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ON MONOCHROMATIC SETS OF INTEGERS WHOSE DIAMETERS FORM A MONOTONE SEQUENCE

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Abstract

Let $g(m, t)$ denote the minimum integer s such that for every 2-coloring of the integers in the interval $[1, s]$, there exist t subsets $A_1, A_2, \ldots, A_t$, of size m satisfying: (i) A_i for every $i = 1, 2, \ldots, t$ is monochromatic (not necessarily the same color) (ii) $\max(A_i) \leq \min(A_{i+1})$ for every $i = 1, 2, \ldots, t-1$, and (iii) either $\text{diam}(A_i) \leq \text{diam}(A_{i+1})$ for every $i = 1, 2, \ldots, t-1$ or $\text{diam}(A_i) \geq \text{diam}(A_{i+1})$ for every $i = 1, 2, \ldots, t-1$. We prove that $2(m-1)(t+1)+1 \leq g(m, t) \leq [(t-1)^2+1](2m-1)$ for every integer m and t, where $m \geq 2$ and $t \geq 3$. Furthermore, we determine that $g(m, 3) = 8m - 5$.

1. Introduction

This paper deals with Ramsey theory on the integers, see [8], where the classical approach of looking at a system of equations has been modified to a system of inequalities. Motivated by [4] and [1], the authors of [2] introduced the theme of sets of non-decreasing diameter, in conjunction with generalizations in the sense of the Erdős-Ginzbirg-Ziv theorem. Numerous papers have followed; see [5], [7], [6], [11], [3], [9], [10].

For integers a and b, we use the closed interval notation $[a, b]$ to denote the set of integers x such that $a \leq x \leq b$. An *r-coloring* of $[a, b]$ is a function $\Delta : [a, b] \to \{1, 2, \ldots, r\}$ and a subset X of $[a, b]$ is called *monochromatic* if $\Delta(y) = \Delta(w)$ for all $y, w \in X$. For two sets of integers X and Y we use the notation $X \prec Y$, if $\max(X) < \min(Y)$. Furthermore, X and Y are said to be *non-overlapping* if either $X \prec Y$ or $Y \prec X$. Finally, the *diameter* of a set X is $\max(X) - \min(X)$ and is denoted by *diam*(X).

Let m, r, t be positive integers. We recall a definition from [2]. Let $f(m, r, t)$ be the minimum integer s such that for every r-coloring of $[1, s]$, there are t pairwise non-overlapping subsets of $[1, s]$, say, $A_1, A_2, \ldots, A_t$ such that (i) $|A_i| = m$ for $i = 1, 2, \ldots, t$, (ii) each A_i is monochromatic and

$$\text{(iii) } \text{diam}(A_1) \leq \text{diam}(A_2) \leq \cdots \leq \text{diam}(A_t).$$

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It was proved that $f(m,2,2) = 5m - 3$ for $m \geq 2$ and $f(m,2,3) = 8m - 5 + \left\lceil \frac{2m-2}{3} \right\rceil$, for $m \geq 5$, in [2] and [7], respectively. The proof in [7] is quite intricate. The difficulty of the determination of $f(m,r,t)$ suggests a relaxation of condition (iii) above. We define $g(m,r,t)$ similarly to $f(m,r,t)$ where the condition (iii) is modified as follows: Either $\text{diam}(A_1) \leq \text{diam}(A_2) \leq \cdots \leq \text{diam}(A_t)$, or $\text{diam}(A_1) \geq \text{diam}(A_2) \geq \cdots \geq \text{diam}(A_t)$. As we assume throughout the paper that $r = 2$, we will denote $g(m,2,t)$ by $g(m,t)$. We prove that $g(m,3) = 8m - 5$ and provide upper and lower bounds for $g(m,t)$ for all integers t greater than 3.

First, we show that the value of $g(m,2)$ is trivial.

