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MULTIPLICITIES OF INTEGER ARRAYS

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Abstract

We prove a general theorem about the multiplicity of the entries in certain integer arrays which is best possible in general. As an application we give non-trivial bounds for the multiplicities of several well-known combinatorial arrays including the binomial coefficients, Narayana numbers and the Eulerian numbers. For the binomial coefficients we obtain the result of Singmaster.

1. Introduction

An *integer array* or *array* is a function $a : \mathbb{N}^2 \to \mathbb{N}$ where $a(n, 0) = 1 \quad (n \in \mathbb{N})$. For a function $b : \mathbb{P}^2 \to \mathbb{N}$ where $b(n, 1) = 1 \quad (n \in \mathbb{P})$, consider the array $a(n, k) = b(n, k + 1) \quad (n \in \mathbb{P}, k \in \mathbb{N})$ where $a(0, 0) = 1$ and $a(0, k) = 0 \quad (k \in \mathbb{P})$. We write $a = \operatorname{shift} b$ and say *a results from shifting b*. Here $\mathbb{N}$ denotes the non-negative integers, $\mathbb{P}$ denotes the positive integers and $[n] = \{1, \ldots, n\} \quad (n \in \mathbb{P})$. The cardinality of a set S is denoted $\# S$ or $|S|$.

Suppose a is an array. Then

(D1) a is *semi-triangular* if and only if there exists a strictly increasing function $d : \mathbb{N} \mapsto \mathbb{N}$ such that $a(n, k) \neq 0 \;\Leftrightarrow\; 0 \leq k \leq d(n) \quad (n \in \mathbb{N})$.

Suppose $a = (a, d)$ is a semi-triangular array. Then

(D2) a is *increasing* if and only if $a(n + 1, k) > a(n, k)$ for $1 \leq k \leq d(n) \quad (n \in \mathbb{N})$.

(D3) a is *semi-unimodal* if and only if there exists a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ with $0 \leq f(n) \leq d(n), \lim_{n\to\infty} f(n) = \infty$, and $a(n, 0) \leq \cdots \leq a(n, f(n) - 1) < a(n, f(n))$ which includes every non-zero value of the $a(n, k) \quad (n \in \mathbb{N})$. Then the largest value $a(n, f(n))$ of the $a(n, k)$ first occurs at $k = f(n)$.

The non-zero entries of a semi-triangular, increasing, semi-unimodal array $a = (a, d, f)$ have the general form given in Figure 1.

Suppose $a = (a, d, f)$ is a semi-triangular, semi-unimodal array. Then

(D4) a has *multiplicity* r if and only if at most r of $a(n,0), \ldots, a(n,d(n))$ assume any identical value ($n \in \mathbb{N}$). Then $ra(n, f(n)) \geq d(n)$ ($n \in \mathbb{N}$) by the Pigeonhole Principle.

(D5) a is Δ-*bounded* if and only if $f(n) \leq f(n-1) + \Delta \quad (n \in \mathbb{P})$.

(D6) a has *growth function* $g : [0, \infty) \to [\tau, \infty)$ where $\tau \in \mathbb{R}$, which is continuous, strictly increasing, surjective and satisfies $g(f(n)) \leq a(n, f(n)) \quad (n \in \mathbb{N})$.

Since f is non-decreasing with $\lim_{n\to\infty} f(n) = \infty$ and g is strictly increasing, $g(f(n))$ is non-decreasing with $\lim_{n\to\infty} g(f(n)) = \infty$. Here g has continuous, strictly increasing, surjective, inverse function $g^{-1} : [\tau, \infty) \to [0, \infty)$. The growth function g is not unique: the larger the g, the smaller the g^{-1}, and the better our bound in Theorem 2.

A semi-triangular, increasing, semi-unimodal, Δ-bounded array $a = (a, d, f, r, \Delta, g)$ with multiplicity r and growth function g is called *normal*.

For $k \geq 1$, take the smallest n_0 with $k \leq d(n_0)$. Then $a(n, k)$ is a strictly increasing function of $n \geq n_0$. Hence, $a(n+m, k) \geq a(n, k) + m \geq m + 1$ for $n+m \geq n \geq n_0$. Here $f(n)$ is non-decreasing (illustrated for $f(n)$ strictly increasing) and $d(n)$ is strictly increasing.

