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INFINITELY OFTEN DENSE BASES FOR THE INTEGERS WITH A PRESCRIBED REPRESENTATION FUNCTION

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Abstract

Nathanson constructed asymptotic bases for the integers with prescribed representation functions, then asked how dense they can be. We can easily obtain an upper bound using a simple argument. In this paper, we will see this is indeed the best bound when we prescribe an arbitrary representation function.

1. Introduction

We will use the following notation and terminology. For sets of integers A and B, any integer t and a positive integer h, we define the *sumset* $A + B = \{a + b : a \in A, b \in B\}$, the *translation* $A + t = \{a + t : a \in A\}$, and the *dilation* $h * A = \{ha : a \in A\}$. In particular, $2A = A + A$. Then we define the representation function of A as

$$r_A(n) = \#\{(a, b) : a, b \in A,\ a \leq b,\ a + b = n\},$$

where n is an integer. A set of integers A is called an *additive basis* for the integers if $r_A(n) \geq 1$ for every integer n. If all but finitely many integers can be written as a sum of two elements from A, then A is called an *asymptotic basis* for the integers. A set of integers A is called a *unique representation basis* for the integers if $r_A(n) = 1$ for every integer n. Also, the *counting function* for the set A is

$$A(y, x) = \#\{a \in A : y \leq a \leq x\}$$

for real numbers x and y.

For bases of the integers, Nathanson obtained the following:

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Theorem 1. ([7, 6]) *Let $\mathbb{N}_0$ be the set of nonnegative integers. Let $f\colon \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be any function such that the set $f^{-1}(0)$ is a finite set.*

(i) *Let $\phi : \mathbb{N}_0 \to \mathbb{R}$ be any nonnegative function such that $\lim_{x\to\infty} \phi(x) = \infty$. Then there exist uncountably many asymptotic bases A of the integers such that $r_A(n) = f(n)$ for every integer n and $A(-x, x) \le \phi(x)$ for every $x \ge 0$.*

(ii) *There exist uncountably many asymptotic bases A of the integers such that $r_A(n) = f(n)$ for every integer n and $A(-x, x) \gg x^{1/3}$ for every sufficiently large x.*

Note that, under our assumption, a finite set $f^{-1}(0)$ just means A is an asymptotic basis. Cilleruelo and Nathanson [2] later improved the exponent 1/3 in the second statement of Theorem 1 to $\sqrt{2} - 1 + o(1)$. Also, Łuczak and Schoen [4] proved the second statement of Theorem 1 by showing that a set with a condition of Sidon type can be extended to a unique representation basis.

When a representation function is arbitrarily given, the first statement of Theorem 1 means an asymptotic basis for the integers can be as sparse as we want, and the second statement of Theorem 1 means we can achieve a certain thickness. Therefore, we want to consider the following general question: *Given an arbitrary representation function, what are the possible thicknesses for asymptotic bases for the integers?* This question was posed by Nathanson [6].

We can obtain an upper bound quite easily, as shown in [5]. Let A be any set of integers with a bounded representation function $r_A(n) \le r$ for some $r > 0$ for every integer n. Take $k = A(-x, x)$. Then there are $\frac{k(k+1)}{2}$ ways to make $a_i + a_j$, where a_i, a_j are in A with $|a_i|, |a_j| \le x$. All of these sums belong to the interval $[-2x, 2x]$ and each number in that interval is represented by $a_i + a_j$ at most r times. Therefore,

$$\frac{k(k+1)}{2} \le r(4x+1)$$

and solving this for $k = A(-x, x)$ gives us $A(-x, x) \ll x^{1/2}$ for all $x > 0$.

Since we are considering possible thicknesses for asymptotic bases with an arbitrary representation function given, the above argument provides us with an upper bound $x^{1/2}$ for possible thicknesses. Then the next question we might ask is, can we find a better upper bound, or is $x^{1/2}$ the best upper bound we can get? One way to find a better upper bound is to find a representation function which requires a better upper bound, just as above.

We might want to start with unique representation bases. Nathanson in [5] asked the following: Does there exist a number $\theta < 1/2$ such that $A(-x, x) \le x^{\theta}$

for every unique representation basis A and for every sufficiently large x? This question was answered negatively by Chen [1].

