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ON RAINBOW SOLUTIONS TO AN EQUATION WITH A QUADRATIC TERM

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Abstract

In this paper, we prove that every 3-coloring of the positive integers such that the upper density of each color is greater than $\frac{1}{4}$ contains a rainbow solution to $a-b=c^2$. A solution is *rainbow* if all of its elements are of different colors. Furthermore, the $\frac{1}{4}$ bound is sharp. We also prove two results for rainbow solutions of $a-b=c^2$ in $\mathbb{Z}_n$. One stipulates that if $\mathbb{Z}_n$, for an odd n, is partitioned into three color classes $\mathcal{R}, \mathcal{B}, \mathcal{G}$ with $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} > \frac{n}{r_1}$, where r_1 is the smallest prime factor of n, then there must always exist a rainbow solution to $a-b \equiv c^2 \bmod n$. Our second theorem in $\mathbb{Z}_n$ extends this, demonstrating that if we have $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} > \frac{n}{2r_1}$, then there exists a rainbow solution to $a-b \equiv c^2 \bmod n$ except in a very specific case, which we classify.

1. Introduction

Ramsey Theory conjectures that, in the words of T. Motzkin, "complete disorder is impossible" (reference within [11]). Problems in Ramsey Theory generally involve demonstrating that there exists some n such that if the set $\{1, 2, ..., n\}$ is partitioned into k sets, for a finite k, then one of the sets contains a specific interesting property. Because of its focus on finding patterns within disarray, Ramsey Theory has wide applications in theoretical computer science. It is currently being used in developing faster algorithms as well as more effecient computer networks, and also has vast applications in other fields of mathematics, including geometry, number theory, and graph theory [12].

Many famous results in Ramsey Theory employ the concept of the *partition regular* equation, which means that any partition of the natural numbers into finitely many color classes will always contain a monochromatic solution to said equation [7]. Schur [14] proved arguably the first major theorem in this field in 1916, demonstrating that the equation $x+y=z$ is partition regular. However, nonlinear equations were not considered until the 1970s, when Erdös and Graham conjectured that the equation $x^2+y^2=z^2$ was partition regular [7]. Their conjecture remains unresolved, but some results have been proven, such as Rödl's theorem (reference within [7]) that the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ is partition regular.

Around the same time, Schönheim [13] proved the first multicolor counterpart to traditional Ramsey Theory, that every partition of $\{1, 2, ..., n\}$ into three color classes, each containing more than $\frac{n}{4}$ numbers, has a solution to $x + y = z$ with x, y, z belonging to different color classes. The authors of [10] called such solutions *rainbow* and, in addition, proved that any 3-coloring of the natural numbers where each color appeared more than one-sixth of the time contained a rainbow solution to $x + y = 2z$. Axenovich and Fon-Der-Flaass [2] developed a similar result for the partition of the set $\{1, 2, ..., n\}$, and [5] yielded the $\frac{1}{6}$ density bound for the "Sidon" equation, $w + x = y + z$, in four colors.

So inspired by these results, we combined both of these developments and prove the first results in Rainbow Ramsey Theory for an equation with a quadratic term.

In particular, we prove the following result.

Theorem 1. *Every 3-coloring of the set of natural numbers with the upper density of each color class greater than $\frac{1}{4}$ contains a rainbow solution to $a - b = c^2$.*

The search for rainbow solutions can also be extended to the modular version of $a - b = c^2$, and a corollary of Theorem 1.1 states that if $\mathbb{Z}_n$ is partitioned into three color classes each having cardinality greater than $\frac{n}{4}$, then there exists a rainbow solution to $a - b = c^2 \bmod n$. Here, however, the $\frac{n}{4}$ bound can be significantly improved, as we will discuss in Section 3.

Previous work concerning these modular Rainbow Ramsey Theory problems have been concentrated on equations with fully linear terms. For example, the authors of [10] proved a strong bound for the modular equation $a + b \equiv 2c \bmod n$. Later, [11] conjectured the exact bound for the same equation. In 2006, Conlon [4] obtained results about the fully generalized linear equation of the form $a_1x_1 + a_2x_2 + ... + a_kx_k \equiv b \bmod p$ with p being a prime and $a_1, a_2, ..., a_k, b$ being integer constants. In this paper, we extend this repertoire by first showing that an exact bound for the modular case of $a - b = c^2$ follows as a simple corollary to Theorem 1, and then by significantly strengthening the bound in almost all cases, and moreover, classifying all of the exceptions.

2. The Infinite Version of $a - b = c^2$

Before proving our main results, we define some terms. Let $\bar{c} : \mathbb{N} \to \{R, B, G\}$ be a 3-coloring of the set of natural numbers and let $\mathcal{R}, \mathcal{B}, \mathcal{G}$ be the three corresponding color classes, red, blue, and green. For any subset S of $\mathbb{N}$ and any $n \in \mathbb{N}$, define $S(n) = |S \setminus \{1, 2, ..., n\}|$; hence, $\mathcal{R}(n) = |\mathcal{R} \setminus \{1, 2, ..., n\}|$, and similarly for $\mathcal{B}, \mathcal{G}$. A *rainbow* solution to the equation $a - b = c^2$ in the coloring $\bar{c}$ is any ordered triple of positive integers (i_1, i_2, i_3), all of different colors, such that $i_1 - i_2 = i_3^2$. We will say that $\bar{c}$ is a *rainbow-free coloring* if there is no rainbow solution to the equation $a - b = c^2$ in $\bar{c}$.

We now proceed to the main result of this section (a more precise statement of Theorem 1):

Theorem 2. *Suppose* $\mathbb{N}$ *is partitioned into three color classes* $\mathcal{R}, \mathcal{B}, \mathcal{G}$*, such that*

$$\limsup_{n\to\infty} \left(\min \{\mathcal{R}(n), \mathcal{B}(n), \mathcal{G}(n)\} - \frac{n}{4} \right) = \infty.$$

Then there exists a rainbow solution to $a - b = c^2$ *in* $\bar{c}$*.*

Let $(i_1, i_2, ..., i_s)$ denote the greatest common divisor of the integers $i_1, i_2, ..., i_s$, $s \in \mathbb{N}$. A *string* of *length* l at *position* i consists of the numbers $i, i+1, i+2, ..., i+l-1$, $i, l \in \mathbb{N}$, and is termed *monochromatic* if it contains only one color, and *bichromatic* if it contains exactly two colors. A color is *dominant* if every bichromatic string contains that color, and *nondominant* otherwise. In particular, if a dominant color exists then it must be unique, and no two nondominant colors can ever appear next to each other.

We begin by proving several preliminary results.

