
$$\subseteq \sum\left(((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2})\prod_{k\neq i-n+2, j-n+2}(\mathbf{e_1} + z_k\mathbf{e_2})\right).$$

Since $z_1 \neq z_2$, we have that $b_i - d_j + z_1 + \sum_{k\in I} z_k \neq b_i - d_j + z_2 + \sum_{k\in I} z_k$. Therefore

$$\begin{aligned}\mathbf{0} &\in \sum\left(\mathbf{e_2}^{n-2}\left(b_i - d_j + z_1 + \sum_{k\in I} z_k\right)\mathbf{e_2}\right)\\ &\qquad \bigcup \sum\left(\mathbf{e_2}^{n-2}\left(b_i - d_j + z_2 + \sum_{k\in I} z_k\right)\mathbf{e_2}\right)\\ &\subseteq \sum\left(\mathbf{e_2}^{n-2}\left((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\right)\prod_{k\neq i-n+2, j-n+2}(\mathbf{e_1} + z_k\mathbf{e_2})\right)\\ &\subseteq \sum\left(S_1 - \tau(S_2)\right),\end{aligned}$$

a contradiction. This proves Claim 1.

Note that $a_i - c_j + a_j - c_i = (a_i - c_i) + (a_j - c_j) = 2$. This forces $a_i - c_j = 1$ for any pair $i, j \in [n+1, 2n-2]$ with $i \neq j$. Therefore

$$\begin{aligned} a_{n+1} &= a_{n+2} = \cdots = a_{2n-2} = a \text{ (say)},\\ c_{n+1} &= c_{n+2} = \cdots = c_{2n-2} = a - 1.\end{aligned}$$

Since $\mathsf{h}(S_1 - S_2) = n - 2$, we have that $z_{k-n+2} \neq z_1$ holds for some $k \in [n + 1, 2n-2]$. Let $j \in [n+1, 2n-2] \setminus \{k\}$, and let $i = n$. Then repeating the proof above we obtain that

$$\begin{aligned} a_n &= a_{n+1} = \cdots = a_{2n-2} = a,\\ c_n &= c_{n+1} = \cdots = c_{2n-2} = a - 1.\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} a_{n-1} &= a_{n+1} = \cdots = a_{2n-2} = a,\\ c_{n-1} &= c_{n+1} = \cdots = c_{2n-2} = a - 1.\end{aligned}$$

Hence

$$a_{n-1} = a_n = \cdots = a_{2n-2} = c_{n-1} + 1 = c_n + 1 = \cdots = c_{2n-2} + 1 = a. \qquad (1)$$

Claim 2. $a_i - c_j \in \{0, 1\}$ holds for every $i \in [1, n-2]$ and every $j \in [n+1, 2n-2]$.

Let $i \in [1, n-2]$, $j \in [n+1, 2n-2]$, and let θ be the transposition $(i, j) \in P_{2n-2}$. Then

$$S_1 - \theta(S_2) = \mathbf{e_2}^{n-3} \left((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\right) \left((a_j - c_i)\mathbf{e_1} + (b_j - d_i)\mathbf{e_2}\right) \times \prod_{k \neq j-n+2} (\mathbf{e_1} + z_k \mathbf{e_2}).$$

Assume to the contrary that $a_i - c_j \in [2, n-1]$. Let $I \subseteq [1, n] \setminus \{j - n + 2\}$ be any subset with $|I| = n - (a_i - c_j)$. Let $J = [1, n] \setminus \{\{j - n + 2\} \cup I\}$. Then $a_i - c_j + \sum_{k \in I} 1 = 0$ and $a_j - c_i + \sum_{k \in J} 1 = 0$. Therefore

$$\sigma\left(\left((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\right) \prod_{k \in I} (\mathbf{e_1} + z_k \mathbf{e_2})\right) = \left(b_i - d_j + \sum_{k \in I} z_k\right) \mathbf{e_2},$$

and

$$\sigma\left(\left((a_j - c_i)\mathbf{e_1} + (b_j - d_i)\mathbf{e_2}\right) \prod_{k \in J} (\mathbf{e_1} + z_k \mathbf{e_2})\right) = \left(b_j - d_i + \sum_{k \in J} z_k\right) \mathbf{e_2}.$$

Since $\mathbf{0} \notin \sum \left(\mathbf{e_2}^{n-3} \left(\left(b_i - d_j + \sum_{k \in I} z_k\right) \mathbf{e_2}\right)\right)$, we infer that

$$b_i - d_j + \sum_{k \in I} z_k \in \{1, 2\}.$$

Similarly

$$b_j - d_i + \sum_{k \in J} z_k \in \{1, 2\}.$$

Note that $a_i - c_j + a_j - c_i + (n - 1) = 0$. Similarly to above we have

$$b_i - d_j + b_j - d_i + \sum_{k \in I} z_k + \sum_{k \in J} z_k \in \{1, 2\}.$$

These conditions force that $b_i - d_j + \sum_{k \in I} z_k = b_j - d_i + \sum_{k \in J} z_k = 1$ holds for every $I \subseteq [1, n] \setminus \{j - n + 2\}$ with $|I| = n - (a_i - c_j)$, which implies $z_1 = z_2$, a contradiction. This proves Claim 2.

Since $a_i - c_j + a_j - c_i = 1$, we have $a_j - c_i \in \{0, 1\}$. Therefore

$$a_i - c_j = 0,\ a_j - c_i = 1 \text{ or } \quad a_i - c_j = 1,\ a_j - c_i = 0 \qquad (2)$$

holds for every pair of i, j with $i \in [1, n-2]$ and $j \in [n+1, 2n-2]$.