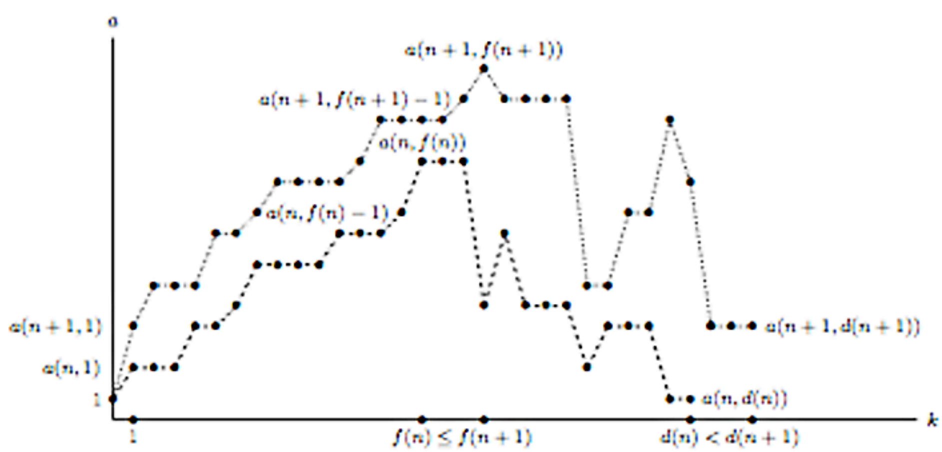

Figure 1: General form of non-zero entries of a semi-triangular, increasing, semi-unimodal array

Suppose $a = \text{shift}\, b$ and $a = (a, d, f, r, \Delta, g)$ is a normal array. Then

$a(n, k) \neq 0 \;\Leftrightarrow\; 0 \leq k \leq d(n)$ if and only if $b(n, k) \neq 0 \;\Leftrightarrow\; 1 \leq k \leq d(n) + 1$;

$a(n + 1, k) > a(n, k)$ for $1 \leq k \leq d(n)$ if and only if
$b(n + 1, k) > b(n, k)$ for $2 \leq k \leq d(n) + 1$;

$a(n, 0) \leq \cdots \leq a(n, f(n - 1) < a(n, f(n))$ if and only if
$b(n, 1) \leq \cdots \leq b(n, f(n)) < b(n, f(n) + 1)$;

a has multiplicity r if and only if b has multiplicity r;

$g(f(n)) \leq a(n, f(n))$ if and only if $g(f(n)) \leq b(n, f(n) + 1)$.

In this case we give the parameters d, f, r, Δ, g for b, not for $a = \text{shift}\, b$, and we say that b is a normal array, not that $a = \text{shift}\, b$ is a normal array.

An array a is *combinatorial* if and only if the $a(n, k)$ enumerate mathematical structures. For example, the binomial coefficients $\binom{n}{k}$ enumerate the k-subsets of an n-set; the Narayana numbers $N(n, k)$ enumerate the Catalan paths from $(0, 0)$ to (n, n) with k peaks; and the Eulerian numbers $A(n, k)$ enumerate the permutations of $[n]$ with $k - 1$ ascents. These are normal combinatorial arrays as are many other combinatorial arrays.

2. Results

Suppose function $c : X^2 \to \mathbb{N}$ where $X = \mathbb{N}$ or $\mathbb{P}$. For $t \in \mathbb{N}$, let

$$N_c(t) = \#\left\{(n, k) \in X^2 : c(n, k) = t\right\}.$$

Observation 1. Suppose $a = \text{shift}\, b$. Then $N_a(t) = N_b(t)$ for all $t \in \mathbb{P}$.

If $a = (a, d)$ is a semi-triangular array, then $N_a(0) = N_a(1) = \aleph_0$ by definition. Suppose $a = (a, d, f)$ is a semi-triangular, increasing, semi-unimodal array. Since a is semi-triangular and increasing, $d(n), a(n, 1) \geq n$. Since a is semi-unimodal, $a(n, k) \geq a(n, 1) \geq n$ or $a(n, k) = 1$ for $1 \leq k \leq d(n)$. Consequently, $N_a(t) \leq d(0) + \cdots + d(t)$ for $t \geq 2$.