Lemma 3. *Let* $\bar{c}$ *be a rainbow-free coloring of* $\mathbb{N}$*. Then there exists a dominant color.*

Proof. Without loss of generality, assume that $\bar{c}(1) = R$. If for some i_1, we have $\bar{c}(i_1) = B$, $\bar{c}(i_1 + 1) = G$ or $\bar{c}(i_1) = G$, $\bar{c}(i_1 + 1) = B$, then $i_1 + 1, i_1, 1$ would form a rainbow solution. Hence, red is dominant, and green and blue never appear next to each other. □

Lemma 4. *Let* $\bar{c}$ *be a rainbow-free coloring of* $\mathbb{N}$ *such that* R *is dominant. If* i_1 *is colored in a nondominant color, then for any string of the other nondominant color at position* i *of length* l*, the strings at positions* $i \pm i_1^2$ *of length* l *must be monochromatic and colored either* B *or* G*.*

Proof. Assume that $\bar{c}(i_1) = B$, and that the string of length l at position i is colored green. Suppose that for some j, $0 \le j \le l-1$, $\bar{c}\left(i + j + i_1^2\right) = R$. Then $i + j + i_1^2, i+j, i_1$ form a rainbow. Hence, the string at position $i+i_1^2$ of length l contains at most two colors, B and G. Without loss of generality, assume that $\bar{c}\left(i + i_1^2\right) = G$. Repeated use of Lemma 2.2 shows that $\bar{c}\left(i + i_1^2 + 1\right) = \bar{c}\left(i + i_1^2 + 2\right) = \cdots = \bar{c}\left(i + i_1^2 + l - 1\right) = G$, as desired. A similar argument shows that the string at position $i - i_1^2$ of length l is also colored blue or green. □

Lemma 5. *Suppose some set* $S \subset \mathbb{N}$ *satisfies* $\limsup_{n\to\infty} \left(S(n) - \dfrac{n}{n_0} \right) = \infty$ *for some positive integer* $n_0 \ge 2$*. Then there exists a* $k \le n_0 - 1$ *such that for any* i*, there exists* $j > i$ *such that* j *and* $j + k$ *are both elements of* S*.*

Proof. Suppose not; then there exists an n_0' such that all integers in S greater than n_0' are at least n_0 apart. Hence, $S(n) \le n_0' + \frac{n}{n_0}$, which contradicts the lemma conditions. □

We now employ the following classical result [3]:

Theorem (Frobenius). Let i_1, j_1 be two relatively prime integers. Then all integers greater than $i_1 j_1 - i_1 - j_1$ can be written in the form $i_1 u + j_1 v$ for some nonnegative u, v.

From this, we immediately obtain the following:

Corollary 6. Suppose two integers i_1 and j_1 satisfy $(i_1, j_1) = k$. Then there exists an integer n_0 such that all numbers greater than n_0 divisible by k can be written in the form $ui_1 + vj_1$, for nonnegative u, v.

Lemma 7. *Let $\bar{c}$ be a rainbow-free coloring of $\mathbb{N}$ satisfying the density condition in Theorem 2.1, with R being the dominant color. Then both $\mathcal{B}$ and $\mathcal{G}$ must contain a pair of relatively prime integers.*

Proof. Write $\mathcal{A} = \mathcal{B} \cup \mathcal{G}$. Suppose that $\mathcal{A}$ contains no consecutive integers. Then for any $i \in \mathbb{N}$, at least one of i and $i+1$ must be in $\mathcal{R}$. But then $\liminf_{n\to\infty} \left(\mathcal{R}(n) - \frac{n}{2}\right) \geq 0$, so $\limsup_{n\to\infty} \left(\min\{\mathcal{B}(n), \mathcal{G}(n)\} - \frac{n}{4}\right) \leq 0$, contradicting the density assumption in Theorem 2. So there exists an i_1 such that $i_1, i_1 + 1 \in \mathcal{A}$; by Lemma 3, we may assume without loss of generality that $i_1, i_1 + 1 \in \mathcal{B}$.

Assume that $\mathcal{G}$ does not contain a pair of relatively prime integers. Let k'_1 be the minimal positive difference between any two elements of $\mathcal{G}$. Since we have $\limsup_{n\to\infty} \left(\mathcal{G}(n) - \frac{n}{4}\right) = \infty$, by Lemma 5 we get $k'_1 \leq 3$. Now consider an i'_1 such that $i'_1, i'_1 + k'_1 \in \mathcal{G}$.

If $k'_1 = 2$ then $(i'_1, i'_1 + 2) = 2$. By the assumption in the first paragraph of this lemma, there exists a blue string of length $l_0 \geq 2$ at position i_1. Corollary 6 tells us that there exists an integer n_0 such that all integers greater than n_0 that are divisible by $\left(i'^2_1, (i'_1 + 2)^2\right) = 4$ can be expressed in the form $i'^2_1 u + (i'_1 + 2)^2 v$. Hence, all integers greater than $i_1 + n_0$ which are congruent to $i_1 \bmod 4$ can be expressed in the form $i_1 + i'^2_1 u + (i'_1 + 2)^2 v$. From Lemma 4 and our assumption that $k'_1 = 2$ it follows that the strings at $i_1 + i'^2_1$ and $i_1 + (i'_1 + 2)^2$ of length 2 are colored blue. By induction we obtain that, for any nonnegative integers u and v, the string at $i_1 + i'^2_1 u + (i'_1 + 2)^2 v$ of length 2 is colored blue. Therefore, there exists an integer n'_0 such that there exists a blue string of length greater than or equal to l_0 at all integers of the form $n'_0 + 4k$ for all nonnegative k. By definition of l_0, $\bar{c}(n'_0 + 4k) = \bar{c}(n'_0 + 4k + 1) = B$, and by Lemma 3, $\bar{c}(n'_0 + 4k + 2) \neq G$ and $\bar{c}(n'_0 + 4k + 3) \neq G$. But this implies that the number of greens is finite, a contradiction.

Finally, suppose that $k'_1 = 3$; we must therefore have $3 | i'_1$, so $(i'_1, i'_1 + 3) = 3$. By Corollary 6, there exists an integer n_1 such that all integers greater than or equal to n_1 that are divisible by $\left(i'^2_1, (i'_1 + 3)^2\right) = 9$ can be expressed in the form $i'^2_1 u + (i'_1 + 3)^2 v$. Using a similar analysis as in the case above, there is an integer n'_1 such that there exists a blue string of length at least l_0 at all integers of the form $n'_1 + 9k$ for all nonnegative k. By Lemma 3 and the definition of l_0, no numbers of the form $n'_1 + 9k$,

$n_1'+9k+1, n_1+9k+2, n_1'+9k+8$ are colored green. Since k_1' is minimal, it follows that for any fixed k', at most two of the numbers $n_1'+9k'+3, n_1'+9k'+4, ..., n_1'+9k'+7$ are colored green. But then $\mathcal{G}(n) \leq n_1' + \frac{2n}{9}$, contradicting the density assumption in Theorem 2. □

Lemma 8. *Let $\bar{c}$ be a rainbow-free coloring of $\mathbb{N}$ satisfying the density condition in Theorem 2.1. Then the length of every monochromatic string colored B or G is bounded above.*

Proof. First, note that no string has infinite length; otherwise, the density condition in Theorem 2 would be violated. Hence, suppose that i_1 is the smallest integer that is colored blue, and suppose for the sake of contradiction that there exist arbitrarily long green strings. Then there must exist integers j_1 and l_0 such that $l_0 \geq i_1^2$ and that there is a green string of length l_0 at position j_1. It follows that the ordered triple $\left(j_1 + i_1^2 - 1, j_1 - 1, i_1\right)$ is a rainbow solution. □

Before proceeding, we will define some more terms. An infinite arithmetic progression with initial term i and common difference d is termed monochromatic if all of its elements are of one color. We say that $j \in \mathbb{N}$ has the *A-property* if it is colored in a nondominant hue and there exists a monochromatic infinite arithmetic progression of the other nondominant color with common difference j^2.