If $a_j - c_i = 0$ then $a_j = a_i$ follows from $a_i - c_i = 0$. By (1), $a_i = a_{n-1} = a_n = \cdots = a_{2n-2}$. Let $t \in [n-1, 2n-2]$. Let γ be the transposition $(i,t) \in P_{2n-2}$. Then

$$S_1 - \gamma(S_2) = \mathbf{e_2}^{n-3}\left((a_i - c_t)\mathbf{e_1} + (b_i - d_t)\mathbf{e_2}\right)\left((a_t - c_i)\mathbf{e_1} + (b_t - d_i)\mathbf{e_2}\right) \times \prod_{k \neq t-n+2} (\mathbf{e_1} + z_k\mathbf{e_2}).$$

By (1) we have $a_i - c_t = 1$, $a_t - c_i = 0$. Therefore

$$\sigma\left(((a_i - c_t)\mathbf{e_1} + (b_i - d_t)\mathbf{e_2}) \prod_{k \neq t-n+2} (\mathbf{e_1} + z_k\mathbf{e_2})\right) = \left(b_i - d_t + \left(\sum_{k=1}^{n} z_k\right) - z_{t-n+2}\right)\mathbf{e_2},$$

and

$$(a_t - c_i)\mathbf{e_1} + (b_t - d_i)\mathbf{e_2} = (b_t - d_i)\mathbf{e_2}.$$

Hence

$$\begin{aligned}\mathbf{0} &\notin \sum\left(\mathbf{e_2}^{n-3}((b_t - d_i)\mathbf{e_2})\left(\left(b_i - d_t + \left(\sum_{k=1}^{n} z_k\right) - z_{t-n+2}\right)\mathbf{e_2}\right)\right) \\ &\subseteq \sum(S_1 - \gamma(S_2)).\end{aligned}$$

This forces that

$$b_t - d_i = b_i - d_t + \left(\sum_{k=1}^{n} z_k\right) - z_{t-n+2} = 1.$$

Since $b_i - d_i = 1$ we have $b_i = b_t$. Therefore, $a_i\mathbf{e_1} + b_i\mathbf{e_2} = a_t\mathbf{e_1} + b_t\mathbf{e_2}$ for every $t \in [n-1, 2n-2]$.

Now we have proved that if $a_j - c_i = 0$ for some $i \in [1, n-2]$ and $j \in [n+1, 2n-2]$, then

$$a_i\mathbf{e_1} + b_i\mathbf{e_2} = a_{n-1}\mathbf{e_1} + b_{n-1}\mathbf{e_2} = \cdots = a_{2n-2}\mathbf{e_1} + b_{2n-2}\mathbf{e_2}. \tag{3}$$

Similarly, if $a_i - c_j = 0$ for some $i \in [1, n-2]$ and some $j \in [n+1, 2n-2]$, then

$$c_i\mathbf{e_1} + d_i\mathbf{e_2} = c_{n-1}\mathbf{e_1} + d_{n-1}\mathbf{e_2} = \cdots = c_{2n-2}\mathbf{e_1} + d_{2n-2}\mathbf{e_2}. \tag{4}$$

From (2), (3) and (4) we infer that there are three possibilities:

(i) $a_1 = a_2 = \cdots = a_{2n-2} = a$, which implies

$$a_1\mathbf{e_1} + b_1\mathbf{e_2} = a_2\mathbf{e_1} + b_2\mathbf{e_2} = \cdots = a_{2n-2}\mathbf{e_1} + b_{2n-2}\mathbf{e_2}.$$

(ii) $c_1 = c_2 = \cdots = c_{2n-2} = a-1$, which implies

$$c_1\mathbf{e_1} + d_1\mathbf{e_2} = c_2\mathbf{e_1} + d_2\mathbf{e_2} = \cdots = c_{2n-2}\mathbf{e_1} + d_{2n-2}\mathbf{e_2}.$$

(iii) $a_i = a_{n-1} = \cdots = a_{2n-2} = a$ and $c_j = c_{n-1} = \cdots = c_{2n-2} = a-1$ for some $i, j \in [1, n-2]$ with $i \neq j$, which implies

$$a_i\mathbf{e_1} + b_i\mathbf{e_2} = a_{n-1}\mathbf{e_1} + b_{n-1}\mathbf{e_2} = \cdots = a_{2n-2}\mathbf{e_1} + b_{2n-2}\mathbf{e_2},$$

and

$$c_j\mathbf{e_1} + d_j\mathbf{e_2} = c_{n-1}\mathbf{e_1} + d_{n-1}\mathbf{e_2} = \cdots = c_{2n-2}\mathbf{e_1} + d_{2n-2}\mathbf{e_2}.$$

But we always get a contradiction. This completes the proof of Case 1.

`Case 2:` $\mathsf{h}(S_1 - \delta(S_2)) = n-1$ holds for some $\delta \in P_{2n-2}$. Since the proof is similar to and much easier than Case 1, we omit it here. □

Lemma 17 *Let $n \geq 3$ have Property B, and let $G = C_n \oplus C_n$. Let $S \in \mathcal{F}(G)$ be a zero-sumfree sequence of length $|S| = 2n-2$. Then for any $g \in G \setminus \{\mathbf{0}\}$, either* $\mathrm{v}_g(S) = n-1$ *or there exists a subsequence T of S such that $|T| \geq 2$ and $g = \sigma(T)$.*

Proof. By Lemma 13, for any $g \in G \setminus \{\mathbf{0}\}$, $(-g)S$ contains a nonempty zero-sum subsequence S_1. Since S is zero-sumfree, we have $(-g)|S_1$. Let $S_2 = S_1(-g)^{-1}$. Then $g = \sigma(S_2)$. If g is not a term of S then $|S_2| \geq 2$. Let $T = S_2$ and we are done. So we may assume that g is a term of S. Clearly, it suffices to prove that either $\mathrm{v}_g(S) = n-1$, or there is a subsequence W of S such that g is not a term of W and $g \in \sum(W)$.

By Lemma 15 there is an automorphism ϕ over G such that

$$\phi(S) = \mathbf{e_2}^r \prod_{i=1}^{2n-2-r} (\mathbf{e_1} + a_i\mathbf{e_2}),$$

where $r = \mathsf{h}(S) = n-1$ or $n-2$. Without loss of generality let $\phi = \mathrm{id}$.