We now prove the main result of the paper. The infinite families of normal combinatorial arrays given in Examples 1 and 2 demonstrate our result is best possible in general apart from the constant $r\Delta$.

Theorem 2. *Suppose that $a = (a, d, f, r, \Delta, g)$ is a normal array. For all integers $t \geq 2$,*

$$N_a(t) < r\big(g^{-1}(t) + \Delta\big).$$

Proof. Fix $t \geq 2$. Let $m = m(t)$ be the smallest positive integer satisfying $a(m, f(m)) \geq t$ which exists since $a(m, f(m)) \geq g(f(m))$ and $g(f(m))$ is non-decreasing with $\lim_{m\to\infty} g(f(m)) = \infty$. Then $(m-1, f(m-1) \in \mathbb{N})$

$$t > a(m-1, f(m-1)) \geq g(f(m-1)), \text{ i.e., } f(m-1) < g^{-1}(t).$$

Suppose $a(n, k) = t$. Since a is increasing and semi-unimodal, $n \geq m$. We may assume that $1 \leq k \leq f(n)$ since a is semi-unimodal. Suppose $k \geq f(m) + 1$, hence, $f(n) \geq k > f(m)$. Then $n > m$ since $n \geq m$. Since a is semi-unimodal, increasing, $f(n) \geq k > f(m)$ and $n > m$,

$$t = a(n, k) \geq a(n, f(m)) > a(m, f(m)) \geq t,$$

which is a contradiction. Hence, $1 \leq k \leq f(m)$ (see Figure 2). For each such k, there is at most one n with $a(n, k) = t$ since a is increasing. For each such n and k, there are at most $r - 1$ other values $0 \leq \ell \leq d(n)$ with $a(n, \ell) = t$ since a has multiplicity r. Since a is Δ-bounded, there are at most

$$r\, f(m) \leq r\, \big(f(m-1) + \Delta\big) < r\big(g^{-1}(t) + \Delta\big)$$

pairs (n, k) with $a(n, k) = t$. □

The following is a special case of Theorem 2.

Corollary 3. *Suppose $a = (a, d, f, r, \Delta, g)$ is a normal array. If $g(x) = \tau^{x-c}$ where $\tau \in (1, \infty)$ and $c \in \mathbb{R}$, then*

$$N_a(t) < r\big(\log_\tau t + c + \Delta\big)\,.$$

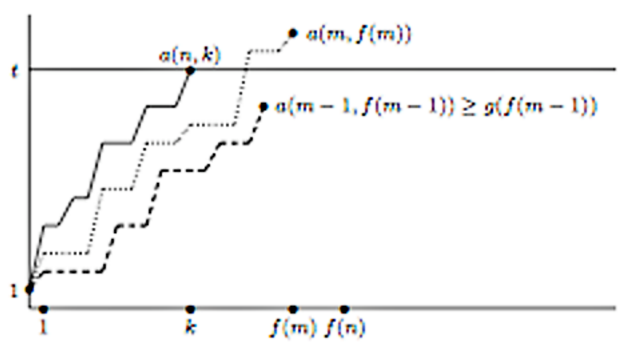

Figure 2: Part of the proof of Theorem 2

Proof. Apply Theorem 2 with $g^{-1}(x) = \log_\tau x + c$. $\square$

More generally we have the following result.

Corollary 4. *Suppose $a = (a, d, f, r, \Delta, g)$ is a normal array. If $g(x) = \Omega(\tau^x)$ where $\tau \in (1, \infty)$, then*

$$N_a(t) = O(\log_\tau t)\,.$$

Proof. If $g(x) = \Omega(\tau^x)$, then $g^{-1}(x) = O(\log_\tau x)$. Theorem 2 implies $N_a(t) = O(\log_\tau t)$. $\square$

Example 5. (Polynomial Growth) Fix integer $s \geq 2$. Let $a(n,k)$ denote the number of functions $f : [s] \to [2n]$ with $Image(f) \subseteq [n+k]$ for $1 \leq k \leq \lfloor n/2 \rfloor$, and, $Image(f) \subseteq [2n] - [k-1]$ for $\lfloor n/2 \rfloor + 1 \leq k \leq n$. Set $a(n,0) = 1$ for $n \in \mathbb{N}$. Then

$$a(n,k) = \begin{cases} 1, & k = 0; \\ (n+k)^s, & 1 \leq k \leq \lfloor n/2 \rfloor; \\ (2n+1-k)^s, & \lfloor n/2 \rfloor + 1 \leq k \leq n; \\ 0, & \text{otherwise}\,. \end{cases}$$