Theorem 2. ([1]) *For any $\epsilon > 0$, there exists a unique representation basis A for the integers such that for infinitely many positive integers x, we have*

$$A(-x, x) \geq x^{1/2-\epsilon}.$$

In this paper, we show that it is impossible to find a better upper bound for any representation function. Therefore, if we consider $A(-x, x)$ for infinitely many integers x instead of all large x, $x^{1/2}$ is indeed the best upper bound.

2. Preliminary Lemmas

From now on, f will denote a function $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ such that the set $f^{-1}(0)$ is a finite set. Then there exists a positive integer d_0 such that $f(n) \geq 1$ for every integer n with $|n| \geq d_0$. Nathanson proved the following:

Lemma 3. ([7]) *Given a function f as above, there exists a sequence $U = \{u_k\}_{k=1}^{\infty}$ of integers such that, for every $n \in \mathbb{Z}$,*

$$f(n) = \#\{k \geq 1 : u_k = n\}. \tag{1}$$

Lemma 4. *Let A be a finite set of integers with $r_A(n) \leq f(n)$ for every integer n. Also assume that $0 \notin A$, and for every integer n,*

$$r_A(n) \geq \#\{i \leq m : u_i = n\}$$

for some integer m which depends only on the set A. Then there exists a finite set of integers B such that $A \subseteq B$, $0 \notin B$, $r_B(n) \leq f(n)$ for every integer n, and

$$r_B(n) \geq \#\{i \leq m+1 : u_i = n\}$$

for every integer n.

Proof. If $r_A(n) \geq \#\{i \leq m+1 : u_i = n\}$ for all n, then take $B = A$ and we are done. Otherwise, note that

$$\#\{i \leq m : u_i = n\} = \#\{i \leq m+1 : u_i = n\}$$

for all $n \neq u_{m+1}$ and

$$\#\{i \leq m : u_i = u_{m+1}\} + 1 = \#\{i \leq m+1 : u_i = u_{m+1}\}.$$

If

$$r_A(n) < \#\{i \le m+1 : u_i = n\}$$

for some n as we are assuming now, since

$$r_A(n) \ge \#\{i \le m : u_i = n\}$$

for all n, we must have

$$r_A(u_{m+1}) < \#\{i \le m+1 : u_i = u_{m+1}\} \le f(u_{m+1}).$$

Let $d = \max\{d_0,\ |u_{m+1}|,\ |a| \text{ where } a \in A\}$. Choose an integer $c > 4d$ if $u_{m+1} \ge 0$ and $c < -4d$ if $u_{m+1} < 0$. Note that $|c| > 4d$.

Let $B = A \cup \{-c,\ c + u_{m+1}\}$. Then $2B$ has three parts:

$$2A, \quad A + \{-c,\ c + u_{m+1}\}, \quad \{-2c,\ u_{m+1},\ 2c + 2u_{m+1}\}.$$

If $b \in 2A$, then $-2d \le b \le 2d$. Let $a \in A$. Then if $u_{m+1} \ge 0$, we have $c > 0$, and thus

$$\begin{aligned} a - c &\le d - 4d = -3d, \\ a + c + u_{m+1} &\ge -d + 4d + u_{m+1} \ge 3d + u_{m+1} \ge 3d\,; \end{aligned}$$

and if $u_{m+1} < 0$, we have $c < 0$, and thus

$$\begin{aligned} a - c &\ge -d + 4d = 3d, \\ a + c + u_{m+1} &\le d - 4d + u_{m+1} = -3d + u_{m+1} \le -3d\,. \end{aligned}$$