Lemma 9. *Let $\bar{c}$ be a rainbow-free coloring of $\mathbb{N}$ with R being the dominant color. Suppose i_1, i_1' are colored in the same nondominant hue, and $(i_1, i_1') = 1$. Then i_1, i_1' cannot both have the A-property.*

Proof. Assume instead that i_1, i_1' both have the A-property and are colored green. Suppose that there exist two blue infinite arithmetic progressions, one with common difference i_1^2 and initial term i_2, and another with common difference $i_1'^2$ and initial term i_2'. Let the blue string at position i_2 have length l_0. By definition and by using Lemma 4, there exist blue strings of length at least l_0 at all integers of the form $i_2 + ki_1^2$ or $i_2' + ki_1'^2$, with k a nonnegative integer. Since $\left(i_1^2, i_1'^2\right) = 1$, there exist positive integers u_0, v_0 such that $u_0 i_1^2 - v_0 i_1'^2 = i_2' - i_2$, or $i_2 + u_0 i_1^2 = i_2' + v_0 i_1'^2$. Define $i_2'' = i_2 + u_0 i_1^2$ and consider the blue string at position i_2''; assume that it has length l_0'. Since i_2'' is a common element of both infinite progressions, then again by Lemma 2.3, there exist blue strings with length at least l_0' at positions $i_2'' + ki_1^2$ and $i_2'' + ki_1'^2$ for all nonnegative k. Additionally, we can find positive integers u_0', v_0' such that $u_0' i_1^2 - v_0' i_1'^2 = 1$. But this means that there is a blue string of length $l_0' + 1$ at position $i_2'' + v_0' i_1'^2$. Repeating this argument, we can generate arbitrarily long blue strings, contradicting Lemma 8. □

In proving our final lemma, we will use some more notation. Let the *magnitude* function be given by $M\left(u, v, w, i, D\right) = i + ud_1^2 + vd_2^2 + wd_3^2$, where u, v, w, i are integers and $D = (d_1, d_2, d_3)$. Also, define a *path* of *complexity* m to be a sequence of distinct lattice points $P = \{\langle a_{P,1}, b_{P,1}\rangle, \langle a_{P,2}, b_{P,2}\rangle, ..., \langle a_{P,m}, b_{P,m}\rangle\}$ such that $|a_{P,i} - a_{P,i-1}| + |b_{P,i} - b_{P,i-1}| = 1$ for all $2 \le i \le m$, in the ab-plane.

Lemma 10. *Assume that R is the dominant color in a coloring $\bar{c}$, and suppose that there exist $i_1, i_1' \in \mathcal{B}$, $(i_1, i_1') = 1$ with $j_1, j_1' \in \mathcal{G}$, $(j_1, j_1') = 1$. Then there must exist a rainbow solution to $a - b = c^2$ in $\bar{c}$.*

Proof. Suppose instead that $\bar{c}$ is rainbow-free. By Lemma 9, at least one of i_1, i_1' and one of j_1, j_1' cannot have the A-property. So without loss of generality, assume that i_1, j_1 do not have this property. Since $(i_1', i_1, j_1) = 1$, it is then well-known that there exist integers u_0, v_0, w_0 with $u_0 > 0$, $v_0, w_0 < 0$ satisfying $u_0 i_1'^2 + v_0 i_1^2 + w_0 j_1^2 = -1$, so take $D_0' = (i_1', i_1, j_1)$.

Consider any i_2 with $c(i_2) \neq R$ and $i_2 > \max\left\{-v_0 i_1^2, -w_0 j_1^2\right\}$, and assume that the string there has length l_0. Take the pair $(u', u'') = (0, 0)$, and repeated perform the following algorithm: if $M(u', 0, u'', i_2, D_0')$ is colored green, then increase u' by 1; conversely, if $M(u', 0, u'', i_2, D_0')$ is colored blue, increase u'' by 1. From Lemma 4 we obtain that there is a non-red string of length at least l_0 at all such integers $M(u', 0, u'', i_2, D_0')$. We must reach a pair (u', u'') with $u' = u_0$; if not, then after some point we must increase u'' infinitely many times consecutively. But then there exists a blue arithmetic progression of common difference j_1^2, contradicting our choice of j_1. So at the point when $u' = u_0$, assume that $u'' = k_1$ for some $k_1 \ge 0$, and that there exists a non-red string of length $l_0' \ge l_0$ at position $i_2 + u_0 i_1'^2 + k_1 j_1^2$.

Construct a path P_0 by first letting $\langle v_{P_0,1}, w_{P_0,1}\rangle = \langle 0, k_1\rangle$. For $i \ge 2$, if we have that $M(u_0, v_{P_0,i-1}, w_{P_0,i-1}, i_2, D_0')$ is colored green, set $\langle v_{P_0,i}, w_{P_0,i}\rangle = \langle v_{P_0,i-1} + 1, w_{P_0,i-1}\rangle$. Conversely, if $M(u_0, v_{P_0,i-1}, w_{P_0,i-1}, i_2, D_0')$ is colored blue, set $\langle v_{P_0,i}, w_{P_0,i}\rangle = \langle v_{P_0,i-1}, w_{P_0,i-1} - 1\rangle$. By Lemma 2.3 and our method of constructing P_0, there must exist a non-red string of length at least l_0' at all positions of the form $M(u_0, v_{P_0,i}, w_{P_0,i}, i_2, D_0')$ if this integer is positive.

Since i_1 does not have the A-property, there cannot exist an i'' such that for all $i > i''$, $\langle v_{P_0,i}, w_{P_0,i}\rangle = \langle v_{P_0,i-1} + 1, w_{P_0,i-1}\rangle$. Hence, for a P_0 of sufficiently large complexity, there must exist at least $k_1 - w_0$ integers i with $\langle v_{P_0,i}, w_{P_0,i}\rangle = \langle v_{P_0,i-1}, w_{P_0,i-1} - 1\rangle$. Therefore, there exists an m_0 such that $w_{P_0,m_0} = w_0$. Terminate P_0 at the point $\langle v_{P_0,m_0}, w_{P_0,m_0}\rangle$; hence, P_0 has complexity m_0. Then all points $\langle v_{P_0,i}, w_{P_0,i}\rangle$, $1 \le i \le m_0$, must satisfy $M(u_0, v_{P_0,i-1}, w_{P_0,i-1}, i_2, D_0') > 0$ since $w_{P_0,i} \ge w_0$ for all $1 \le i \le m_0$ and $i_2 > -w_0 j_1^2$, and therefore there exist non-red strings of length at least l_0' at all positions of the form $M(u_0, v_{P_0,i}, w_{P_0,i}, i_2, D_0') > 0$.