`Case 1:` $S = \mathbf{e_2}^{n-1} \prod_{i=1}^{n-1}(\mathbf{e_1} + a_i\mathbf{e_2})$.

`Subcase 1.1:` $a_1 = a_2 = \cdots = a_{n-1}$. Since g is a term of S, $g = \mathbf{e_2}$ or $\mathbf{e_1} + a_1\mathbf{e_2}$. Therefore, $\mathrm{v}_g(S) = n-1$.

`Subcase 1.2:` $a_1 = a_2 = \cdots = a_{n-1}$ does not hold. Without loss of generality let $a_1 \neq a_2$. If $g = \mathbf{e_2}$ then $\mathrm{v}_g(S) = n-1$. Now assume $g = \mathbf{e_1} + a_i\mathbf{e_2}$ for some $i \in [1, n-1]$. Note that either $a_i \neq a_1$ and we have $g = \mathbf{e_1} + a_i\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-1}(\mathbf{e_1} + a_1\mathbf{e_2})\right)$, or $a_i \neq a_2$ and we have $g = \mathbf{e_1} + a_i\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-1}(\mathbf{e_1} + a_2\mathbf{e_2})\right)$.

`Case 2:` $S = \mathbf{e_2}^{n-2}\prod_{i=1}^{n}(\mathbf{e_1} + a_i\mathbf{e_2})$ and $\mathsf{h}(S) = n-2$. By rearranging the subscripts, if necessary, we can assume that $a_1 \neq a_2$. By Lemma 15, we have $\mathbf{e_2} = \sigma\left(\prod_{i=1}^{n}(\mathbf{e_1} + a_i\mathbf{e_2})\right)$. So it remains to check the case that $g = \mathbf{e_1} + a_i\mathbf{e_2}$ for some $i \in [1, n]$.

`Subcase 2.1:` There are three distinct elements among of $a_1, \ldots, a_n$. Then there are two indices $j, k \in [1, n] \setminus \{i\}$ such that a_i, a_j, a_k are pairwise distinct. Since $[a_j, a_j + n - 2] \cup [a_k, a_k + n - 2] = [0, n-1] \setminus \{a_j + n - 1\} \cup [0, n-1] \setminus \{a_k + n - 1\} = [0, n-1]$, we infer that $\{\mathbf{e_1}, \mathbf{e_1} + \mathbf{e_2}, \ldots, \mathbf{e_1} + (n-1)\mathbf{e_2}\} \subseteq \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_j\mathbf{e_2})\right) \cup \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_k\mathbf{e_2})\right)$. Hence

$$g = \mathbf{e_1} + a_i\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_j\mathbf{e_2})\right) \cup \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_k\mathbf{e_2})\right).$$

`Subcase 2.2:` There are exactly two distinct elements among $a_1, \ldots, a_n$. Let $j \in [1, n]$ with $a_j \neq a_i$. If $a_i \neq a_j + n - 1$ then $g = \mathbf{e_1} + a_1\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_j\mathbf{e_2})\right)$. Otherwise $a_i = a_j + n - 1$. Let r be the number of $k \in \{1, \ldots, n\}$ such that $a_k = a_i$. By Lemma 15, $a_1 + a_2 + \cdots + a_n \equiv 1 \pmod{n}$, that is, $ra_i + (n-r)(a_i + 1) \equiv 1 \pmod{n}$. Hence, $r = n - 1$ contradicting $\mathsf{h}(S) = n - 2$. □

Lemma 18 *Let $n \geq 3$ have Property B, and let $G = C_n \oplus C_n$. Let $S \in \mathcal{F}(G)$ be a zero-sumfree sequence of length $|S| = 2n - 3$, and let $W \in \mathcal{F}(G)$ be a nonempty zero-sum sequence. If W contains no $\mathbf{0}$ then there exist $W_1|W$ and $S_1|S$ such that $\sigma(W_1) = \sigma(S_1)$ and $1 \leq |W_1| \leq |S_1|$.*

Proof. It is easy to check the lemma for $n \in \{3, 4\}$.

Let $n \geq 5$. We may assume that W is a minimal zero-sum sequence. Let

$$W = g_1 \cdot \ldots \cdot g_w, \text{ where } w = |W| \geq 2.$$

If $(-g_i)S$ contains a nonempty zero-sum subsequence S_1' (say) for some $i \in [1, w]$, then $-g_i|S_1'$ follows from S is zero-sumfree. Let $S_1 = S_1'(-g_i)^{-1}$ and $W_1 = g_i \in \mathcal{F}(G)$. Then $S_1|S$, $g_i = \sigma(S_1)$ and we are done.

Now we may assume that, for any $i \in [1, w]$, $(-g_i)S$ is zero-sumfree. By Lemma 15, there exists an automorphism ϕ over G such that

$$\phi((-g_1)S) = \mathbf{e_2}^r \prod_{i=1}^{2n-2-r}(\mathbf{e_1} + z_i\mathbf{e_2}),$$

where $\mathsf{h}(\phi((-g_1)S)) = r = n - 1$ or $n - 2$. Without loss of generality let $\phi = \mathrm{id}$. Then

$$(-g_1)S = \mathbf{e_2}^r \prod_{i=1}^{2n-2-r}(\mathbf{e_1} + z_i\mathbf{e_2}),$$

where $\mathsf{h}((-g_1)S) = r = n - 1$ or $n - 2$. By rearranging the subscripts, if necessary, we may assume that

$$-g_1 = \mathbf{e_2}, \text{ or } -g_1 = \mathbf{e_1} + z_1\mathbf{e_2}.$$

`Case 1:` $w = 2$. Then $g_1 + g_2 = \mathbf{0}$.