Theorem 1. *Let $m \geq 2$ be an integer. Then $g(m,2) = 4m - 2$.*

Proof. The coloring of $[1, 4m-3]$ represented by the alternating string $1212\ldots121$ establishes that $g(m,2) \geq 4m-2$ for $m \geq 2$. Let Δ be a 2-coloring of $[1, 4m-2]$. Then there exist two monochromatic m-subsets of $[1, 4m-2]$, say A_1 and A_2, such that $A_1 \subset [1, 2m-1]$ and $A_2 \subset [2m, 4m-2]$. Now, either $\text{diam}(A_1) \geq \text{diam}(A_2)$ or $\text{diam}(A_1) \leq \text{diam}(A_2)$, and in either case A_1 and A_2 have monotone diameters. □

2. Preliminaries

Lemma 2 *Let m and x be positive integers satisfying $m \leq x + 1$. If Δ is a 2-coloring of $[1, x+m]$, then one of the following holds:*

(i) *there exists a monochromatic m-subset of $[1, x+m]$, say A, satisfying the inequality $\text{diam}(A) \geq x$, or*

(ii) *there exists two monochromatic m-subsets of $[1, x+m]$, say A_1 and A_2, satisfying $A_1 \prec A_2$ and $\text{diam}(A_1) = \text{diam}(A_2) = m - 1$.*

Proof. If $\left|\Delta^{-1}(1)\right| \leq m-1$, then $\left|\Delta^{-1}(2)\right| \geq x$, yielding an m-subset A of $\Delta^{-1}(2)$ which satisfies (i). Therefore we can assume that $\left|\Delta^{-1}(1)\right| \geq m$ and similarly we can assume that $\left|\Delta^{-1}(2)\right| \geq m$. Suppose, without loss of generality, that $\Delta(1) = 1$ and let t be the largest integer satisfying $\Delta(t) = 1$. If $t > x$, then there is a monochromatic m-subset A of $[1, t]$ which satisfies (i). Otherwise $t \leq x$, hence $\Delta(v) = 2$ for every $v \in [x+1, x+m]$. Since $\Delta(x+m) = 2$, by applying a similar argument we obtain $\Delta(w) = 1$ for every $w \in [1, m]$. Set $A_1 = [1, m]$ and $A_2 = [x+1, x+m]$. We see that A_1 and A_2 are monochromatic m-subsets of $[1, x+m]$, and $A_1 \prec A_2$ follows as long as $m < x+1$, satisfying (ii). If $m = x+1$ then the given interval is $[1, 2m-1]$, and since we have assumed that $\left|\Delta^{-1}(i)\right| \geq m$ for $i = 1, 2$, we get a contradiction and the proof is complete. □

Lemma 3 *Let $m \geq 2$ be an integer. If Δ is a 2-coloring of $[1, 4m-3]$, then one of the following holds:*

(i) *There exist two monochromatic* m*-subsets of* $[1, 4m-3]$*, say* A_1 *and* A_2*, satisfying* $A_1 \subset [1, 2m-1]$*,* $A_2 \subset [2m-1, 4m-3]$*, and* $A_1 \prec A_2$*, or*

(ii) $\left|\Delta^{-1}(i) \cap [1, 2m-2]\right| = m-1$ *and* $\left|\Delta^{-1}(i) \cap [2m, 4m-3]\right| = m-1$ *for* $i = 1, 2$.

Proof. If (ii) does not hold, then, without loss of generality, we can assume that $\left|\Delta^{-1}(1) \cap [1, 2m-2]\right| \geq m$, and hence the interval $[1, 2m-2]$ contains a monochromatic m-subset, say A_1. Since the complement of $[1, 2m-2]$ in $[1, 4m-3]$ is the interval $[2m-1, 4m-3]$ having $2m-1$ integers, it follows that it contains a monochromatic m-subset, say A_2, and (i) follows. □

Lemma 4 *Let* $m \geq 2$ *be an integer. If* Δ *is a 2-coloring of* $[1, 6m-4]$ *such that* $[1, \lfloor \frac{5m-3}{2} \rfloor]$ *contains a monochromatic* m*-subset, say* B*, satisfying* $2m-2 \leq \operatorname{diam}(B) \leq \lfloor \frac{5m-5}{2} \rfloor$*, then one of the following holds:*