Take another path P_0' satisfying $\left\langle v_{P_0',1}, w_{P_0',1}\right\rangle = \langle 0, k_1\rangle$. For $i \ge 2$, if we have that $M\left(u_0, v_{P_0',i-1}, w_{P_0',i-1}, i_2, D_0'\right)$ is colored green, set $\left\langle v_{P_0',i}, w_{P_0',i}\right\rangle = \left\langle v_{P_0',i-1} - 1, w_{P_0',i-1}\right\rangle$. But if $M\left(u_0, v_{P_0',i-1}, w_{P_0',i-1}, i_2, D_0'\right)$ is blue, set $\left\langle v_{P_0',i}, w_{P_0',i}\right\rangle = \left\langle v_{P_0',i-1}, w_{P_0',i-1} + 1\right\rangle$. Again, if $v_{P_0',i} > v_0$, $M\left(u_0, v_{P_0',i-1}, w_{P_0',i-1}, i_2, D_0'\right)$ must be positive: again, there exists a blue or green string of length $\ge l_0'$ at all positions of the form $M\left(u_0, v_{P_0',i}, w_{P_0',i}, i_2, D_0'\right)$ if this value is positive. We can conclude that there exists an m_0' such that $v_{P_0',m_0'} = v_0$. Terminate P_0' at the point $\left\langle v_{P_0',m_0'}, w_{P_0',m_0'}\right\rangle$.

Now consider the union P_1 of P_0 and P_0'; it is contiguous (connected) and has complexity $m_0 + m_0' - 1$. Define $\langle v_{P_1,1}, w_{P_1,1}\rangle = \left\langle v_{P_0',m_0'}, w_{P_0',m_0'}\right\rangle$ and therefore it follows that $\left\langle v_{P_1,m_0+m_0'-1}, w_{P_1,m_0+m_0'-1}\right\rangle = \langle v_{P_0,m_0}, w_{P_0,m_0}\rangle$. Let P_1' be the path formed when P_1 is shifted $-v_0$ units to the right and $-w_0$ units up in the vw-plane. Then $\left\langle v_{P_1',1}, w_{P_1',1}\right\rangle = \left\langle 0, w_{P_0',m_0'} - w_0\right\rangle$ and $\left\langle v_{P_1',m_0+m_0'-1}, w_{P_1',m_0+m_0'-1}\right\rangle = \langle v_{P_0,m_0} - v_0, 0\rangle$.

Finally, take the path P_2 satisfying $\langle v_{P_2,1}, w_{P_2,1}\rangle = \langle 0, 0\rangle$. For $i \geq 2$, if we have that $M(0, v_{P_2,i-1}, w_{P_2,i-1}, i_2, D_0')$ is colored green, then set $\langle v_{P_2,i}, w_{P_2,i}\rangle = \langle v_{P_2,i-1} + 1, w_{P_2,i-1}\rangle$. But if $M(0, v_{P_2,i-1}, w_{P_2,i-1}, i_2, D_0')$ is colored blue, then set $\langle v_{P_2,i}, w_{P_2,i}\rangle = \langle v_{P_2,i-1}, w_{P_2,i-1} + 1\rangle$. As with path P_0, there exists a non-red string of length $\geq l_0$ at all positions of the form $M(0, v_{P_2,i}, w_{P_2,i}, i_2, D_0')$. Figure 1 shows the three paths P_1, P_1', and P_2 transposed onto the Cartesian plane.

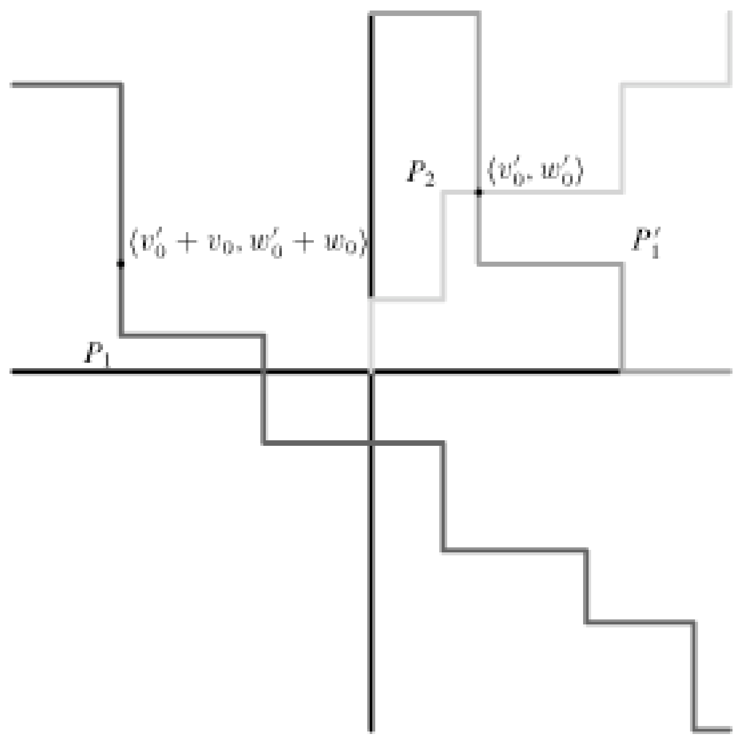

Figure 1: Paths P_1, P_1', P_2 and the two crucial points.

By the construction of P_1' and P_2, P_1' must intersect P_2 at some point $\langle v_0', w_0'\rangle$ and that $v_0' + w_0' > 0$. This corresponds to the point $\langle v_0' + v_0, w_0' + w_0\rangle$ on P_1, which in turn corresponds to the magnitude $i_2 + u_0 i_1'^2 + v_0' i_1^2 + v_0 i_1^2 + w_0' j_1^2 + w_0 j_1^2$.

On the other hand, $\langle v_0', w_0' \rangle$ on P_2 corresponds to the magnitude $i_2+v_0'i_1^2+w_0'j_1^2$. The difference between these is $u_0 i_1'^2 + v_0 i_1^2 + w_0 j_1^2 = -1$. Since the strings at positions $i_2+v_0'i_1^2+w_0'j_1^2$ and $i_2+u_0i_1'^2+v_0'i_1^2+v_0i_1^2+w_0'j_1^2+w_0j_1^2 = i_2+v_0'i_1^2+w_0'j_1^2-1 \geq i_2$ are non-red, adjacent, and have length at least l_0, it follows that there must exist a non-red string of length at least l_0+1 at position $i_2+v_0'i_1^2+w_0'j_1^2-1$. Repeating the above arguments, we can generate arbitrarily long non-red strings at positions greater than or equal to i_2, contradicting Lemma 8. □

Proof of Theorem 2. Assume there is a rainbow-free coloring of $\mathbb{N}$ satisfying the density condition in Theorem 2. Lemma 3 tells us that there exists a dominant color, and without loss of generality assume that it is R. By Lemma 7, there exist integers $i_1, i_1' \in \mathcal{B}$ and $j_1, j_1' \in \mathcal{G}$ with $(i_1, i_1') = 1$ and $(j_1, j_1') = 1$. Then by Lemma 10, there must exist a rainbow solution to $a - b = c^2$, contradiction. □