`Subcase 1.1:` $-g_1 = \mathbf{e_1} + z_1\mathbf{e_2}$. Then $g_2 = -g_1 = \mathbf{e_1} + z_1\mathbf{e_2}$. If $r = n-1$, it is easy to see that $g_2 \in \sum\left((\mathbf{e_1} + z_2\mathbf{e_2})\mathbf{e_2}^{n-1}\right) \subseteq \sum(S)$ and we are done. If $r = n-2$ then $\mathsf{h}(z_2 z_3 \cdot \ldots \cdot z_n) \le n-2$. By rearranging the subscripts, if necessary, we assume that $z_2 \ne z_3$. Furthermore, we may assume that $z_1 \ne z_2 + (n-1)$. Thus $g_2 \in \sum\left((\mathbf{e_1} + z_2\mathbf{e_2})\mathbf{e_2}^{n-2}\right) \subseteq \sum(S)$ and we are done.

`Subcase 1.2:` $-g_1 = \mathbf{e_2}$. Then $g_2 = -g_1 = \mathbf{e_2}$. Letting $S_1 = \mathbf{e_2} \in \mathcal{F}(G)$ and $W_1 = g_2 \in \mathcal{F}(G)$ verify the lemma.

`Case 2:` $w \ge 3$. Let $i, j \in [1, w]$ be an arbitrary pair with $i \ne j$. By Lemma 13, $(-g_i)(-g_j)S$ contains a nonempty zero-sum subsequence S_2' (say). Since both sequences $(-g_i)S$ and $(-g_j)S$ are zero-sumfree, we have $(-g_i)(-g_j)|S_2'$. Let $S_2 = S_2'(-g_i)^{-1}(-g_j)^{-1}$. Then $S_2|S$ and $|S_2| \ge 1$. If $|S_2| \ge 2$, setting $S_1 = S_2$ and $W_1 = g_i g_j$ verifies the lemma. So, we may assume that $|S_2| = 1$. Therefore, for any $i, j \in [1, w]$ with $i \ne j$,

$$g_i + g_j$$

is a term of S.

`Subcase 2.1:` $-g_1 = \mathbf{e_1} + z_1\mathbf{e_2}$. Then $g_1 = (n-1)\mathbf{e_1} + (n - z_1)\mathbf{e_2}$. For any $2 \le i \le w$, since $g_1 + g_i$ is a term of S, we infer that $g_i = \mathbf{e_1} + z\mathbf{e_2}$ or $2\mathbf{e_1} + z\mathbf{e_2}$ for some $z \in C_n$. Therefore, for any $i, j \in [2, w]$ with $i \ne j$ we have $g_i + g_j = a\mathbf{e_1} + b\mathbf{e_2}$ for some $a \in \{2, 3, 4\}$, a contradiction of $g_i + g_j$ is a term of S.

`Subcase 2.2:` $-g_1 = \mathbf{e_2}$. Then $g_1 = (n-1)\mathbf{e_2}$. For any $2 \le i \le w$, since $g_1 + g_i$ is a term of S, we infer that $g_i = 2\mathbf{e_2}$ or $\mathbf{e_1} + z\mathbf{e_2}$ for some $z \in C_n$. If $g_i = 2\mathbf{e_2}$, letting $W_1 = g_i$ and $S_1 = \mathbf{e_2}^2$ verify the lemma. So we may assume that $g_i = \mathbf{e_1} + z\mathbf{e_2}$ for every $2 \le i \le w$. Therefore, for any $i, j \in [2, w]$ with $i \ne j$ we have $g_i + g_j = 2\mathbf{e_1} + z'\mathbf{e_2}$, it is not a term of S, a contradiction. This completes the proof. □

Lemma 19 ([7], Theorem 1) *Let G be a finite abelian group, and let $S \in \mathcal{F}(G)$. If $|S| = |G| + \mathrm{D}(G) - 1$ then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge 1$.*

We also need the following technical results.

Lemma 20 *Let $n \ge 3, k, p_1, \ldots, p_k$ be positive integers. If $p_1 + p_2 + \cdots + p_k \ge 3n-2$ and $2 \le p_i \le 2n-3$ for every $i \in [1, k]$, then $p_1 p_2 \cdots p_k \ge n^2 + 1$.*

Proof. Since $2 \le p_i \le 2n-3$ for every $i \in [1, k]$, we have

$$p_1 p_2 \cdots p_k \ge p_1(p_2 + \cdots + p_k) \ge p_1(3n - 2 - p_1) \ge (2n-3)(n+1) \ge n^2 + 1. \quad \square$$

Lemma 21 *Let $A_1, \ldots, A_l$ be subsets of $[1, k]$ with $|A_1| = \cdots = |A_l| = 2$. If $l \le k$ then there exist a subset $A \subseteq [1, k]$ such that $|A| \le \frac{k}{2} + \frac{l}{4}$ and $A \cap A_i \ne \emptyset$ holds for every $i \in [1, l]$.*

Proof. By rearranging the subscripts, if necessary, we may assume that $A_1 \cap A_2 \neq \emptyset, A_3 \cap A_4 \neq \emptyset, \ldots, A_{2t-1} \cap A_{2t} \neq \emptyset$, and $A_{2t+1}, \ldots, A_l$ are pairwise disjoint. Put $r = l - 2t$. Clearly, $0 \le t \le \frac{l}{2}$ and $r \le \frac{k}{2}$. Now take one element x_i from $A_{2i-1} \cap A_{2i}$ for every $i \in [1, t]$ (note that $x_1, \ldots, x_t$ are not necessarily distinct), and take one element x_{2t+j} from A_{2t+j} for every $j \in [1, r]$. Let

$$A = \{x_1, \ldots, x_t, x_{2t+1}, \ldots, x_l\}.$$

Then, $A \cap A_i \neq \emptyset$ for every $i \in [1, l]$.