(i) *there exists a monochromatic* m*-subset of* $[\lfloor \frac{5m-1}{2} \rfloor, 6m-4]$*, say* A*, satisfying* $\operatorname{diam}(A) \geq \operatorname{diam}(B)$*, or*

(ii) *there exist two monochromatic* m*-subsets of* $[\lfloor \frac{5m-1}{2} \rfloor, 6m-4]$*, say* A_1 *and* A_2*, satisfying* $A_1 \prec A_2$ *and* $\operatorname{diam}(A_1) = \operatorname{diam}(A_2) = m-1$.

Proof. Since the interval $I = [\lfloor \frac{5m-1}{2} \rfloor, 6m-4]$ is a translation of $[1, \lceil \frac{5m-5}{2} + m \rceil]$, and since $m \leq \lceil \frac{5m-5}{2} \rceil$, we can apply Lemma 2.1 to I with $x = \lceil \frac{5m-5}{2} \rceil$ to obtain that either I contains a monochromatic m-subset, say A, satisfying $\operatorname{diam}(A) \geq \lceil \frac{5m-5}{2} \rceil \geq \lfloor \frac{5m-5}{2} \rfloor \geq \operatorname{diam}(B)$ yielding (i) or it contains two monochromatic m-subsets, say A_1 and A_2, satisfying $A_1 \prec A_2$ and $\operatorname{diam}(A_1) = \operatorname{diam}(A_2) = m-1$, yielding (ii). □

3. Evaluation of $\mathbf{g(m, 3)}$

Theorem 5 *Let* $m \geq 2$ *be an integer. Then* $g(m, 3) = 8m-5$.

Proof. The equality $g(2, 3) = 11$ can be checked separately. The coloring of $[1, 8m-6]$ represented by the string $12^{m-1}1^{m-1}2^{m-1}1^{2m-2}2^{m-1}1^{m-1}2^{m-1}1$ establishes that $g(m, 3) \geq 8m-5$ for $m \geq 2$. Let Δ be a 2-coloring of $[1, 8m-5]$. In order to prove that $g(m, 3) \leq 8m-5$ for $m \geq 3$, we begin with a claim and proceed with a case-analysis of two cases.

Claim. *If there exists a monochromatic* m*-subset of* $[2m, 6m-4]$*, say* A*, satisfying diam*$(A) = 2m-2$*, then the conclusion of Theorem 5 follows.*

Proof of Claim. Let A be the subset of $[2m, 6m-4]$ stated in the claim. Since $\operatorname{diam}(A) = 2m-2$, either $\min(A) > 3m-2$ or $\max(A) < 5m-2$. First suppose that $\min(A) > 3m-2$. Then there exists a monochromatic m-subset of $[6m-3, 8m-5]$, say C, satisfying $\operatorname{diam}(C) \leq 2m-2$. In addition, by applying Lemma 2 with

$x = 2m-2$ to the interval $[1, 3m-2]$ we obtain one of two cases. If conclusion (i) of Lemma 2 holds, then there exists a monochromatic m-subset of $[1, 3m-2]$, say B, satisfying $\text{diam}(B) \geq 2m-2$. The sets B, A, and C satisfy $\text{diam}(B) \geq \text{diam}(A) \geq \text{diam}(C)$, hence satisfy the conclusion of Theorem 5. If conclusion (ii) of Lemma 2 holds, then there exist two m-subsets of $[1, 3m-2]$, say B_1 and B_2, satisfying $B_1 \prec B_2$ and $\text{diam}(B_1) = \text{diam}(B_2) = m-1$. The sets B_1, B_2, and A satisfy $\text{diam}(B_1) \leq \text{diam}(B_2) \leq \text{diam}(A)$, hence satisfy the conclusion of Theorem 5. Next, suppose that $\max(A) < 5m-2$. The proof proceeds in a similar fashion to the previous case. Thus the proof of the claim is complete. $\diamond$

There exist monochromatic m-subsets $A_1 \subset [1, 2m-1]$ and $A_4 \subset [6m-3, 8m-5]$ with $diam(A_1) \leq 2m-2$ and $diam(A_4) \leq 2m-2$. We consider $[2m, 6m-4]$ as a translation of $[1, 4m-3]$ and proceed by considering the two cases in the conclusion of Lemma 3:

`Case 1:` Conclusion (i) of Lemma 3 holds.

