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DISTANCE GRAPHS FROM *P*-ADIC NORMS

Jeong-Hyun Kang
Department of Mathematics, University of West Georgia, Carrollton, GA 30118
jkang@westga.edu

Hiren Maharaj
Department of Math Sciences, Clemson University, Clemson, SC 29634-0975
hmahara@clemson.edu

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Abstract

Given a set $D = \{d_1, d_2, \ldots\}$ of positive integers, one defines a *distance graph* with the set of integers $\mathbb{Z}$ as the vertex set and xy an edge iff $|x-y| \in D$. We approach distance graphs by using p-adic methods. This allows us to give general bounds on the chromatic number that depend on the divisibility properties of the numbers d_i. Furthermore, the chromatic number is determined for large classes of distance graphs.

1. Introduction

The Hadwiger–Nelson problem asks for the minimum number of colors needed for coloring the real plane such that no two points at distance 1 receive the same color. The best known lower and upper bounds are 4 and 7, with no improvement inthe last 50 years. The same question can be asked for any metric space in any dimension; n-dimensional real space $\mathbb{R}^n$ under any norm, n-dimensional integer grid $\mathbb{Z}^n$ under the ℓ_1-norm, and even the integer line under Archimedean or non-Archimedean norms. The natural generalization of the Hadwiger–Nelson problem in high dimensions has been studied for many years. The first breakthrough lower and upper bounds under the Euclidean norm are by Frankl and Wilson [4], and Larman and Rogers [9]. The lower bound was improved by Raigorodskii [10, 11, 12]. The problem has been generalized to other metric spaces by Benda and Perles [1], Raigorodskii [13], and Woodall [16], and to any normed space by Füredi and Kang [5, 6].

While a coloring of a metric space in high dimensions has the flavor of combinatorial geometry, an analogous question asked for the integer line has more of a flavor of combinatorial number theory. The integer distance graph $G(\mathbb{Z}, D)$ with distance set $D = \{d_1 < d_2 < \ldots\}$ has the set of integers $\mathbb{Z}$ as the vertex set and two vertices $x, y \in \mathbb{Z}$ are adjacent if and only if $|x-y| \in D$. Integer distance graphs were first systematically studied by Eggleton, Erdős and Skilton [2] and have been studied by others since then [14, 15, 17].

In this paper, we consider the distance graph on the set of integers $\mathbb{Z}$ under the p-adic norms. Let p be a prime number. Then any non-zero rational number x can be uniquely written in the form $x = \frac{r}{s}p^{\ell}$ where $\ell \in \mathbb{Z}$ and r, s are integers not divisible by p. One defines the p-adic norm of x by $||x||_p := 1/p^{\ell}$. This gives rise to a non-Archimedean norm on the rational numbers $\mathbb{Q}$. Some basic properties of p-adic norms will be given in Section 2. We define a p-adic distance graph $G(\mathbb{Z}, \mathcal{D})$ with the vertex set of $\mathbb{Z}$ and distance set $\mathcal{D} \subset \mathbb{Q}$ such that two integers x, y are adjacent if and only if $||x-y||_p \in \mathcal{D}$ for some prime p. Here, distance sets $\mathcal{D}$ should be reasonably chosen subsets of $\mathbb{Q}$ and the details will be discussed in Section 3.

The work of this paper is related to that of Ruzsa, Tuza and Voigt [14], so we first recall some of their results to put our work in perspective.

Theorem 1 [14, Ruzsa, Tuza and Voigt] *Let $D = \{d_1, d_2, \ldots\}$ be an infinite distance set. The chromatic number $\chi(G(\mathbb{Z}, D))$ is finite whenever*

$$\inf \frac{d_{i+1}}{d_i} > 1.$$

Moreover, this result is tight in the sense that every growth speed smaller than this admits a distance set D with infinite chromatic number.