The combinatorial array $a(n,k)$ is normal (in particular, is increasing) with $d(n) = n$, $f(n) = \lceil n/2 \rceil$, $g(x) = (3x-1)^s$ with $\tau = -1$ (odd s) and $\tau = 0$ (even s), $r = 2$ and $\Delta = 1$. Then $g^{-1}(x) = (1/3)x^{1/s} + (1/3)$ and Theorem 2 implies

$$N_a(t) < \frac{2\,t^{1/s}}{3} + \frac{8}{3}\,.$$

Suppose $n \in \mathbb{P}$. If n is even, then $a(n,1), \ldots, a(n,n)$ is the set $(n+1)^s < \cdots < (n + \lfloor n/2 \rfloor)^s$ where each is assumed twice. If n is odd, then $a(n,1), \ldots, a(n,n)$ is the set $(n+1)^s < \cdots < (n + \lfloor n/2 \rfloor)^s < (n + \lfloor n/2 \rfloor + 1)^s = a(n, \lfloor n/2 \rfloor + 1)$ where each is assumed twice except $(n + \lfloor n/2 \rfloor + 1)^s$ which is assumed once.

Suppose $t = u^s$ where non-zero $u \equiv 0 \bmod 3$.

If $n \geq u$, then all non-zero $a(n,k) \geq (u+1)^s > t$ except $a(n,0) = 1 < t$. If $n \leq (2u-3)/3$, then all $a(n,k) \leq (3n/2+1)^s \leq (u - 1/2)^s < t$. For each $2u/3 \leq n \leq u-1$, there are precisely two $0 \leq k \leq n$ with $a(n,k) = t$. Hence,

$$N_a(t) = \frac{2u}{3} = \frac{2t^{1/s}}{3}\,.$$

Consequently, for infinitely-many t,

$$N_a(t) = \frac{2t^{1/s}}{3}\,.$$

Example 6. (Exponential Growth) Fix integer $s \geq 2$. Let $a(n,k)$ denote the number of functions $f : [2n] \to [s]$ where $f(n+k+1) = \cdots = f(2n) = 1$ for $1 \leq k \leq \lfloor n/2 \rfloor$, and, $f(2n+2-k) = \cdots = f(2n) = 1$ for $\lfloor n/2 \rfloor + 1 \leq k \leq n$. Set $a(n,0) = 1$ for $n \in \mathbb{N}$. Then

$$a(n,k) = \begin{cases} 1, & k = 0; \\ s^{(n+k)}, & 1 \leq k \leq \lfloor n/2 \rfloor; \\ s^{(2n+1-k)}, & \lfloor n/2 \rfloor + 1 \leq k \leq n; \\ 0, & \text{otherwise}. \end{cases}$$

The combinatorial array $a(n,k)$ is normal (in particular, is increasing) with $d(n) = n, f(n) = \lceil n/2 \rceil, g(x) = s^{3x-1}$ with $\tau = s^{-1}$, $r = 2$ and $\Delta = 1$. Then $g^{-1}(x) = (1/3)\log_s x + (1/3)$ and Theorem 2 implies

$$N_a(t) < \frac{2\,\log_s t}{3} + \frac{8}{3}\,.$$

Suppose $n \in \mathbb{P}$. If n is even, then $a(n,1), \ldots, a(n,n)$ is the set $s^{n+1} < \cdots < s^{n+\lfloor n/2 \rfloor}$ where each is assumed twice. If n is odd, then $a(n,1), \ldots, a(n,n)$ is the set $s^{n+1} < \cdots < s^{n+\lfloor n/2 \rfloor} < s^{n+\lfloor n/2 \rfloor+1} = a(n, \lfloor n/2 \rfloor + 1)$ where each is assumed twice except $s^{n+\lfloor n/2 \rfloor+1}$ which is assumed once.

Suppose $t = s^u$ where non-zero $u \equiv 0 \bmod 3$.