It remains to show that $|A| \le t + r \le \frac{k}{2} + \frac{l}{4}$. Note that

$$2t + r = l \text{ and } r \le \frac{k}{2}.$$

If $r \le k - \frac{l}{2}$ then $|A| \le t + r = r + \frac{l-r}{2} = \frac{l+r}{2} \le \frac{k}{2} + \frac{l}{4}$. Now assume that $r > k - \frac{l}{2}$. Then, $t = \frac{l-r}{2} < \frac{l-k+\frac{l}{2}}{2} \le \frac{l}{4}$. Therefore, $|A| \le r + t \le \frac{k}{2} + \frac{l}{4}$. This completes the proof. □

3. Proof of Theorem 3

Let $n \ge 3$. Note that $\mathsf{N}_g^{|G|}(S) = \mathsf{N}_g^{|G|}(-x+S)$ holds for every $g \in G$, we may assume that $\mathrm{v}_{\mathbf{0}}(S) = \mathsf{h}(S)$. Let $g \in G \setminus \{\mathbf{0}\}$. Suppose $\mathsf{N}_g^{|G|}(S) \ge 1$, we need to show that $\mathsf{N}_g^{|G|}(S) \ge n$.

By rearranging the subscripts we may assume that

$$S = S_1 S_2,$$

where

$$\begin{aligned} S_1 =& a_1 a_2 \cdot \ldots \cdot a_{n^2-r} \mathbf{0}^r, \\ S_2 =& b_1 b_2 \cdot \ldots \cdot b_{2n-2-\mathsf{h}(S)+r} \mathbf{0}^{\mathsf{h}(S)-r}, \\ g =& \sigma(S_1) = a_1 + a_2 + \cdots + a_{n^2-r}. \end{aligned}$$

We first assume that $\mathsf{h}(S) \le 2n-3$. By Lemma 16 there exist $T_1 | a_1 a_2 \cdot \ldots \cdot a_{2n-2}$ and $T_1' | S_2$ such that $\sigma(T_1) = \sigma(T_1')$ and $|T_1| = |T_1'| \ge 1$. By rearranging the subscripts of S_1 we may assume that $a_1 | T_1$. Again by Lemma 16 there exist $T_2 | a_2 a_3 \cdot \ldots \cdot a_{2n-1}$ and $T_2' | S_2$ such that $\sigma(T_2) = \sigma(T_2')$ and $|T_2| = |T_2'| \ge 1$. Clearly, T_1 and T_2 are different. Similarly, we can obtain subsequences $T_3, \ldots, T_n$ of S_1 and subsequences $T_3', \ldots, T_n'$ of S_2 satisfying $|T_i| = |T_i'|$, $\sigma(T_i) = \sigma(T_i')$ for any $i \in [1, n]$, and $T_1, T_2, \ldots, T_n$ are pairwise different. Therefore, $S_1 T_1^{-1} T_1', S_1 T_2^{-1} T_2', \ldots, S_1 T_n^{-1} T_n'$ are pairwise different subsequences of S with sum g and length n^2. So we have $\mathsf{N}_g^{|G|}(S) \ge n$.

Now suppose that $\mathsf{h}(S) \geq 2n-2$. We distinguish four cases.

Case 1: $1 \leq r \leq \mathsf{h}(S)-1$. Then $\mathsf{N}_g^{|G|}(S) \geq \binom{\mathsf{h}(S)}{r} \geq \binom{\mathsf{h}(S)}{1} = \mathsf{h}(S) > n$.

Case 2: $r = 0$. Then $\mathsf{h}(S) = 2n-2$. Since $|S_1| = n^2 \geq 3n-2$, by Lemma 14, there is a short zero-sum subsequence T of S_1. So $S_1T^{-1}\mathbf{0}^{|T|}$ is a sequence with sum g and length n^2. Replace S_1 by $S_1T^{-1}\mathbf{0}^{|T|}$ and it reduces to Case 1.

Case 3: $r = \mathsf{h}(S)$ and S_2 is not zero-sumfree. Assume that $T|S_2$ and $\sigma(T) = \mathbf{0}$. Replace S_1 by $S_1\mathbf{0}^{-|T|}T$ and it reduces to Case 1 or Case 2.

Case 4: $r = \mathsf{h}(S)$ and S_2 is zero-sumfree. Since $g \neq \mathbf{0}$, there is at least one term of S_1 is not zero. Let $g'|S_1$ and $g' \neq \mathbf{0}$. By Lemma 17 we have that either $\mathrm{v}_{g'}(S_2) = n-1$ or there exists a subsequence T of S_2 such that $|T| \geq 2$ and $g' = \sigma(T)$. If $\mathrm{v}_{g'}(S_2) = n-1$ then $\mathsf{N}_g^{|G|}(S) \geq \binom{\mathrm{v}_{g'}(S_1)+\mathrm{v}_{g'}(S_2)}{1} \geq \binom{n}{1} = n$. Now assume that $g' = \sigma(T)$ for some $T|S_2$ with $|T| \geq 2$. Replace S_1 by $S_1g'^{-1}\mathbf{0}^{-|T|+1}T$ and it reduces to Case 1 or Case 2.

It is easy to check the case $n = 2$ directly and we omit it here. Now the proof is completed. □

4. Proof of Theorem 5

Let $n \geq 526$. Without loss of generality let $\mathsf{h}(S) = \mathrm{v}_{\mathbf{0}}(S)$. From Lemma 19 and Lemma 13 we know that $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq 1$. Assume that $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq 2$. We have to show $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq n^2+1$.

By rearranging the subscripts we may assume that

$$S = S_1S_2,$$

where

$$\begin{aligned} S_1 &= a_1a_2 \cdot \ldots \cdot a_{n^2-r}\mathbf{0}^r, \\ S_2 &= b_1b_2 \cdot \ldots \cdot b_{2n-2-\mathsf{h}(S)+r}\mathbf{0}^{\mathsf{h}(S)-r}, \\ \mathbf{0} &= \sigma(S_1) = a_1 + a_2 + \cdots + a_{n^2-r}. \end{aligned}$$

We distinguish between the values taken by $\mathsf{h}(S)$.