Let the two resulting sets be A_2 and A_3, with $A_2 \subset [2m, 4m-2]$ and $A_3 \subset [4m-2, 6m-4]$. Next we consider the cases for which of the sets A_i, for $i \in \{1,2,3,4\}$ has the largest diameter. The case $\max_{i\in\{1,2,3,4\}}\text{diam}(A_i) = \text{diam}(A_1)$ is symmetric to the case $\max_{i\in\{1,2,3,4\}}\text{diam}(A_i) = \text{diam}(A_4)$, and the case $\max_{i\in\{1,2,3,4\}}\text{diam}(A_i) = \text{diam}(A_2)$ is symmetric to the case $\max_{i\in\{1,2,3,4\}}\text{diam}(A_i) = \text{diam}(A_3)$. Hence we will consider only two cases:

`Subcase 1.1:` $\max_{i\in\{1,2,3,4\}} \text{diam}(A_i) = \text{diam}(A_1)$.

Either there exists an $i \in \{2,3\}$ such that $\text{diam}(A_i) \geq \text{diam}(A_{i+1})$ yielding $\text{diam}(A_1) \geq \text{diam}(A_i) \geq \text{diam}(A_{i+1})$ or $\text{diam}(A_2) \leq \text{diam}(A_3) \leq \text{diam}(A_4)$. In either case the proof is complete.

`Subcase 1.2:` $\max_{i\in\{1,2,3,4\}} \text{diam}(A_i) = \text{diam}(A_2)$.

By Lemma 2, the interval $[4m-1, 7m-3]$ contains either two monochromatic m-subsets B_1, B_2 satisfying $\text{diam}(B_1) = \text{diam}(B_2) = m-1$ and $B_1 \prec B_2$, yielding $\text{diam}(A_1) \geq \text{diam}(B_1) \geq \text{diam}(B_2)$, or the interval $[4m-1, 7m-3]$ contains a monochromatic m-subset B with $\text{diam}(B) \geq 2m-2$. Recall that $A_2 \subset [2m, 4m-2]$ hence $\text{diam}(A_2) \leq 2m-2$. Thus $\text{diam}(A_1) \leq \text{diam}(A_2) \leq \text{diam}(B)$, completing the proof of case 1.

`Case 2:` Conclusion (ii) of Lemma 3 holds.

By Lemma 3, $\left|\Delta^{-1}(i) \cap [2m, 4m-3]\right| = m-1$ and $\left|\Delta^{-1}(i) \cap [4m-1, 6m-4]\right| = m-1$ for $i = 1, 2$. Assume, without loss of generality, that $\Delta(2m) = 1$. If $\Delta(4m-2) = 1$, then $\left|\Delta^{-1}(1) \cap [2m, 4m-2]\right| = m$, yielding a monochromatic m-subset of $[2m, 4m-2]$, say A, with $\text{diam}(A) = 2m-2$, completing the proof in view of the claim. Thus, we assume that $\Delta(4m-2) = 2$. If $\Delta(6m-4) = 2$, then $\left|\Delta^{-1}(2) \cap [4m-2, 6m-4]\right| = m$, yielding a monochromatic m-subset of $[4m-2, 6m-4]$, say A, with $\text{diam}(A) = 2m-2$, completing the proof in view of the claim. Thus we assume that $\Delta(6m-4)=1$. Consequently, $\Delta(2m)=\Delta(6m-4)=1$, and $\Delta(4m-2)=2$.