Therefore,

$$2A \,\cap\, A + \{-c,\ c + u_{m+1}\} = \emptyset\,.$$

Also, each element of $A + \{-c,\ c + u_{m+1}\}$ has a unique representation in the form of $a + \{-c,\ c + u_{m+1}\}$, $a \in A$, and the same is true for $\{-2c,\ u_{m+1},\ 2c + 2u_{m+1}\}$ in the form of $\{-c,\ c + u_{m+1}\} + \{-c,\ c + u_{m+1}\}$. And

$$\begin{aligned} |-2c| &= 2|c| > 8d, \\ |2c + 2u_{m+1}| &= 2|c + u_{m+1}| \ge 2|c| \ge 8d\,, \end{aligned}$$

and thus $2A \cap \{-2c, 2c + 2u_{m+1}\} = \emptyset$ (recall that c and u_{m+1} have the same sign). Also note that $A + \{-c,\ c + u_{m+1}\}$ and $\{-2c,\ u_{m+1},\ 2c + 2u_{m+1}\}$ are disjoint. To see this, for example, if $a - c = 2c + 2u_{m+1}$ for some $a \in A$, then $a = 3c + 2u_{m+1}$ so $|a| = |3c + 2u_{m+1}| \ge |3c| > 12d$, giving us a contradiction. Other cases are similar.

Thus, we have

$$r_B(n) = \begin{cases} r_A(n)+1 & \text{if } n = u_{m+1} \\ r_A(n) & \text{if } n \in 2A \setminus \{u_{m+1}\} \\ 1 & \text{if } n \in 2B \setminus \big\{2A \cup \{u_{m+1}\}\big\}. \end{cases}$$

Now, we have $r_A(u_{m+1}) < f(u_{m+1})$, so

$$r_B(u_{m+1}) = r_A(u_{m+1}) + 1 \ \le f(u_{m+1}).$$

If $n \in 2B \setminus \big\{2A \cup \{u_{m+1}\}\big\}$, then $|n| \ge d_0$, so $f(n) \ge 1 = r_B(n)$. Thus, $r_B(n) \le f(n)$ for all n.

Now, $r_A(u_{m+1}) \ge \#\{i \le m : u_i = u_{m+1}\}$, so

$$\begin{aligned} r_A(u_{m+1}) + 1 &\ge \#\{i \le m : u_i = u_{m+1}\} + 1 \\ &= \#\{i \le m+1 : u_i = u_{m+1}\}, \end{aligned}$$

and therefore,

$$r_B(u_{m+1}) \ge \#\{i \le m+1 : u_i = u_{m+1}\}.$$

If $n \in 2A \setminus \{u_{m+1}\}$, then

$$r_B(n) = r_A(n) \ge \#\{i \le m : u_i = n\} = \#\{i \le m+1 : u_i = n\}.$$

If $n \in 2B \setminus \big\{2A \cup \{u_{m+1}\}\big\}$, then

$$0 = r_A(n) \ge \#\{i \le m : u_i = n\} = \#\{i \le m+1 : u_i = n\},$$

so

$$0 = \#\{i \le m+1 : u_i = n\} \le 1 = r_B(n).$$

Thus,

$$r_B(n) \ge \#\{i \le m+1 : u_i = n\}$$

for all n. □

Lemma 5. *Let A be a finite set of integers with $r_A(n) \le f(n)$ for all n, and $0 \notin A$. Let $\phi(x)\colon \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function such that $\lim_{x\to\infty} \phi(x) = \infty$. Then for any $M > 0$, there exists an integer $x > M$ and a finite set of integers B with $0 \notin B$, $A \subseteq B$, $r_B(n) \le f(n)$ for all n, and $B(-x, x) > \sqrt{x}/\phi(x)$.*

Proof. It is well-known that there exists a Sidon set $D \subseteq [1, n]$ such that $|D| = n^{1/2} + o(n^{1/2})$ for all $n \ge 1$ (see, for example, [3, section 2.3]). Choose an integer x which satisfies the following:

1. $\phi(x) > M + \sqrt{20T}$ where $T = \max\{d_0,\ |a| \text{ where } a \in A\}$,
2. x is a multiple of $5T$,
3. $x > M$,
4. If n is large enough, $|D| = \sqrt{n} + o(\sqrt{n}) > \sqrt{n}/2$. Let x be large enough so that $n = x/5T$ would be large enough to satisfy the above.

