#A31 INTEGERS 10 (2010), 365-368

ON VANISHING SUMS OF DISTINCT ROOTS OF UNITY

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Received: 1/25/10, Accepted: 3/24/10, Published: 7/9/10

Abstract

We show that for any integers $n \geq 2$ and $0 \leq k \leq n$, one may find k distinct nth roots of unity whose sum is 0 if and only if both k and $n-k$ are expressible as linear combinations of prime factors of n with nonnegative coefficients.

1. The Main Result

The inspiration for this problem is the following question: given a centrifuge with n buckets, for which k can we fill k of the buckets such that the centrifuge is balanced? An equivalent formulation asks for which k may we pick k distinct nth roots of unity whose sum is 0; we will say that n is *k-balancing* when this is possible.

T. Y. Lam and K. H. Leung proved in [2] that, if we allow repetition of roots (which we will not), then n is k-balancing if and only if k is expressible as a sum of the prime factors of n. Since we restrict to a subset of these cases, this is again a necessary condition. Additionally, since the sum of all nth roots vanishes, if a subset of nth roots has vanishing sum, so does its complement. Using the notation of Lam and Leung, and adopting their convention that $0 \in \mathbb{N}$, the complement condition implies the following:

Proposition 1 *Write $n = p_1^{e_1} \cdots p_r^{e_r}$, with each p_i prime and each e_i positive. Then n is k-balancing only if both k and $n-k$ are in $\mathbb{N}p_1 + \ldots + \mathbb{N}p_r$.*

Our goal is to show that this condition is also sufficient; i.e.,

Theorem 2 (Main Theorem) *Write $n = p_1^{e_1} \cdots p_r^{e_r}$ as above. Then n is k-balancing if and only if both k and $n-k$ are in $\mathbb{N}p_1 + \ldots + \mathbb{N}p_r$.*

This will be proven in several stages, depending on the prime factorization of n. This first lemma will be useful in all cases:

Lemma 3 *If* $\gcd(n,k) > 1$*, then* n *is* k*-balancing.*

Proof. For convenience here and for the rest of this paper, let $\zeta_n := \exp \frac{2\pi i}{n}$. The cases $k = 0$ and $k = n$ are trivial, so assume $1 \leq k \leq n-1$, and take any prime p dividing both n and k. The pth roots of unity are also nth roots of unity and have sum 0, and multiplying them all by some other root of unity preserves the sum while effectively rotating the roots. Since $k < n$, we have $k/p < n/p$, so we may rotate the pth roots of unity by ζ_n k/p times without seeing any one root twice. Thus the set $\{\zeta_p^a \cdot \zeta_n^b \mid 0 \leq a < p,\ 0 \leq b < k/p\}$ has k distinct elements with sum 0. □

This, combined with Proposition 1, proves the main theorem for all prime powers. Next, we look at products of two primes.

Proposition 4 *The main theorem holds for any product* $n = pq$ *of two distinct primes.*

Proof. Fix $k \leq n$ and suppose we may write $k = ap + bq$ and $n - k = cp + dq$ for some $a, b, c, d \in \mathbb{N}$. Then $n = (a+c)p + (b+d)q = pq$, so p divides $b + d$ and q divides $a + c$. If both coefficient sums are positive, then we must have $n = (a+c)p + (b+d)q \geq pq + pq = 2n$, a contradiction, so either $a + c = 0$ or $b + d = 0$, hence either $a = c = 0$ or $b = d = 0$. Thus k is either a multiple of p or a multiple of q, and n is k-balancing by Lemma 3. □

To complete the proof of the main theorem, we introduce two more lemmas that will aid in the construction of vanishing sums.

Lemma 5 *If* $\gcd(p,q) = 1$ *and* $k \geq (p-1)(q-1)$*, then* $k \in \mathbb{N}p + \mathbb{N}q$*.*

Proof. This is a classical result, dating back to [4]. □

Lemma 6 *Suppose two concentric disks are each divided into* n *equal wedges. If* a *wedges are colored in the first disk and* b *in the second, and* $ab < n$*, then the disks may be rotated such that no two colored wedges overlap.*

Proof. Fix one of the b wedges on the second disk. In the course of the n rotated positions, it overlaps a colored wedge from the first disk a different times. Considering each wedge and each rotation, we see a total of ab overlaps. Since $ab < n$, there must be some rotation which has no overlaps. □

The application of this lemma will be to vanishing sets of roots of size a and b, since with $ab < n$ we can rotate the second set so that the rotated set does not intersect the first set, and then the union gives a set of $a + b$ roots whose sum vanishes. In practice, we may restrict to the case where b is a prime dividing n and the roots correspond to a rotation of the bth roots of unity, in which case the exponents of ζ_n in the set of a roots cannot cover all possible residues modulo n/b and we may use the b roots in the remaining residue class.