Suppose there exists a set $B\subset[2m,\lfloor\frac{9m-5}{2}\rfloor]$ satisfying $2m-2\le diam(B)\le\lfloor\frac{5m-5}{2}\rfloor$. Replacing $[1,6m-4]$ by $[2m,8m-5]$ the hypotheses of Lemma 4 hold and one of the conlusions follows. If conclusion (i) of Lemma 4 follows, then there exists a monochromatic m-subset of $[\lfloor\frac{9m-3}{2}\rfloor,8m-5]$, say A, with $\text{diam}(A)\ge\text{diam}(B)$. Hence we obtain m-subsets A_1,A, and B of $[1,8m-5]$ satisfying $\text{diam}(A_1)\le\text{diam}(A)\le\text{diam}(B)$ and $A_1\prec A\prec B$, completing the proof. Otherwise, conclusion (ii) of Lemma 4 follows. Hence there exist two monochromatic m-subsets of $[\lfloor\frac{9m-3}{2}\rfloor,8m-5]$, say B_1 and B_2, with $\text{diam}(B_1)=\text{diam}(B_2)=m-1$, yielding $\text{diam}(A_1)\ge\text{diam}(B_1)\ge\text{diam}(B_2)$ and $A_1\prec B_1\prec B_2$, completing the proof.

Hence we can assume that $[2m,\lfloor\frac{9m-5}{2}\rfloor]$ does not contain a monochromatic m-subset B satisfying $2m-2\le\text{diam}(B)\le\frac{5m-5}{2}$. Since $\Delta(2m)=1$ and since $\left|\Delta^{-1}(1)\cap[2m,4m-3]\right|=m-1$, it follows that $\Delta(v)=2$ for every $v\in[4m-2,\lfloor\frac{9m-5}{2}\rfloor]$, as otherwise there exists a $w\in[4m-2,\lfloor\frac{9m-5}{2}\rfloor]$ with $\Delta(w)=1$ and $\Delta^{-1}(1)\cap[2m,4m-3]\cup\{w\}$ is a monochromatic m-subset, say B, satisfying $2m-2\le\text{diam}(B)\le\lfloor\frac{5m-5}{2}\rfloor$.

Similarly, Let $[1,6m-4]$ represent a reversal of the interval $[1,6m-4]$ in Lemma 4 - that is, 1 is represented by $6m-4$ and vice versa. Applying Lemma 4 to the reversed interval, it follows that either the proof is complete as before, or $\Delta(v)=2$ for every $v\in[\lceil\frac{7m-3}{2}\rceil,4m-2]$. Thus, assume the following:

$$\begin{aligned}\Delta(v)=2 \qquad &\text{for every } v\in[4m-2-\alpha,4m-2+\alpha],\\ &\text{where } \alpha=\left\lfloor\frac{9m-5}{2}\right\rfloor-(4m-2)\end{aligned} \tag{1}$$

It can be seen that $\alpha>0$ for $m\ge3$ and that $|[4m-2-\alpha,4m-2+\alpha]|\ge m-2$. Since $\left|\Delta^{-1}(2)\cap[2m,4m-3]\right|=m-1$, define β to be the smallest integer satisfying $\beta\in[2m+1,4m-3]$ and $\Delta(\beta)=2$. Similarly, since $\left|\Delta^{-1}(2)\cap[4m-1,6m-4]\right|=m-1$ define γ to be the largest integer satisfying $\gamma\in[4m-1,6m-5]$ and $\Delta(\gamma)=2$. We consider three cases:

`Subcase 2.1:` $\beta\in[2m+1,2m+\alpha]$. As can be seen from (1), we get $\Delta(\beta)=2$ and $\Delta(\beta+2m-2)=2$. Let $A=\Delta^{-1}(2)\cap[2m,4m-2]\cup\{\beta+2m-2\}$, and since $\left|\Delta^{-1}(2)\cap[2m,4m-2]\right|=m$, we get $|A|=m+1$. Deleting any element of A excluding the minimum and maximum, we get a monochromatic m-subset $A^{'}$ of $[2m,6m-4]$ satisfying $\text{diam}(A^{'})=2m-2$, completing the proof in view of the claim.