Let $B = A \cup \{5Td : d \in D\}$ where $D \subseteq [1, n]$ with $n = x/5T$ as above. Then

$$\begin{aligned} B(-x,x) &\geq B(0,x) \geq D(1,n) = |D| \\ &> \frac{\sqrt{n}}{2} = \frac{1}{2}\sqrt{\frac{x}{5T}} = \frac{\sqrt{x}}{\sqrt{20T}} > \frac{\sqrt{x}}{\phi(x) - M} > \frac{\sqrt{x}}{\phi(x)}. \end{aligned}$$

Now, note that $2B$ has three parts:

$$2A,\ A + 5Td \text{ for } d \in D, \text{ and } 5T(d_1 + d_2) \text{ for } d_1, d_2 \in D.$$

As earlier, we have $2A \cap (A+5T{*}D) = \emptyset$ and $2A \cap \{5T(d_1+d_2) : d_1, d_2 \in D\} = \emptyset$. Now, if $a + 5Td_1 = 5T(d_2 + d_3)$ for $a \in A$, $d_i \in D$, then $|a| = 5T\,|d_2 + d_3 - d_1|$. If $|\,d_2 + d_3 - d_1| = 0$, then $a = 0 \in A$, a contradiction. And if $|\,d_2 + d_3 - d_1| \geq 1$, then $|\,a| \geq 5T$, a contradiction. Thus

$$(A + 5T{*}D) \cap \{5T(d_1 + d_2) : d_1, d_2 \in D\} = \emptyset.$$

If $a_1 + 5Td_1 = a_2 + 5Td_2$ for $a_1, a_2 \in A,\ d_1, d_2 \in D$, then $|\,a_1 - a_2| = 5T|\,d_1 - d_2|$. As above, this cannot happen unless $a_1 = a_2$ and $d_1 = d_2$. And if $5T(d_1 + d_2) = 5T(d_3 + d_4)$ for $d_i \in D$ with $d_1 \leq d_2$, $d_3 \leq d_4$, then $d_1 + d_2 = d_3 + d_4$. Since D is a Sidon set, we have $d_1 = d_3,\ d_2 = d_4$. Thus we have

$$r_B(n) = \begin{cases} r_A(n) & \text{if } n \in 2A \\ 1 & \text{if } n \in 2B \setminus 2A. \end{cases}$$

If $n \in 2B \setminus 2A$, $n \geq T \geq d_0$, so $f(n) \geq 1 = r_B(n)$. Thus, $r_B(n) \leq f(n)$ for all n. $\square$

3. Main Result

Theorem 6. *Let $f\colon \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be a function where $f^{-1}(0)$ is a finite set. Let $\phi\colon \mathbb{N}_0 \to \mathbb{R}$ be any nonnegative function with $\lim_{x\to\infty} \phi(x) = \infty$. Then there exists an asymptotic basis A for the integers such that $r_A(n) = f(n)$ for all n, and for infinitely many positive integers x, we have $A(-x,x) > \sqrt{x}/\phi(x)$.*

Proof. Recall that given such a function f, we have d_0 and $\{u_k\}$ as defined in Section 2. We use induction to get an infinite sequence of finite sets of integers $A_1 \subseteq A_2 \subseteq \cdots$ and a sequence of positive integers $\{x_i\}_{i=1}^{\infty}$ with $x_{i+1} > x_i$ such that, for every j,

(i) $r_{A_j}(n) \leq f(n)$ for all n,

(ii) $r_{A_{2j-2}}(n),\, r_{A_{2j-1}}(n) \;\geq\; \#\{i \leq j : u_i = n\}$ for all n,

(iii) $A_{2j-1}(-x_j, x_j) > \sqrt{x_j}/\phi(x_j)$,

(iv) $0 \notin A_j$.

If $u_1 \geq 0$, take $c = 4d_0 > 0$. If $u_1 < 0$, take $c = -4d_0 < 0$. Let $\alpha = |2c + 2u_1| > 0$ (thus $\alpha > |c|, |u_1|, d_0$). As before, if n is large enough, there exists a Sidon set $D \subseteq [1, n]$ such that $|D| > \sqrt{n}/2$. Take such an integer n which also satisfies $\phi(3\alpha n) > 2\sqrt{3\alpha}$.