Via p-adic norms, we will give bounds on the chromatic number of distance sets that are quite dense and have divisibility constraints. Consequently, the chromatic numbers are applicable even in the case that

$$\inf \frac{d_{i+1}}{d_i} = 1.$$

For example, Theorem 16 in Section 6 gives us a sufficient condition for the distance graph $G(\mathbb{Z}, D)$ to have finite chromatic number:

Theorem 16 *Let $D := \{d_1, d_2, \ldots\}$ be a distance set. For each prime number p, let $D(p)$ be the set of all powers p^n of p such that p^n divides d_i but p^{n+1} does not divide d_i for some i. Then*

$$\chi(G(\mathbb{Z}, D)) \leq \min\{p^{|D(p)|} : p \text{ is prime}\}.$$

Theorem 16 can be viewed as complementing Theorem 1 of Ruzsa, Tuza and Voigt which is also a sufficient (but not necessary) condition for $\chi(G(\mathbb{Z}, D))$ to be finite. For example, let $p_1 < p_2 < \ldots$ be an enumeration of the prime numbers. Set $D = \{d_1, d_2, \ldots\}$ where $d_i := (p_1 p_2 \ldots p_i)^i$ for each i. Then by Theorem 1, $\chi(G(\mathbb{Z}, D))$ is finite but Theorem 16 is inconclusive. On the other hand, if D is the set of all positive integers not divisible by a fixed prime number p (so $D(p) = \{1\}$), then Theorem 16 implies that $\chi(G(\mathbb{Z}, D)) \leq p$ while Theorem 1 is inconclusive.

In Section 2, we give a brief overview of results from p-adic methods which we will find useful. In Section 3, we discuss p-adic distance graphs and their relationship to Euclidean distance graphs. In Section 4, we determine the chromatic number of distance graphs with various classes of infinite distance sets. A generalization of p-adic distance sets and its characterization to have finite chromatic number are discussed in Section 5. As an application, we show how effectively p-adic results can be applied to Euclidean distance graphs in Section 6. We conclude the paper with suggestions on possible future research.

2. p-Adic Norms

In this section, we state the basic properties of p-adic norms that we will use. As excellent introductions to this subject we refer the reader to [7, 8]. The p-adic norm of 0 is defined to be 0. Equivalently, the p-adic norm can be formulated as follows. Any rational number x can be uniquely represented in the form $\sum_{i=\ell}^{\infty} a_i p^i$ where $0 \leq a_i \leq p-1$ and $a_\ell \neq 0$. Then the p-adic norm of x is given by $||x||_p := 1/p^\ell$. Note that the infinite expansion $\sum_{i=\ell}^{\infty} a_i p^i$ of a rational only makes sense in the p-adic topology.

Note that the set of possible p-adic norms of integers belongs to the set $\{1/p^\ell : \ell = 0, 1, 2 \ldots\} \cup \{0\}$. Furthermore, if $x \in \mathbb{Z}$ then $||x||_p = 1/p^i$ if and only if $x \in \{ap^i : a \in \mathbb{Z} \text{ with } p \nmid a\}$.

Let S be a non-empty set of distinct powers of the prime number p. We will say that x has *support* in S if the p-adic expansion of x is of the form $x = \sum_i a_i p^i$ $(0 \leq a_i \leq p-1)$ where $a_i = 0$ whenever $p^i \notin S$. Observe that if x and y are distinct and have support in S, then the p-adic norm of $x - y$ must be of the form $1/p^i$ for some $p^i \in S$.

We will repeatedly use the fact that if $x \equiv y \bmod p^n$ then $x - y$ has support contained in $\{p^n, p^{n+1}, p^{n+2}, \ldots\}$. Furthermore, we will make use of the result that

$$||x - y||_p \leq \max(||x||_p, ||y||_p)$$

with equality when $||x||_p \neq ||y||_p$. In general, one has

$$||x_1 + x_2 + \ldots + x_t||_p \leq \max(||x_i||_p : 1 \leq i \leq t)$$

with equality if there exists j such that $||x_j||_p > ||x_i||_p$ whenever $i \neq j$.

3. Distance Graphs Arising From p-Adic Norms

Throughout the rest of this paper, a script $\mathcal{D}$ signifies that the distance set arises from the non-Archimedean p-adic distances.