The following proposition shows that the constant $\frac{1}{4}$ in the density assumption cannot be weakened:

Proposition 11. *There exists a rainbow-free coloring of* $\mathbb{N}$ *such that for every* n

$$\min\{\mathcal{R}(n), \mathcal{B}(n), \mathcal{G}(n)\} = \left\lfloor \frac{n}{4} \right\rfloor.$$

Proof. Consider the following coloring of $\mathbb{N}$:

$$\bar{c}(i) = \begin{cases} R & \text{if } i \equiv 1 \bmod 2, \\ B & \text{if } i \equiv 2 \bmod 4, \\ G & \text{if } i \equiv 0 \bmod 4. \end{cases}$$

It is not difficult to see that this coloring does not contain a rainbow solution to $a - b = c^2$ and that $\min\{\mathcal{R}(n), \mathcal{B}(n), \mathcal{G}(n)\} = \left\lfloor \frac{n}{4} \right\rfloor$. □

3. The Modular Version of $\mathbf{a - b = c^2}$

A rainbow solution to $a - b = c^2$ in $\mathbb{Z}_n$ is an ordered triple of positive integers (i_1, i_2, i_3), all of different colors, such that $i_1 - i_2 \equiv i_3^2 \bmod n$. For a 3-coloring $\bar{c} : \mathbb{Z}_n \to \{R, B, G\}$, we define the color class $\mathcal{R}$ to be the set of congruence classes mod n which are colored red. Define $\mathcal{B}$ and $\mathcal{G}$ similarly. Also from $\bar{c}$, define a 3-coloring $\bar{c}'$ of $\mathbb{N}$ as follows: for every $i \in \mathbb{N}$, $\bar{c}'(i) = \bar{c}(i \bmod n)$, and denote the corresponding color classes by $\mathcal{R}', \mathcal{B}', \mathcal{G}'$. Therefore, if there exists a rainbow solution in every 3-coloring $\bar{c}'$ of $\mathbb{N}$, then there must exist a rainbow solution in all 3-colorings $\bar{c}$ in $\mathbb{Z}_n$.

A natural question is if the density constant of $\frac{1}{4}$ in Theorem 2.1 can be weakened when working in $\mathbb{Z}_n$. For n divisible by 4, the construction given in Proposition 2.10 shows that this bound is tight. If n is divisible by 2 but not by 4, we can consider the following proposition:

Proposition 12. *There exists a rainbow-free coloring of $\mathbb{Z}_n$ such that*

$$\min\{\mathcal{R}(n), \mathcal{B}(n), \mathcal{G}(n)\} = \left\lfloor \frac{n}{4} \right\rfloor.$$

Proof. The following coloring of $\mathbb{Z}_n$ is rainbow-free:

$$\bar{c}(i) = \begin{cases} R & \text{if } i \equiv 1 \bmod 2, \\ B & \text{if } i \equiv 2 \bmod 4, \\ G & \text{if } i \equiv 0 \bmod 4. \end{cases}$$

□

However, for odd n, we have two much stronger results:

Theorem 13. *Let n be odd, and let r_1 be the smallest prime factor of n. If $\mathbb{Z}_n$ is partitioned into three color classes $\mathcal{R}, \mathcal{B}, \mathcal{G}$ such that* $\min\{\mathcal{R}(n), \mathcal{B}(n), \mathcal{G}(n)\} > \dfrac{n}{r_1}$ *then there exists a rainbow solution to $a - b = c^2$ in $\mathbb{Z}_n$.*

Theorem 14. *Let n be an odd number, and let r_1 be the smallest prime factor of n. Suppose $\mathbb{Z}_n$ is partitioned into three color classes $\mathcal{R}, \mathcal{B}, \mathcal{G}$ such that $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} > \dfrac{n}{2r_1}$. Then there exists a rainbow solution to $a - b = c^2$ in $\mathbb{Z}_n$ if, for each prime factor of n from the interval $[r_1, 2r_1)$, each nondominant class contains at least one integer not divisible by that prime.*

Consider the coloring $\overline{c}'$ defined at the beginning of this section. The assumption $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} > \dfrac{n}{r_1}$ implies that

$$\lim_{n' \to \infty} \sup \left(\min\{\mathcal{R}'(n'), \mathcal{B}'(n'), \mathcal{G}'(n')\} - \frac{n'}{r_1} \right) = \infty.$$

Repeating this analysis with Theorem 14's condition, we get a similar interpretation with the density constant $\frac{1}{2r_1}$. We will proceed by proving that there is a rainbow solution to $a - b = c^2$ in $\overline{c}'$ for both theorems, from which the two results follow by our earlier observations.

We also need the following classical theorems, listed in [1]:

Theorem (Dirichlet). Let a, b be two relatively prime positive integers. Then the set $\{ax + b | x \in \mathbb{N}\}$ contains infinitely many primes.

Theorem (Chinese Remainder). Let $p_1, p_2, ..., p_j$ be distinct primes, and suppose that $a_1, a_2, ..., a_j$ are integers. Then there exists an integer k_1 such that $k_1 \equiv a_1 \bmod p_1$, $k_1 \equiv a_2 \bmod p_2$,...,$k_1 \equiv a_j \bmod p_j$.

Lemma 15. *If two integers i_1, i_2 satisfy $(|i_1 - i_2|, n) = 1$, then there exists a pair of relatively prime integers i'_1, i'_2 with $\overline{c}'(i'_1) = \overline{c}'(i_1)$, $\overline{c}'(i'_2) = \overline{c}'(i_2)$, and $|i'_1 - i'_2| = |i_1 - i_2|$.*

Proof. If $(i_1, i_2) = 1$, then take $i'_1 = i_1$ and $i'_2 = i_2$ to complete the proof; otherwise, assume $(i_1, i_2) > 1$. Without loss of generality, assume $i_2 > i_1$ and we may write $i_2 = i_1 + k_1$ with $(k_1, n) = 1$. Let $Q = \{q_1, q_2, ..., q_s\}$ be the set of primes that divide k_1. Since for each j, $1 \leq j \leq s$, $(q_j, n) = 1$, there exists a c_j such that $q_j | (i_1 + c_j n)$. Since $(k_1, n) = 1$, $q_j \not| n$, so only integers k of the form $c_j + k' q_j$, where k' is any nonnegative integer, can satisfy the divisibility condition $q_j | (i_1 + kn)$. Using the Chinese Remainder Theorem, we can show that exists a k'_1 satisfying $k'_1 \not\equiv c_j \bmod q_j$ for all $1 \leq j \leq s$. We then have $q_j \not| (i_1 + k'_1 n)$ for all $1 \leq j \leq s$, but also $(i_1 + k'_1 n) | k_1$. It follows that $(i_1 + k'_1 n) = 1$. Take $i'_1 = i_1 + k'_1 n$ and $i'_2 = i_1 + k'_1 n + k_1$. It is clear that $\bar{c}'(i'_1) = \bar{c}'(i_1)$, $\bar{c}'(i'_2) = \bar{c}'(i_2)$, and $|i'_1 - i'_2| = |i_1 - i_2|$. $\square$