`Subcase 2.2:` $\gamma\in[6m-4-\alpha,6m-5]$. As can be seen from (1), we get $\Delta(\gamma)=2$ and $\Delta(\gamma-(2m-2))=2$. Let $A=\Delta^{-1}(2)\cap[4m-2,6m-4]\cup\{\gamma-(2m-2)\}$, and since $\left|\Delta^{-1}(2)\cap[4m-2,6m-4]\right|=m$, we get $|A|=m+1$. Deleting any element of A excluding the minimum and maximum, we get a monochromatic m-subset $A^{'}$ of $[2m,6m-4]$ satisfying $diam(A^{'})=2m-2$, completing the proof in view of the claim.

Subcase 2.3: $\beta \notin [2m+1, 2m+\alpha]$ and $\gamma \notin [6m-4-\alpha, 6m-5]$. Let $S_1 = [2m+\alpha+1, 4m-\alpha-3]$ and $S_2 = [4m+\alpha-1, 6m-\alpha-5]$ and consider $S = S_1 \cup S_2$. We have, by previous arguments, that $\Delta^{-1}(1) \cap S = 2m-2\alpha-4$ and $\Delta^{-1}(2) \cap S = 2m-2\alpha-2$. If $x \in S_1$ then $x+(2m-2) \in S_2$. Hence we cannot have $\Delta(x) = \Delta(x+2m-2) = 2$; indeed, if they were then we are done by the fact that the interval in (1) has at least $m-1$ integers, completing the proof in view of the claim. Thus $\Delta^{-1}(1) \cap S \geq \Delta^{-1}(2) \cap S$, contradicting the deduced number of integers of each color. □

4. Upper and Lower Bounds for $\mathbf{g(m,t)}$

Theorem 6. *If $t \geq 3$, then $2(m-1)(t+1)+1 \leq g(m,t) \leq [(t-1)^2+1](2m-1)$.*

Proof. The upper bound follows from the Erdős-Szekeres Theorem, which states that a sequence of $(n-1)^2+1$ integers has either a decreasing subsequence of length n or an increasing subsequence of length n. Since every interval of $2m-1$ integers must contain a monochromatic m-subset, we get $g(m,t) \leq [(t-1)^2+1](2m-1)$.

Denote $X = 1^{m-1}$ and $Y = 2^{m-1}$. To prove that $g(m,t) \geq 2(m-1)(t+1)+1$, we consider separate cases for t even and odd:

If $t \geq 3$ is odd, then let $t = 2a+1$. It can be verified that the coloring of $[1, 2(m-1)(t+1)]$ represented by the string $Y(XY)^a XX(YX)^a Y$ contains no monotone sequences with length t of diameters of monochromatic m-subsets.

If $t \geq 4$ is even, then let $t = 2a$. The coloring of $[1, 2(m-1)(t+1)]$ represented by the string $Y(XY)^{a-1}X^2Y^2(XY)^{a-1}X$ contains no monotone sequences with length t of diameters of monochromatic m-subsets. □

5. Concluding Remarks

The solution of the following related conjecture would make the proof of $g(m,3) = 8m-5$ trivial.

Conjecture. Let $m \geq 2$ be an integer. If Δ is a 2-coloring of $[1, 6m-4]$, then there exist two non-overlapping monochromatic m-subsets of $[1, 6m-4]$, say A_1 and A_2, satisfying $\text{diam}(A_1) = \text{diam}(A_2)$.

The conjecture has been proven true for $m \leq 5$ by computer search. The coloring of $[1, 6m-5]$ represented by the string $(12)^{2m-2}1(12)^{m-1}$ proves that there exists a 2-coloring of $[1, 6m-5]$ avoiding two non-overlapping monochromatic m-subsets, say A_1 and A_2, satisfying $\text{diam}(A_1) = \text{diam}(A_2)$.

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