If $n \geq u$, then all non-zero $a(n,k) \geq s^{(u+1)} > t$ except $a(n,0) = 1 < t$. If $n \leq (2u-3)/3$, then all $a(n,k) \leq s^{3n/2+1} \leq s^{u-1/2} < t$. For each $2u/3 \leq n \leq u-1$, there are precisely two $0 \leq k \leq n$ with $a(n,k) = t$. Hence,

$$N_a(t) = \frac{2u}{3} = \frac{2\log_s t}{3}\,.$$

Consequently, for infinitely-many t,

$$N_a(t) = \frac{2\log_s t}{3}\,.$$

In passing we mention the Infinite Pigeonhole Principle: If κ is a regular cardinal, λ is a cardinal with $\lambda < \kappa$, and $a : \kappa \to \lambda$, then there exists $t \in \lambda$ with $|a^{-1}(t)| \geq \kappa$. In particular, $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph_0$ is a regular cardinal. An array $a : \mathbb{N}^2 \to \mathbb{N}$ may be viewed as a function $a : \aleph_0 \to \aleph_0$ by traversing successive diagonals of the displayed form of a. If a is semi-triangular, then $|a^{-1}(0)| = |a^{-1}(1)| = \aleph_0$ by definition. For a normal array $a = (a, d, f, r, \Delta, g)$, Theorem 2 gives the upper bound $|a^{-1}(t)| < r(g^{-1}(t) + \Delta)$ for all $t \geq 2$. Hence Theorem 2 is, in a sense, a complement of the Infinite Pigeonhole Principle which provides a lower bound for some $|a^{-1}(t)|$.

3. Applications

Suppose $a = (a, d, f, r, \Delta, g)$ is a normal array. Theorem 2 gives $N_a(t) < r\big(g^{-1}(t) + \Delta\big)$. The normal combinatorial arrays in Examples 1 and 2 show this bound is best possible apart from the constant $r\Delta$.

We apply Theorem 2 to several of the numerous normal combinatorial arrays including the well-known binomial coefficients $\binom{n}{k}$, Narayana numbers $N(n,k)$ and Eulerian numbers $A(n,k)$. For the binomial coefficients we obtain the result of Singmaster [11]. For all the other arrays a we obtain non-trivial bounds for $N_a(t)$ which to our knowledge are new. The results of Examples $3-6$ are summarized in the following table.

$a(n,k)$	$d(n)$	$f(n)$	$g(x)$	r	Δ	$N_a(t)$		
$\binom{n}{k}$	n	$\lfloor n/2 \rfloor$	2^x	2	1	$< 2\,\log_2 t + 2$	$(t \geq 2)$	[11]
$N(n,k)$	n	$\lceil n/2 \rceil$	$4^{x-1}/x$	2	1	$< 2\,\log_3 t$	$(t \geq B)$	
$A(n,k)$	n	$\lceil n/2 \rceil$	$(x/e)^x$	2	1	$< 3\,\ln t/\ln\ln t$	$(t \geq B)$	
$Q(n,k)$	n	$\lceil n/2 \rceil$	ρ^x	2	1	$< 2\,\log_\rho t + 2$	$(t \geq 2)$	

Table 1: Results from Examples 3–6

One would hope that a better bound than that provided by the general Theorem 2 could be obtained for a particular normal array by using its own special properties. Even slight improvements of this general bound can require deep results from, say, number theory (see Example 7).

3.1. Binomial Coefficients

Example 7. The array of binomial coefficients $a(n,k) = \begin{pmatrix} n \\ k \end{pmatrix}$.

The maximum of the $\binom{n}{k}$ is $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} \geq 2^{\lfloor n/2 \rfloor}$ for $n \in \mathbb{N}$. The array $b(n,k)$ is normal with $d(n) = n$, $f(n) = \lfloor n/2 \rfloor$, $g(x) = 2^x$ with $\tau = 1$, $r = 2$ and $\Delta = 1$. Then $g^{-1}(x) = \log_2 x$ and Theorem 2 implies

$$N_a(t) < 2\log_2 t + 2 \qquad (t \geq 2)\,.$$

This is the result of Singmaster [11].