Case 1. $\mathsf{h}(S) \geq n^2+1$. Since $1 \leq r \leq n^2$, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq \binom{\mathsf{h}(S)}{r} \geq \binom{n^2+1}{1} \geq n^2+1$.

Case 2. $\mathsf{h}(S) = n^2$. We have $n^2-2n+2 \leq r \leq n^2-2$ or $r = n^2$. If $n^2-2n+2 \leq r \leq n^2-2$ then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq \binom{n^2}{r} \geq \binom{n^2}{2} \geq n^2+1$. So we may assume that $r = n^2$. If S_2 is zero-sumfree then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1$, a contradiction. If S_2 has a zero-sum subsequence T of length at least 2 then $T\mathbf{0}^{n^2-|T|}$ is a zero-sum sequence of length n^2. Therefore, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq \binom{n^2}{n^2-|T|} \geq \binom{n^2}{2} \geq n^2+1$.

Case 3. $2n-2 \le \mathsf{h}(S) \le n^2-1$. We distinguish four subcases according to the value taken by r.

Subcase 3.1: $2 \le r \le \mathsf{h}(S)-2$. Then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge \binom{\mathsf{h}(S)}{r} \ge \binom{2n-2}{2} \ge n^2+1$.

Subcase 3.2: $0 \le r \le 1$. Then $\mathsf{h}(S)-r \ge n+2$. Since $n^2-r \ge n^2-1 \ge 3n-2$, by Lemma 14, there is a zero-sum subsequence T of $a_1a_2\cdot\ldots\cdot a_{n^2-r}$ with $2 \le |T| \le n$. Now replace S_1 by $S_1T^{-1}\mathbf{0}^{|T|}$ and it reduces to Subcase 3.1.

Subcase 3.3: $r = \mathsf{h}(S)-1$. Let $S_1' = S_1\mathbf{0}^{-\mathsf{h}(S)+1}$ and $S_2' = S_2\mathbf{0}^{-1}$. If S_2' contains a nonempty zero-sum subsequence T, then replace S_1 by $S_1T\mathbf{0}^{-|T|}$ and it reduces to Subcase 3.1 or Subcase 3.2. So we assume that S_2' is zero-sumfree.

If there exist $T|S_1$ and $U|S_2'$ such that $|T| < |U|$ and $\sigma(T) = \sigma(U)$ then replace S_1 by $S_1UT^{-1}\mathbf{0}^{|T|-|U|}$. Note that $|U| \le 2n-3$ and it reduces to Subcase 3.1 or Subcase 3.2. So we may assume that $T|S_1'$, $U|S_2'$ and $\sigma(T) = \sigma(U)$ imply

$$|T| \ge |U|. \tag{5}$$

If $\mathsf{h}(S) \ge \frac{n^2+1}{2}$, then by Lemma 18 and (5) there exist $T|S_1'$ and $U|S_2'$ such that $|T| = |U|$ and $\sigma(T) = \sigma(U)$. Therefore, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge 2\binom{\mathsf{h}(S)}{1} \ge n^2+1$.

Now we may assume that $\frac{n^2+1}{2} \ge \mathsf{h}(S) \ge 2n-2$. Since $|S_1'| = n^2-\mathsf{h}(S)+1 \ge 2n-1$, by Lemma 18 and (5), there exist $T_1|S_1'$ and $U_1|S_2'$ such that $\sigma(T_1) = \sigma(U_1)$ and $|T_1| = |U_1|$. Without loss of generality let $a_1|T_1$. Since $\left|S_1'a_1^{-1}\right| \ge n^2-\mathsf{h}(S)+1-1 \ge 2n-1$, by Lemma 13, there is a zero-sum subsequence of $S_1'a_1^{-1}$. Now by Lemma 18 and (5), there exist $T_1|S_1'a_1^{-1}$ and $U_1|S_2'$ such that $|T_2| = |U_2|$ and $\sigma(T_2) = \sigma(U_2)$. Clearly, T_1 and T_2 are different. Assume that $a_2|T_2$. Similarly we can obtain subsequences $T_3, \ldots, T_n$ of S_1' and subsequences $U_3, \ldots, U_n$ of S_2' satisfying $|T_i| = |U_i|$ and $\sigma(T_i) = \sigma(U_i)$ for for every $i \in [1,n]$, and $T_1, \ldots, T_n$ are pairwise different. Note that for every $i \in [1,n]$, $S_1'U_iT_i^{-1}0^{\mathsf{h}(S)-1}$ has sum zero and length n^2; we infer that $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge n\binom{\mathsf{h}(S)}{1} \ge n \times (2n-2) \ge n^2+1$.

Subcase 3.4: $r = \mathsf{h}(S)$. If S_2 has a zero-sum subsequence T with $|T| \ge 2$, then replace S_1 by $S_1T\mathbf{0}^{-|T|}$ and it reduces to Subcase 3.1 or Subcase 3.2.

Now we assume that S_2 is zero-sumfree. Suppose $S_1 = g_1^{\mathrm{v}_{g_1}(S_1)} \cdots g_k^{\mathrm{v}_{g_k}(S_1)}\mathbf{0}^{\mathsf{h}(S)}$, where $g_1, \ldots, g_k, \mathbf{0}$ are distinct elements in G. If there exists a subsequence T of S_2 such that $|T| \ge 2$ and $g_i = \sigma(T)$ for some i, then replace S_1 by $S_1g_i^{-1}\mathbf{0}^{-|T|+1}T$ and it reduces to Subcase 3.1 or Subcase 3.2 or Subcase 3.3. So by Lemma 17 we may suppose that $\mathrm{v}_{g_i}(S_2) = n-1$ holds for any $i \in [1,k]$. Since $|S_2| = 2n-2$, we have $k \le 2$.

If $k = 1$ then $\mathrm{v}_{g_1}(S_1) \ge n$. Therefore, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge \binom{\mathrm{v}_{g_1}(S_1)+\mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_2)} \ge \binom{n+n-1}{n-1} \ge n^2+1$.