Proof of the Main Theorem. The cases where n is a prime power or a product of two distinct primes have already been proven, so assume we are in neither case. Let $p < q$ be the two smallest distinct primes dividing n, and write $n = pqm$, where $m \geq p \geq 2$. If n is even, then the case $k = n/2$ follows from Lemma 3. If n is arbitrary and $k > n/2$, then $(p-1)(q-1) < n/m \leq n/2 < k$, so $k \in \mathbb{N}p + \mathbb{N}q$ by Lemma 5. Thus we may reduce the problem to showing that n is k-balancing if k is a sum of prime factors of n and $k < n/2$.

First, suppose $k < (p-1)(q-1)$. Writing the prime factorization as $n = p_1^{e_1} \cdots p_r^{e_r}$ with $p_1 < p_2 < \cdots < p_r$, so that $p_1 = p$ and $p_2 = q$, and writing $k = a_1p_1 + \ldots + a_rp_r$ with each $a_i \in \mathbb{N}$, we can show by induction on $\sum a_i$ that n is k-balancing. The base case corresponds to a single prime factor of n and follows from Lemma 3. Assume that n is k-balancing whenever $\sum a_i = s$, and now consider $k < (p-1)(q-1)$ with $\sum a_i = s + 1$. Choose the least i such that a_i is positive, and by hypothesis n is $(k - p_i)$-balancing. Pick distinct roots $\{\zeta_n^{b_1}, \ldots, \zeta_n^{b_{k-p_i}}\}$ whose sum vanishes, and take separately the balanced set $\{1, \zeta_{p_i}, \ldots, \zeta_{p_i}^{p_i-1}\}$. The first set has cardinality less than $(p-1)(q-1)$, and the second set has cardinality p_i, so if $(p-1)(q-1)p_i < n$ then by Lemma 6 we are done. Observe that p_i must divide at least one of p, q, and m. If $p_i = p$ or $p_i | m$ then $p_i \leq m$, so $(p-1)(q-1)p_i < pqm = n$. If $p_i = q$ then either $q \leq m$ and we proceed as before, or $m < q$, in which case m is a power of p. Then $n = p^{e_1}q$ $(e_1 > 1)$, and we may write $k = ap + bq$; by minimality of i we have $a = 0$, and n is k-balancing by Lemma 3, completing the induction.

Next, suppose $(p-1)(q-1) \leq k < n/2$. By Lemma 5 we may write $k = ap + bq$ with $a, b \in \mathbb{N}$; this will simplify the construction of a balanced set of k roots of unity. If $b \geq p$ we may also write $k = (a+q)p + (b-p)q$ and repeat; it then suffices to assume $0 \leq b < p$. Take the set of pqth roots of unity, and from those select b rotations of the qth roots; call their union S. Now there are at most q rotations of the pth roots of unity among the nth roots which intersect S, leaving at least $n/p - q = qm - q$ disjoint collections of p balanced roots which do not intersect S. Then we can add these sets one at a time to find a balanced set of as many as $bq + p(qm - q) = bq + n - pq$ roots. Since $m \geq 2$, $n - pq \geq n/2$, so $bq + p(qm - q) \geq n/2 > k$ and we can balance k roots by including fewer sets of p, and the theorem follows. □

References

[1] Conway, J. H. and Jones, A. J., Trigonometric Diophantine equations (On vanishing sums of roots of unity), *Acta Arith.* **30** (1976), 229–240.

[2] Lam, T. Y. and Leung, K. H., On vanishing sums of roots of unity, *J. Algebra* **224** (2000), 91–109.

[3] Mann, Henry B., On linear relations between roots of unity, *Mathematika* **12** (1965), 107–117.

[4] Sylvester, J. J., Question 7382, *Mathematical Questions from the Educational Times* **41** (1884).