Singmaster [11] searched up to $t = 2^{48}$ and found that all $N_a(t) \leq 8$. He also found that $N_a(t) = 6$ only for $t = 120, 210, 1540, 7140, 11628, 24310$ and $N_a(t) = 8$ only for $t = 3003$. Singmaster conjectured there that $N_a(t) = O(1)$. Erdős concurred with this conjecture stating that it must be very hard (see [11; p.385]). Singmaster [12] showed that there are infinitely-many t with $N_a(t) \geq 6$.

He conjectured there that $N_a(t) \leq 10$. Both conjectures are still open. The best result to date is Abbott, Erdős and Hanson [1] who showed that $N_a(t) = O(\log t/\log\log t)$ for $t \geq 2$. Their proof used a deep result of Ingham [5] on the distribution of the primes: If $\alpha \geq 5/8$, then there is a prime between x and $x+x^\alpha$ for all sufficiently large x. See de Weger [16] for some further results on the multiplicities of the binomial coefficients.

3.2. Narayana Numbers

The Narayana numbers are attributed to Narayana [7] and are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \qquad\qquad (n,k \in \mathbb{P}).$$

They have a nice combinatorial definition. A Catalan path P is a lattice path from $(0,0)$ to (n,n) that does not go below the line $y = x$. Then $P = (s_1, \ldots, s_{2n})$ where each $s_j \in \{V = (0,1), H = (1,0)\}$. Here P has a peak at j provided $(s_{j-1}, s_j) = (V, H)$. If $\mathrm{Cat}(n)$ denotes the set of Catalan paths from $(0,0)$ to (n,n), then $|\mathrm{Cat}(n)| = \binom{2n}{n}/(n+1)$ is the Catalan number C_n. If $\mathrm{Cat}(n,k)$ denotes the set of Catalan paths from $(0,0)$ to (n,n) with k peaks, then the statistic $|\mathrm{Cat}(n,k)| = N(n,k)$. Hence, $N(n,1) + \cdots + N(n,n) = C_n$. Sulanke [14], [15] cataloged more than 200 other statistics on $\mathrm{Cat}(n)$ with Narayana distribution. Riordan [10] showed that the number of labelled plane trees with n edges and k leaves is $N(n,k)$. Many other statistics of combinatorial structures have Narayana distribution (cf. [2], [8], [9]). The Narayana numbers are sequence $A001263$ in Sloane [13].

Example 8. The array of Narayana numbers $N(n,k)$.

The maximum of the $N(n,k)$ occurs at $\lceil n/2 \rceil$ for odd n and at $\lceil n/2 \rceil$ and $\lceil n/2 \rceil + 1$ for even n. Further, $N(n, \lceil n/2 \rceil) \geq 4^{\lceil n/2 \rceil - 1}/\lceil n/2 \rceil$ for $n \in \mathbb{P}$. The array $N(n,k)$ is normal with $d(n) = n, f(n) = \lceil n/2 \rceil, g(x) = 4^{x-1}/x$ with $\tau = 1$, $r = 2$ and $\Delta = 1$. Then $g^{-1}(x) < \log_3 x - 1 \quad (x \geq B)$ and Observation 1 and Theorem 2 imply

$$N_N(t) < 2\,\log_3 t \qquad (t \geq B).$$

To our knowledge this non-trivial bound for $N_N(t)$ is new. We conjecture that $N_N(t) = O(1)$. In view of Erdős' comments regarding the multiplicities of the binomial coefficients this conjecture may be hard. We note that $N_N(105) = 4$: $N(7,3) = N(7,5) = N(15,2) = N(15,14) = 105$. The non-zero values of $N(n,k)$ for $1 \leq n \leq 10$ are given in the following table.

	$N(n,k)$	k 1	2	3	4	5	6	7	8	9	10
	1	1									
	2	1	1								
	3	1	3	1							
	4	1	6	6	1						
n	5	1	10	20	10	1					
	6	1	15	50	50	15	1				
	7	1	21	105	175	105	21	1			
	8	1	28	196	490	490	196	28	1		
	9	1	36	336	1176	1764	1176	336	36	1	
	10	1	45	540	2520	5292	5292	2520	540	45	1

Table 2. Small values of $N(n,k)$

3.3. Eulerian Numbers

The Eulerian numbers $A(n,k)$ may be defined by the recurrence relation

$$A(n,k) = (n-k+1)\,A(n-1,k-1) + k\,A(n-1,k) \qquad (n,k \geq 2)$$

with initial conditions $A(n,1) = 1$ for $n \geq 1$ and $A(1,k) = 0$ for $k \geq 2$ (cf. Comtet [3; pps.240–246]). Hence, $A(n,k) \neq 0$ if and only if $1 \leq k \leq n$. Dillon and Roselle [4] first showed that $A(n,k)$ is the number of permutations of $[n]$ with $k-1$ ascents or with $k-1$ descents.