Take $A_1 = 3\alpha{*}D \,\cup\, \{-c,\; c + u_1\}$ and $x_1 = 3\alpha n$. Then $2A_1$ has three parts:

$$2(3\alpha{*}D), \quad 3\alpha{*}D + \{-c,\; c+u_1\}, \quad \{-2c,\; u_1,\; 2c+2u_1\}.$$

It is easy to see that these are pairwise disjoint(for example, if $3\alpha d - c = 2c+2u_1$, then $3\alpha d = 3c + 2u_1$, so $3\alpha \leq |\,3c + 2u_1| < |\,4c + 4u_1| = 2\alpha$, a contradiction). If $3\alpha d_1 - c = 3\alpha d_2 + c + u_1$, then $2c + u_1 = 3\alpha(d_1 - d_2)$. If $d_1 \neq d_2$, then $|2c+u_1| \geq 3\alpha$ but $|\,2c + u_1| < |\,2c + 2u_1| = \alpha$. So $d_1 = d_2$. Then $-c = c + u_1$, so $-2c = u_1$, a contradiction. Thus,

$$r_{A_1}(n) = \begin{cases} 1 & \text{if } n \in 2A_1 \\ 0 & \text{if } n \notin 2A_1. \end{cases}$$

Now, $3\alpha d - c \geq 3\alpha - \alpha = 2\alpha$, and also $3\alpha d + c + u_1 \geq 3\alpha - \alpha - \alpha = \alpha$. So if $n \in 2A_1 \setminus \{u_1\}$ then $|n| \geq d_0$, so $f(n) \geq 1$. And if $n = u_1$, then by the definition of $\{u_k\}$, we have $f(u_1) = \#\{k : u_k = u_1\} \geq 1$. Thus, for all $n \in 2A_1$, $r_{A_1}(n) = 1 \leq f(n)$. If $n \notin 2A_1$, $r_{A_1}(n) = 0 \leq f(n)$. Therefore, for all n, $r_{A_1}(n) \leq f(n)$. Now, we have $1 = r_{A_1}(u_1) \geq \#\{i \leq 1 : u_i = u_1\}$. For other $n \neq u_1$, $r_{A_1}(n) \geq \#\{i \leq 1 : u_i = n\} = 0$. Thus, $r_{A_1}(n) \geq \#\{i \leq 1 : u_i = n\}$ for all n. Also,

$$\begin{aligned} A_1(-x_1, x_1) &\geq A_1(1, 3\alpha n) \geq D(1, n) = |D| \\ &> \frac{\sqrt{n}}{2} = \frac{\sqrt{3\alpha n}}{2\sqrt{3\alpha}} > \frac{\sqrt{3\alpha n}}{\phi(3\alpha n)} = \frac{\sqrt{x_1}}{\phi(x_1)}. \end{aligned}$$

Thus A_1 satisfies all of the Conditions (i) through (iv) above.

Now, suppose we have $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{2l-1}$ and $x_1 < x_2 < \cdots < x_l$. By Lemma 4, there exists A_{2l} such that $A_{2l-1} \subseteq A_{2l}$, with $r_{A_{2l}}(n) \leq f(n)$ for all n and

$$r_{A_{2l}}(n) \geq \#\{i \leq l+1 : u_i = n\}$$

for all n, and $0 \notin A_{2l}$. Thus, A_{2l} satisfies Conditions (i), (ii), and (iv). Now, by Lemma 5, there exists an integer $x_{l+1} > x_l$ and A_{2l+1} with $0 \notin A_{2l+1}$, $A_{2l} \subseteq A_{2l+1}$, $r_{A_{2l+1}}(n) \leq f(n)$ for all n, and

$$A_{2l+1}(-x_{l+1}, x_{l+1}) > \frac{\sqrt{x_{l+1}}}{\phi(x_{l+1})}.$$

Also, $r_{A_{2l+1}}(n) \geq r_{A_{2l}}(n) \geq \#\{i \leq l+1 : u_i = n\}$ for all n. Thus, A_{2l+1} satisfies Conditions 3 through 3.

Now, let $A = \cup_{l=1}^{\infty} A_l$. By Conditions (i) and (ii), $r_A(n) = f(n)$ for all n and

$$A(-x_k, x_k) \geq A_{2k-1}(-x_k, x_k) > \frac{\sqrt{x_k}}{\phi(x_k)}$$

for all k. □

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