Lemma 16. *Let $\bar{c}'$ be a rainbow-free coloring of $\mathbb{N}$ satisfying the conditions of Theorem 3.3, and assume that red is the dominant color. If $\mathcal{B}'$ contains two relatively prime integers i_1, i_2, then there exist two integers $i'_1, i'_2 \in \mathcal{G}'$ such that $r_2 = |i'_1 - i'_2| = (i'_1, i'_2)$ is a prime from $[r_1, 2r_1)$.*

Proof. Since $\limsup_{n' \to \infty} \left(\mathcal{G}'(n) - \frac{n'}{2r_1} \right) = \infty$, by Lemma 5, there exists an i_3 and a $k_3 \leq 2r_1 - 1$ such that i_3 and $i_3 + k_3$ are both colored green. If $(i_3, i_3 + k_3) = 1$, then all the conditions of Lemma 10 are satisfied, so there must exist a rainbow solution, contradicting the fact that $\bar{c}'$ is rainbow-free. So assume $(i_3, i_3 + k_3) > 1$; then $(i_3, i_3 + k_3) \leq k_3$. By Lemma 3.4, $(k_3, n) > 1$. Since the only numbers not relatively prime to n less than $2r_1$ are prime divisors of n, it follows that $(i_3, i_3 + k_3) = r_2$, where r_2 is a prime dividing n in the interval $[r_1, 2r_1)$. Since $r_2 | k_3$, $k_3 < 2r_1$, and $r_2 \geq r_1$, it follows that $k_3 = r_2$. Taking $i'_1 = i_3$ and $i'_2 = i_3 + k_3 = i_3 + r_2$ completes the proof. $\square$

Lemma 17. *Let $\bar{c}'$ be a rainbow-free coloring of $\mathbb{N}$ satisfying the conditions of Theorem 3.3, and assume that red is the dominant color. If $\mathcal{B}'$ contains two relatively prime integers i_1, i_2, and if there exist integers $i'_1 \in \mathcal{B}'$ and $i'_2 \in \mathcal{G}'$ with $(i'_1, i'_2) = 1$, then $i'_1 + ki'^2_2 \in \mathcal{B}'$ for all integers k with $i'_1 + ki'^2_2 > 0$.*

Proof. Assume u' is the smallest positive integer such that either $i'_1 - u'i'^2_2$ or $i'_1 + u'i'^2_2$ is not colored blue; without loss of generality, assume at least that $i'_1 + u'i'^2_2$ is not blue. If $\bar{c}'\left(i'_1 + u'i'^2_2\right) = G$, then $i'_2, i_1 + u'i'^2_2$ would be a pair of relatively prime green integers. But then Lemma 2.9 guarantees a rainbow solution, contradicting the assumption that $\bar{c}'$ is rainbow-free. If $\bar{c}'\left(i'_1 + u'i'^2_2\right) = R$ then $i'_1 + u'i'^2_2, i'_1 + (u' - 1)\, i'^2_2, i'_2$ would form a rainbow solution, again a contradiction. Therefore, no such u' exists. Hence, $i'_1 + ki'^2_2 \in \mathcal{B}'$ for every integer k such that $i'_1 + ki'^2_2 > 0$. $\square$

Lemma 18. *Let $\bar{c}'$ be a rainbow-free coloring of $\mathbb{N}$ satisfying the conditions of Theorem 14, and assume that red is the dominant color. If $d_{\mathcal{A}'} = \min \{|i - j| \,|\, i, j \in \mathcal{A}'\}$,* *then any $i_1, i_2 \in \mathcal{A}'$ satisfying $|i_1 - i_2| = d_{\mathcal{A}'}$ must both be colored in the same nondominant hue.*

Proof. Let $k_1 = d_{\mathcal{A}'}$. Since $\limsup_{n' \to \infty} \left(\mathcal{A}'(n') - \frac{n'}{r_1}\right) = \infty$, Lemma 5 says that $k_1 \leq r_1 - 1$. Choose i'_1 such that $i'_1, i'_1 + k_1 \in \mathcal{A}'$ and write $i'_2 = i'_1 + k_1$. Suppose for the sake of contradiction that $i'_1 \in \mathcal{B}'$ and $i'_2 \in \mathcal{G}'$. Then by Lemma 3, $d_{\mathcal{A}'} \geq 2$. Since $|i'_1 - i'_2| \leq r_1 - 1$, Lemma 15 tells us that there exist $i''_1 \in \mathcal{B}'$ and $i''_2 \in \mathcal{G}'$ with $(i''_1, i''_2) = 1$ and $|i''_1 - i''_2| = d_{\mathcal{A}'}$.

Since $(i''_1, i''_2) = 1$, it follows that $\left(i''_1, i''_1 + i''^2_2\right) = \left(i''_2, i''_1 + i''^2_2\right) = 1$. Clearly $i''_1 + i''^2_2$ and $i''_2 + i''^2_1$ are not red. If $\vec{c}'\left(i''_1 + i''^2_2\right) = B$, then $i''_1, i''_1 + i''^2_2$ are a pair of relatively prime blue integers. Therefore, by Lemma 17, it follows that $i''_1 + ki''^2_2 \in \mathcal{B}'$ for all k with $i''_1 + ki''^2_2 > 0$. Similarly, if $\vec{c}'\left(i''_1 + i''^2_2\right) = G$, then $i''_2, i''_1 + i''^2_2$ are a relatively prime pair of green integers. It follows that $i''_2 + ki''^2_1 > 0$ is also colored green for each k satisfying the given relation, or else we have $i''_1, i''_2 + ki''^2_1$ as a relatively prime pair of blue integers, which together with Lemma 10 guarantees a rainbow solution, contradicting the assumption that $\vec{c}'$ is rainbow-free. Thus, we may assume without loss of generality that $\vec{c}'\left(i''_1 + i''^2_2\right) = B$.

We claim that there exists an $i_3 \in \mathcal{G}'$ with i_3 even satisfying $r_1 \nmid i_3$. Suppose not; by the conditions of Theorem 14, there exists an integer i'_3 in $\mathcal{G}'$ not divisible by r_1. Then write $i_3 = i'_3 + n$, and clearly i_3 is even and is colored green. Additionally, since $\left(i''_1, i''_1 + i''^2_2\right) = 1$ and $i''_1, i''_1 + i''^2_2 \in \mathcal{B}'$, Lemma 3.5 tells us that there exist integers $i_4, i_5 \in \mathcal{G}'$ and a prime divisor r_2 of n from $[r_1, 2r_1)$ with $i_5 = i_4 + r_2$ and $(i_4, i_5) = r_2$. Also, because $\left(i''_1, i''^2_2\right) = 1$, then by Dirichlet's theorem, there exists infinitely many k such that $i''_1 + ki''^2_2$ is prime. Hence, there exist a prime p_0 of the form $i''_1 + ki''^2_2$ which is greater than i_3, i_4, and i_5.