Example 9. The array of Eulerian numbers $A(n,k)$.

The maximum of the $A(n,k)$ occurs at $\lceil n/2 \rceil$ for odd n and at $\lceil n/2 \rceil$ and $\lceil n/2 \rceil + 1$ for even n. The recurrence relation for the $A(n,k)$ implies $A(n,\lceil n/2 \rceil) \geq \lceil n/2 \rceil\, A(n-1,\lceil (n-1)/2 \rceil)$ for $n \geq 3$. Iteration of this inequality and Stirlings Formula gives the weak bound $A(n,\lceil n/2 \rceil) \geq (\lceil n/2 \rceil/e)^{\lceil n/2 \rceil}$ for $n \in \mathbb{P}$. In fact,

$$A(n,\lceil n/2 \rceil) \sim \sqrt{12}\left(\frac{n}{e}\right)^n .$$

The array $A(n,k)$ is normal with $d(n) = n, f(n) = \lceil n/2 \rceil, g(x) = (x/e)^x$ with $\tau = e^{-1}$, $r = 2$ and $\Delta = 1$. Then $g^{-1}(x) < 3\ln x / 2\ln\ln x - 1 \quad (x \geq B)$ and Observation 1 and Theorem 2 imply

$$N_A(t) < \frac{3\ln t}{\ln\ln t} \qquad (t \geq B) .$$

To our knowledge this non-trivial bound for $N_A(t)$ is new. We conjecture that $N_A(t) = O(1)$. The non-zero values of $A(n,k)$ for $1 \leq n \leq 10$ are given in the following table.

	$A(n,k)$	1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1	1	1
	2		1	4	11	26	57	120	247	502	1013
	3			1	11	66	302	1191	4293	14608	47840
	4				1	26	302	2416	15619	88234	455192
k	5					1	57	1191	15619	156190	1310354
	6						1	120	4293	88234	1310354
	7							1	247	14608	455192
	8								1	502	47840
	9									1	1013
	10										1

Table 3. Small values of $A(n,k)$

3.4. Quasi-Eulerian Numbers

Suppose $Q, b, c : \mathbb{P}^2 \to \mathbb{N}$ and $Q(n,k)$ satisfies the recurrence relation

$$Q(n,k) = b(n,k)\,Q(n-1,k-1) + c(n,k)\,Q(n-1,k) \qquad (n,k \geq 2) \tag{1}$$

where $Q(n,1) = 1 \quad (n \in \mathbb{P})$, $Q(1,k) = 0 \quad (k \geq 2)$ and $b(n,k), c(n,k) \neq 0 \Leftrightarrow k \in [n] \quad (n \in \mathbb{P})$. Hence, $Q(n,k) \neq 0$ if and only if $1 \leq k \leq n$. This definition of Q is due to Kurtz [6] who investigated their concavity properties. Taking $b(n,k) = n-k+1$ and $c(n,k) = k$ gives the Eulerian numbers. Hence, we call the $Q(n,k)$ the *Quasi-Eulerian numbers.*

Kurtz [6] proved that the Q satisfying (1) are strictly log-concave, hence have a peak or plateau of width two, provided the following hold for $1 < k < n$ and $n \geq 3$:

$$2\,b(n,k) \geq b(n,k-1) + b(n,k+1) \tag{2}$$

$$2\,c(n,k) \geq c(n,k-1) + c(n,k+1). \tag{3}$$

Suppose $b(n,n) = 1 \quad (n \in \mathbb{P})$ and

$$b(n,k) = c(n,n-k+1) \qquad (k \in [n], n \in \mathbb{P})\,. \tag{4}$$

If (4) holds, then (2) and (3) are equivalent. The Eulerian numbers satisfy (1)$-$(4).