If $k = 2$ then $g_1+g_2 \ne \mathbf{0}$ follows from S_2 is zero-sumfree. Therefore we have $\max\{\mathrm{v}_{g_1}(S_1), \mathrm{v}_{g_2}(S_1)\} \ge 2$. Thus, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge \binom{\mathrm{v}_{g_1}(S_1)+\mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_1)}\binom{\mathrm{v}_{g_2}(S_1)+\mathrm{v}_{g_2}(S_2)}{\mathrm{v}_{g_2}(S_1)} \ge \binom{1+n-1}{1}\binom{2+n-1}{2} \ge n\cdot(n+1) > n^2+1$.

`Case 4.` $\mathsf{h}(S) \le 2n-3$. Now rewrite S_1 and S_2 in the form

$$\begin{aligned} S_1 = & g_1^{\mathrm{v}_{g_1}(S_1)} \cdots g_{r_1}^{\mathrm{v}_{g_{r_1}}(S_1)} g_{r_1+1}^{\mathrm{v}_{g_{r_1+1}}(S_1)} \cdots g_{r_1+r_2}^{\mathrm{v}_{g_{r_1+r_2}}(S_1)}, \\ S_2 = & g_1^{\mathrm{v}_{g_1}(S_2)} \cdots g_{r_1}^{\mathrm{v}_{g_{r_1}}(S_2)} g_{r_1+r_2+1}^{\mathrm{v}_{g_{r_1+r_2+1}}(S_2)} \cdots g_{r_1+r_2+r_3}^{\mathrm{v}_{g_{r_1+r_2+r_3}}(S_2)}, \end{aligned}$$

where $g_1, \ldots, g_{r_1+r_2+r_3}$ are distinct elements in G.

Let

$$S_3 = g_{r_1+1}^{\mathrm{v}_{g_{r_1+1}}(S_1)} \cdots g_{r_1+r_2}^{\mathrm{v}_{g_{r_1+r_2}}(S_1)} = S_1 \left(g_1^{\mathrm{v}_{g_1}(S_1)} \cdots g_{r_1}^{\mathrm{v}_{g_{r_1}}(S_1)} \right)^{-1}.$$

If $\mathrm{v}_{g_1}(S_1) + \cdots + \mathrm{v}_{g_{r_1}}(S_1) \ge 3n-3$, then

$$(\mathrm{v}_{g_1}(S_1) + \mathrm{v}_{g_1}(S_2)) + \cdots + \left(\mathrm{v}_{g_{r_1}}(S_1) + \mathrm{v}_{g_{r_1}}(S_2)\right) \ge 3n-2.$$

By Lemma 20, we have

$$\begin{aligned} \mathsf{N}_{\mathbf{0}}^{|G|}(S) &\ge \binom{\mathrm{v}_{g_1}(S_1) + \mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_1)} \cdots \binom{\mathrm{v}_{g_{r_1}}(S_1) + \mathrm{v}_{g_{r_1}}(S_2)}{\mathrm{v}_{g_{r_1}}(S_1)} \\ &\ge (\mathrm{v}_{g_1}(S_1) + \mathrm{v}_{g_1}(S_2)) \cdots \left(\mathrm{v}_{g_{r_1}}(S_1) + \mathrm{v}_{g_{r_1}}(S_2)\right) \\ &\ge n^2 + 1. \end{aligned}$$

So we may assume that $\mathrm{v}_{g_1}(S_1) + \cdots + \mathrm{v}_{g_{r_1}}(S_1) \le 3n-4$.

Let $N_1 = \binom{\mathrm{v}_{g_1}(S_1)+\mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_1)} \cdots \binom{\mathrm{v}_{g_{r_1}}(S_1)+\mathrm{v}_{g_{r_1}}(S_2)}{\mathrm{v}_{g_{r_1}}(S_1)}$. Let N_2 denote the number of subsequences T_1 of S_3 satisfying

(I) $|T_1| = 2$, and

(II) there is a subsequence T_2 of S_2 such that $|T_2| = 2$ and $\sigma(T_1) = \sigma(T_2)$.

Clearly, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge N_1 + N_2$. So we may assume that

$$N_2 \le n^2.$$

By Lemma 21 there exists a subsequence W of S_3 such that S_3W^{-1} contains no subsequence satisfying both (I) and (II) and such that

$$|W| \le \frac{|S_3|}{2} + \frac{N_2}{4}.$$

Let $\mathcal{N}_3$ denote the set of nonempty subsequences T_1 of S_3W^{-1} such that $|T_2| = |T_1|$ and $\sigma(T_1) = \sigma(T_2)$ for some $T_2|S_2$. By the definition of $W|S_3$ we know that

$$|T_1| \ge 3 \tag{6}$$

holds for every $T_1 \in \mathcal{N}_3$.

Let $k = |S_3W^{-1}|$. Note that

$$\begin{aligned}|S_3W^{-1}| &= |S_3| - |W|\\ &\geq |S_3| - \frac{|S_3|}{2} - \frac{N_2}{4} = \frac{|S_3|}{2} - \frac{N_2}{4}\\ &\geq \frac{1}{2}\left(n^2 - \left(\mathrm{v}_{g_1}(S_1) + \cdots + \mathrm{v}_{g_{r_1}}(S_1)\right)\right) - \frac{1}{4}n^2\\ &\geq \frac{1}{4}n^2 - \frac{3}{2}n + 2.\end{aligned}$$

Therefore

$$k \geq \frac{1}{4}n^2 - \frac{3}{2}n + 2. \tag{7}$$

Note that every $T_1 \in \mathcal{N}_3$ is contained by

$$\binom{k - |T_1|}{2n - 2 - |T_1|} = \binom{k - |T_1|}{k - (2n-2)}$$

subsequences of S_3W^{-1} with length $2n-2$. By Lemma 16 we have

$$\sum_{T_1 \in \mathcal{N}_3} \binom{k - |T_1|}{k - (2n-2)} \geq \binom{k}{k - (2n-2)}. \tag{8}$$