Since $(p_0, i_3) = 1$, Lemma 17 shows that $p_0 + ki_3^2 \in \mathcal{B}'$ for all nonnegative k. Also, since $\left(i_3, i_3 + p_0^2\right) = 1$, then $i_3 + p_0^2 \in \mathcal{B}'$, so Lemma 17 shows that $i_3 + p_0^2 + ki_3^2 \in \mathcal{B}'$ for all nonnegative k. Finally, since $(i_3, i_4, i_5) = 1$, then using a similar method as in the beginning of the proof of Lemma 10, we can find integers u_0, v_0, w_0 with $u_0, v_0 > 0$ and $w_0 < 0$ such that $u_0 i_3^2 + v_0 i_4^2 + w_0 i_5^2 = 1 + p_0 - p_0^2 - i_3$. We now claim that there exists an integer k''_1 such that the numbers $p_0 + (k''_1 - u_0)\, i_3^2 = p_0 - u_0 i_3^2 + k''_1 i_3^2$ and $i_3 + p_0^2 + k''_1 i_3^2$ are both relatively prime to i_4 and i_5. Note that no prime factor of i_3 can divide $p_0 - u_0 i_3^2 + ki_3^2$ since i_3 shares no common divisors with p_0. Similarly, no prime factor of i_3 can divide $i_3 + p_0^2 + ki_3^2$ either.

Now consider the set $Q' = \{q'_1, q'_2, ..., q'_t\}$ of all the prime divisors of i_4 and i_5 that do not also divide i_3. Suppose that for $1 \leq j \leq t$, q'_j divides $p_0 - u_0 i_3^2 + c_j i_3^2$ for some positive integer c_j. Note that $q'_j \neq 2$ as $p_0 - u_0 i_3^2 + c_j i_3^2$ is odd. Then since $\left(i_3, q'_j\right) = 1$, it follows that $q'_j | \left(p_0 - u_0 i_3^2 + ki_3^2\right)$ only if $k = c_j + k' q'_j$ for some nonnegative integer k'. Now assume that for $1 \leq j \leq t$, q'_j divides $i_3 + p_0^2 + c'_j i_3^2$ for some positive integer c'_j. Since $\left(i_3, q'_j\right) = 1$, it follows that $q'_j | \left(i_3 + p_0^2 + ki_3^2\right)$

only if $k = c'_j + k'q'_j$ for some nonnegative integer k'. Thus, it suffices to find an integer k''_1 such that $k''_1 \not\equiv c_j, c'_j \bmod q'_j$ for any integer $1 \leq j \leq t$. The existence of one such k''_1 is guaranteed by the Chinese Remainder Theorem as $q'_j \geq 3$. Write $k_2 = p_0 - u_0 i_3^2 + k''_1 i_3^2$ and $k'_2 = i_3 + p_0^2 + k''_1 i_3^2$.

Since k'_2 is blue and is relatively prime to both i_4 and i_5, it follows from Lemma 15 that $k'_2 + ki_4^2$ is colored blue for all nonnegative k. Hence, $\bar{c}'\left(k'_2 + v_0 i_4^2\right) = B$. At the same time, k_2 is blue and is relatively prime to i_5, so $k_2 + ki_5^2$ is colored blue for all nonnegative k; in particular, $\bar{c}'\left(k_2 - w_0 i_5^2\right) = B$. We have $\left(k'_2 + v_0 i_4^2\right) - \left(k_2 - w_0 i_5^2\right) = u_0 i_3^2 + v_0 i_4^2 + w_0 i_5^2 + p_0^2 - p_0 + i_3 = 1$, so there is a blue string of length 2 at position $k_2 - w_0 i_5^2$. But this contradicts the minimality of $d_{\mathcal{A}'}$. □

We now show that Theorem 13 follows as a simply corollary from Theorem 2 and Lemma 15:

Proof of Theorem 13. Suppose that $\bar{c}'$ is a rainbow-free coloring. By Lemma 2.2, there exists a dominant color; assume that it is red. Since $\lim_{n' \to \infty} \sup \left(\mathcal{B}'(n') - \frac{n'}{r_1}\right) = \infty$, Lemma 5 tells us that there exists an i_1 and a $k_1 \leq r_1 - 1$ such that $i_1, i_1 + k_1 \in \mathcal{B}'$. Then by Lemma 15, we can find integers $i'_1, i'_2 \in \mathcal{B}'$ which are relatively prime. Similarly, we can also find two relatively prime integers in $\mathcal{G}'$. Lemma 10 now guarantees the existence of a rainbow solution to $a - b = c^2$ in $\bar{c}'$ in $\mathbb{N}$, so there also exists a rainbow in $\bar{c}$ in $\mathbb{Z}_n$, contradiction. □

The following proposition demonstrates that the bound given in Theorem 13 is exact in almost all cases:

Proposition 19. *Let n be an odd number, and let r_1 be the smallest prime factor of n. If $r_1 \geq 5$, then there exists a partition of $\mathbb{Z}_n$ into three color classes $\mathcal{R}, \mathcal{B}, \mathcal{G}$, such that $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} = \frac{n}{r_1}$ so that there exists no rainbow solution to $a - b = c^2$ in $\mathbb{Z}_n$.*

Proof. Consider the following 3-coloring of $\mathbb{Z}_n$:

$$\bar{c}(i) = \begin{cases} R & \text{if } i \equiv \pm 1 \bmod r_1, \\ B & \text{if } i \equiv 0 \bmod r_1, \\ G & \text{otherwise.} \end{cases}$$

It is not difficult to see that this coloring is rainbow-free. □

Proof of Theorem 14. Suppose that $\bar{c}'$ is a rainbow-free coloring. Lemma 3 says there is a dominant color; let it be red. By Lemma 18, for any two $i_1, i_2 \in \mathcal{A}'$ with $|i_1 - i_2| = d_{\mathcal{A}'}$, i_1, i_2 are colored in the same nondominant hue. Without loss of generality, assume that $i_1, i_2 \in \mathcal{B}'$. Clearly, $d_{\mathcal{A}'} \leq r_1 - 1$, so by Lemma 3.4, we can find relatively prime integers $i'_1, i'_2 \in \mathcal{B}'$.