If Q satisfies (1) and (4), then Q is symmetric: Here $Q(2,1) = Q(2,2) = 1$. Suppose $Q(n-1,k) = Q(n-1,n-k)$ $\;(k \in [n-1], n \geq 3)$. Then $Q(n,1) = Q(n,n) = 1$ and, for $2 \leq k \leq n-1$,

$$\begin{aligned} Q(n,n-k+1) &= b(n,n-k+1)\,Q(n-1,n-k) \\ &\qquad\qquad + c(n,n-k+1)\,Q(n-1,n-k+1) \\ &= c(n,k)\,Q(n-1,k) + b(n,k)\,Q(n-1,k-1) \\ &= Q(n,k)\,. \end{aligned}$$

Suppose Q satisfies $(1)-(4)$. Then the maximum of the $Q(n,k)$ occurs at $\lceil n/2 \rceil$ for odd n and at $\lceil n/2 \rceil$ and $\lceil n/2 \rceil + 1$ for even n $\;(n \in \mathbb{P})$. Since Q is symmetric, (1) implies

$$Q(n, \lceil n/2 \rceil) \geq c(n, \lceil n/2 \rceil)\, Q(n-1, \lceil (n-1)/2 \rceil). \tag{5}$$

Iteration of (5) gives the weak bound

$$Q(n, \lceil n/2 \rceil) \geq \prod_{k=2}^{n} c(k, \lceil k/2 \rceil) \geq g(\lceil n/2 \rceil) \qquad (n \geq 2)\,.$$

Suppose we can lift $g : \mathbb{P} \to \mathbb{P}$ to $g : [0,\infty) \to [\tau,\infty)$ which is continuous, strictly increasing and surjective. Then the array Q is normal with $d(n) = n, f(n) = \lceil n/2 \rceil, g(x), r = 2$ and $\Delta = 1$. Observation 1 and Theorem 2 imply

$$N_Q(t) < 2\, g^{-1}(t) + 2\,.$$

For the Eulerian numbers, $c(n,k) = k$ and $\prod_{k=2}^{n} \lceil k/2 \rceil \geq (\lceil n/2 \rceil / e)^{\lceil n/2 \rceil}$ and we took $g(x) = (x/e)^x$ in Example 9.

Our final example is one infinite sub-family of the functions Q.

Example 10. Certain arrays of Quasi-Eulerian numbers $Q(n,k)$.

Suppose $h : \mathbb{P} \to \mathbb{P}$ with $2\,h(k) \geq h(k-1) + h(k+1)$ $\;(k \geq 2)$. For example, $h(k) = k$ or $h \equiv \ell \in \mathbb{P}$ are such functions. Define $Q, b, c : \mathbb{P}^2 \to \mathbb{N}$ where Q is given by (1) and where

$$c(n,k) = b(n,n-k+1) = \begin{cases} h(k), & k \in [n], n \in \mathbb{P}; \\ 0, & \text{otherwise}\,. \end{cases}$$

Then Q satisfies $(1)-(4)$. Suppose $\prod_{k=2}^{n} h(k) \geq \rho^{\lceil n/2 \rceil}$ $\;(n \geq 2)$ where $\rho \in (1,\infty)$. For example, if $h \equiv \ell \geq 2$, then $\prod_{k=2}^{n} h(k) = \ell^{\,n-1} \geq \rho^{\lceil n/2 \rceil}$ $\;(n \geq 2)$ where $\rho = \ell$. The array Q is normal with $d(n) = n, f(n) = \lceil n/2 \rceil, g(x) = \rho^x$ with $\tau = \rho$, $r = 2$ and $\Delta = 1$. Observation 1 and Theorem 2 imply

$$N_Q(t) < 2\,\log_\rho t + 2 \qquad (t \geq 2)\,.$$

4. Conclusion

For every normal Δ-bounded array $a = (a, d, f, r, \Delta, g)$ with multiplicity r and growth function g, Theorem 1 gives the non-trivial bound $N_a(t) < r\big(g^{-1}(t) + \Delta\big)$ for all $t \geq 2$. This bound is best possible in general, apart from the constant $r\Delta$, as the combinatorial arrays in Examples 5 and 6 demonstrate. Perhaps special properties of a particular array a can be used to give better upper bounds for $N_a(t)$.

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