Let $N_3 = |\mathcal{N}_3|$. Combining (6), (7) and (8) we obtain that

$$\begin{aligned}N_3 &\geq \frac{\binom{k}{k-(2n-2)}}{\binom{k-3}{k-(2n-2)}} = \frac{\binom{k}{2n-2}}{\binom{k-3}{2n-5}}\\ &= \frac{k(k-1)(k-2)}{(2n-2)(2n-3)(2n-4)}\\ &\geq \frac{\left(\frac{1}{4}n^2 - \frac{3}{2}n + 2\right)\left(\frac{1}{4}n^2 - \frac{3}{2}n + 1\right)\left(\frac{1}{4}n^2 - \frac{3}{2}n\right)}{(2n-2)(2n-3)(2n-4)}\\ &\geq n^2 + 1 \,(\text{since } n \geq 526).\end{aligned}$$

So $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq N_1 + N_2 + N_3 \geq n^2 + 1$. This completes the proof. □

5. Remarks and Open Problems

Conjecture 1.2 and Theorem 3 suggest the following.

Conjecture 22 Let $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ be a finite abelian group, where $n_i | n_{i+1}$ for any $i \in [1, r-1]$. Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = |G| + \mathrm{D}(G) - 1$. Then

$$\mathsf{N}_g^{|G|}(S) = 0 \text{ or } \mathsf{N}_g^{|G|}(S) \geq n_1$$

for every $g \in G \setminus \{\mathbf{0}\}$.

From the following result, it is easy to see that Conjecture 22 is true for all elementary abelian groups.

Proposition 23 *Let p be a prime, and let G be a finite abelain p-group. Let $S \in \mathcal{F}(G)$ with $|S| = |G| + D(G) - 1$. Then $\mathsf{N}_g^{|G|}(S) = 0$ or $\mathsf{N}_g^{|G|}(S) \geq p$ for every $g \in G \setminus \{\mathbf{0}\}$, and either $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1$ or $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq p + 1$.*

Proof. By a result in [10] (or see [13], Theorem 8.3) we know that

$$\mathsf{N}_g^{|G|}(S) \equiv \begin{cases} 1 \pmod p, & \text{if } g = \mathbf{0}, \\ 0 \pmod p, & \text{otherwise.} \end{cases}$$

Now the proposition follows. □

Conjecture 24 Let G be a finite abelian group. Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = |G| + \mathrm{D}(G) - 1$. If $G \neq C_2 \oplus C_2$, then

$$\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 0 \text{ or } \mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq |G| + 1.$$

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References

[1] A. Bialostocki, P. Dierker, D. Grynkiewicz and M. Lotspeich, *On some developments of the Erdős-Ginzburg-Ziv Theorem II*, Acta Arith. **110** (2003) 173-184.

[2] A. Bialostocki and M. Lotspeich, *Some developments of the Erdős-Ginzburg-Ziv Theorem I*, in: Sets, Graphs and Numbers, vol. 60, Colloquia Mathematica Societatis János Bolyai , North-Holland, Amsterdam, New York (1992) 97-117.

[3] B. Bollobás and I. Leader, *The number of k-sums modulo k*, J. Number Theory **78** (1999) 27-35.

[4] Y. Caro, *Remarks on a zero-sum theorem*, J. Combin. Theory Ser. A **76** (1996) 315-322.

[5] P. Erdős, A. Ginzburg and A. Ziv, *Theorem in the additive number theory*, Bull. Res. Council Israel **10F** (1961) 41-43.

[6] Z. Füredi and D.J. Kleitman, *The number of zero sums, in: Combinatorics*, Paul Erdős is Eighty, J. Bolyai Mathematical Society (1993) 159-172.

[7] W.D. Gao, *A combinatorial problem on finite abelian groups*, J. Number Theory **58** (1995) 100-103.

[8] W.D. Gao, *Two addition theorems on groups of prime order*, J. Number Theory **56** (1996) 211-213.

[9] W.D. Gao, *On the number of subsequences with given sum*, Discrete Math. **195** (1999) 127-138.

[10] W.D. Gao, *On zero-sum subsequences of restricted size II*, Discrete Math. **271** (2003) 51-59.

[11] W.D. Gao and A. Geroldinger, *On long minimal zero sequences in finite abelian groups*, Period Math. Hungar. **38** (1999) 179-211.

[12] W.D. Gao and A. Geroldinger, *On zero-sum sequences in* $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, Integers **3** (2003) (Paper A08).

[13] W.D. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: A survey*, Expo. Math. **24** (2006) 337-369.

[14] W.D. Gao, A. Geroldinger, Y.L. Li and H.Y. Zhang, *On Property B*, Preprint, 2007.

[15] A. Geroldinger and F. Halter-Koch, Non-unique factorizations, Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematixe, vol. 278, Chapman & Hall/CRC, London, Boca Raton, Fl,2006.

[16] D.J. Grynkiewicz, *On the number of m-term zero-sum subsequences*, Acta Arith. **121** (2006) 275-298.

[17] Y.O. Hamidoune, O. Ordaz and A. Ortuño, *On a combinatorial theorem of Erdős, Ginzburg and Ziv*, Combin. Probab. Comput. **7** (1998) 403-412.

[18] M. Kisin, *The number of zero sums modulo m in a sequence of length n*, Mathematika **41** (1994) 149-163.

[19] H.B. Mann, *Two addition theorems*, J. Combinatorial Theory **3** (1967) 233-235.

[20] M.B. Nathanson, Additive Number Theory: Inverse Problems and the Geometry of Sumsets, Springer, 1996.

[21] J.E. Olson, *A combinatorial problem on finite abelian groups I*, J. Number Theory **1** (1969) 8-10.

[22] J.E. Olson, *A combinatorial problem on finite abelian groups II*, J. Number Theory **1** (1969) 195-199.