By Lemma 16, there exists an i_3 and a prime divisor r_2 of n from $[r_1, 2r_1)$ such that $i_3, i_3 + r_2 \in \mathcal{G}'$ and $(i_3, i_3 + r_2) = r_2$. By the conditions in Theorem 14, there exists an $i_4 \in \mathcal{G}'$ such that $r_2 \not| i_4$. Write $g = (i_3, i_4)$. Note that $(i_3, i_3 + r_2, i_4) = 1$. Since $\left(i_3^2, (i_3 + r_2)^2\right) = r_2^2$, Corollary 2.5 tells us that there exists an n_0 such that all integers greater than n_0 divisible by r_2^2 can be written in the form $ui_3^2 + v\,(i_3 + r_2)^2$ for some nonnegative integers u, v. Suppose that not all integers of the forms $i_1 + ui_3^2 + v\,(i_3 + r_2)^2$ and $i_2 + ui_3^2 + v\,(i_3 + r_2)^2$ are colored blue; let $u' + v'$ be the smallest integer such that at least one of $i_1 + u'i_3^2 + v'\,(i_3 + r_2)^2$ and $i_2 + u'i_3^2 + v'\,(i_3 + r_2)^2$ is colored red or green. If $i_1 + u'i_3^2 + v'\,(i_3 + r_2)^2$ is red, then it would form a rainbow with $i_1 + (u' - 1)\,i_3^2 + v'\,(i_3 + r_2)^2 \in \mathcal{B}'$ and $i_3 \in \mathcal{G}'$. A similar observation applies for $i_2 + u'i_3^2 + v'\,(i_3 + r_2)^2$, so neither number can be colored red. If both are colored green, then by Lemma 3.4, we can find a relatively prime pair of green integers, contradiction. If one of them is green, then we have two integers of different nondominant colors differing by $d_{\mathcal{A}'}$, contradiction. So it follows that all integers of the form $'i_1 + ui_3^2 + v\,(i_3 + r_2)^2$ and $i_2 + ui_3^2 + v\,(i_3 + r_2)^2$ are colored blue. Therefore, there exists an n_0' such that all integers greater than n_0' that can be written in the form $i_1 + kr_2^2$, $i_2 + kr_2^2$ for some nonnegative k are colored blue. Similarly, since $g^2 = \left(i_3^2, i_4^2\right)$, Corollary 2.5 says that there exists an n_1 such that all integers greater than n_1 divisible by g^2 can be written in the form $ui_3^2 + vi_4^2$ for some nonnegative integers u, v, and using a similar analysis as above, all numbers of the form $i_1 + ui_3^2 + vi_4^2$ and $i_2 + ui_3^2 + vi_4^2$ are colored blue. So there exists an n_1' such that all integers greater than n_1' that can be written in the form $i_1 + kg^2$, $i_2 + kg^2$ for some nonnegative k are colored blue. Since $\left(g^2, r_2^2\right) = 1$, there exist sufficiently large positive integers u_0, v_0 such that $u_0 g^2 - v_0 r_2^2 = 1$ (if u_0, v_0 are not sufficiently large, continually replace u_0 with $u_0 + r_2^2$ and v_0 with $v_0 + g^2$ to get large enough u_0, v_0). Hence, there is a blue string of length at least 2 at position $i_1 + v_0 r_2^2$.

Now consider this blue string with length $l_0 \geq 2$ and position i_3'; by the dominance of red, it follows that $\bar{c}'\,(i_3' + l_0) = R$. Then using a similar analysis as above, there must exist blue strings of length $\geq l_0$ at all integers of the forms $i_3' + kr_2^2$ and $i_3' + kg^2$ for sufficiently large k. Also, because $\left(g^2, r_2^2\right) = 1$, we can find sufficiently large u_0', v_0' with $u_0' g^2 - v_0' r_2^2 = 1$. So there is a string of length $\geq l_0 + 1$ at position $i_3' + v_0' r_2^2$. Working backwards, it follows that there is a blue string of length $\geq l_0 + 1$ at position i_3', but then $\bar{c}'\,(i_3' + l_0) = B$, contradicting the assumption above. Hence, there exists a rainbow solution in $\bar{c}'$ in $\mathbb{N}$, and therefore also in $\bar{c}$ in $\mathbb{Z}_n$, and we are done. □

4. A Note About $\mathbf{a - b = ec^2}$

A natural extension of our above results is to the equation $a - b = ec^2$, where $e \in \mathbb{N}$. It turns out that with a few minor modifications, every lemma in Section 2 except Lemma 7 applies to this general case as well. Therefore, it is likely that we can extend the above results to $a-b=ec^2$. In particular, we need only to show that in this case, both nondominant color classes contain a pair of relatively prime integers. We present our conjecture about this generalized form:

Conjecture 20. *Suppose that* $\mathbb{N}$ *is divided into three equally dense color classes* $\mathcal{R}, \mathcal{B}, \mathcal{G}$. *Then for all* e *not divisible by 3, there exists a rainbow solution to* $a-b=ec^2$.

The following result shows that Conjecture 4.1, if true, gives the best possible density bound as e is taken arbitrarily large. In particular, it also demonstrates that there exists a partition of $\mathbb{N}$ into equally dense color classes with no rainbow solution to $a-b=ec^2$ when $3|e$.

Proposition 21. There is a 3-coloring $\bar{c}:\mathbb{N}\to\{R,B,G\}$ of the natural numbers with

$$\lim_{n\to\infty}\frac{\min\{\mathcal{R}(n),\mathcal{B}(n),\mathcal{G}(n)\}}{n}=\frac{\left\lfloor\frac{e}{3}\right\rfloor}{e}$$

such that no rainbow solution exists to $a-b=ec^2$.

Proof. Color all numbers congruent to $1,2,...,\left\lfloor\frac{e}{3}\right\rfloor \bmod e$ red, color all numbers congruent to $\left\lfloor\frac{e}{3}\right\rfloor+1, \left\lfloor\frac{e}{3}\right\rfloor+2, ..., 2\left\lfloor\frac{e}{3}\right\rfloor \bmod e$ blue, and color all the remaining natural numbers green. Clearly, any rainbow solution to $a-b=ec^2$ must satisfy both $\bar{c}(a)\neq\bar{c}(b)$ and $a\equiv b \bmod e$, but every congruence class mod$e$ is monochromatic, so this is impossible.□

5. Conclusion and Directions for Future Work

In this paper, we have combined some well-known techniques (namely, that of the dominant color, used in [2, 5, 10, 11] among others) with some of our own to prove some interesting results Rainbow Ramsey Theory for nonlinear equations. Here, we offer a few avenues for future work.

One obvious direction would be to extend the ideas presented in section 4, and resolve the generalized equation $a-b=ec^f$, with $e,f\in\mathbb{N}$ and $e+f\geq 4$. A more difficult, but more interesting path, would be to prove results for fully quadratic equations, such as the Pythagorean equation $x^2+y^2=z^2$, or the quadratic counterpart, $w^2+x^2=y^2+z^2$, to the Sidon equation.

Additionally, because our methods combine numerous properties of linear equations, we believe that they will offer some more tools in attacking a question raised in [10]: to prove rainbow results for any linear equation. Conlon has already made progress in this area in the modular case when n is a prime; we hope that his work can be extended to other modulii as well as the infinite case. Generalized versions of our methods, combined with recent results in additive number theory (see [9, 15], and references therein) and previous work on specific linear equations, could potentially lead to much stronger theorems in this direction.

We hope the methods we have developed and the results we have obtained will positively contribute to the currently growing pool of theorems and techniques concerning nonlinear rainbow configurations, which began in 2006 with Frantzikinakis and Kra's result on independent polynomials and ergodic averages [6]. Future investigations with this "nonlinear" Ramsey Theory will certainly produce exciting and powerful results.

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