Let p be a fixed prime number. We will denote by $\mathcal{D}(p)$ a (finite or infinite) set of p-adic distances, i.e., $\mathcal{D}(p) = \{1/p^{n_1}, 1/p^{n_2}, \ldots\}$ for some positive integers $n_1, n_2, \ldots$. The graph $G(\mathbb{Z}, \mathcal{D})$ with

$$\mathcal{D} := \mathcal{D}(p) \tag{1}$$

denotes a p-adic distance graph with vertex set $\mathbb{Z}$, and two vertices $x, y \in \mathbb{Z}$ are adjacent (i.e., $\|x - y\| \in \mathcal{D}$) if $||x - y||_p \in \mathcal{D}(p)$. If $\mathcal{D}(p) = \{1/p^{n_1}, 1/p^{n_2}, \ldots\}$ for nonnegative integers $n_1 < n_2 < \cdots$, then one observes that, from Section 2, it is precisely the same as the distance graph $G(\mathbb{Z}, D)$ where $D = \{ap^{n_i} : i = 1, 2, \ldots$ and $a \in \mathbb{Z}^+$ with $p \nmid a\}$.

Let $p_1, p_2, \ldots$ be distinct prime numbers. The p-adic distance graph $G(\mathbb{Z}, \mathcal{D})$ on vertex set $\mathbb{Z}$ with

$$\mathcal{D} := \mathcal{D}(p_1) \sqcup \mathcal{D}(p_2) \sqcup \cdots \tag{2}$$

has an edge xy if $||x - y||_{p_i} \in \mathcal{D}(p_i)$ for *some* i. Analogously, when

$$\mathcal{D} := \mathcal{D}(p_1) \sqcap \mathcal{D}(p_2) \sqcap \cdots, \tag{3}$$

two vertices $x, y \in \mathbb{Z}$ are adjacent if $||x - y||_{p_i} \in \mathcal{D}(p_i)$ for *each* i. If $\mathcal{D}_1, \mathcal{D}_2, \ldots$ are p-adic distance sets of types in (1), (2) or (3), we define p-adic distance sets $\bigsqcup_i \mathcal{D}_i$ and $\sqcap_i \mathcal{D}_i$, and corresponding graphs $G(\mathbb{Z}, \bigsqcup_i \mathcal{D}_i)$ and $G(\mathbb{Z}, \sqcap_i \mathcal{D}_i)$, with two vertices $x, y \in \mathbb{Z}$ adjacent if xy is an edge in $G(\mathbb{Z}, \mathcal{D}_i)$ for *some* i, and for *each* i, respectively.

Using these notations, we can give a precise description of a given Euclidean distance distance graph in terms of non-Archimedean p-adic norms as follows. Let P denote the set of prime numbers. If $d > 1$ is an integer, the product formula [7, 8] states that

$$|d| \prod_{p \in P} ||d||_p = 1. \tag{4}$$

In other words, $d = p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$ is the factorization of d as a product of distinct prime numbers p_1, p_2, $\ldots$, p_t iff $||d||_{p_i} = 1/p_i^{e_i}$ for each $i = 1, 2, \ldots, t$ and $||d||_p = 1$ for all remaining primes. Now put $\mathcal{D}(p_i) = \{1/p_i^{e_i}\}$ for $i = 1, 2, \ldots, t$, and put $\mathcal{D}(q) = \{1\}$ for primes $q \neq p_1, p_2, \ldots, p_t$. Then the Euclidean distance graph $G(\mathbb{Z}, \{d\})$ is identical to the distance graph

$G(\mathbb{Z}, \sqcap_{p\in P}\mathcal{D}(p))$ arising from p-adic norms. For general cases, let $D := \{d_1, d_2, \ldots\}$ be a distance set of positive integers. For each distance d_i, let $e_i : P \to \mathbb{N}_0$ the function defined by $e_i(p)$ is the power of p that exactly divides d_i (so if a prime p does not divide d_i, then $e_i(p) = 0$). Now put $\mathcal{D}_i := \sqcap_{p\in P}\{1/p^{e_i(p)}\}$ and $\mathcal{D} := \bigsqcup_{i=1}^{\infty} \mathcal{D}_i$. Then the graph $G(\mathbb{Z}, D)$ is identical to the graph $G(\mathbb{Z}, \mathcal{D})$ described using p-adic norms.

Thus, by using the product formula, any distance graph $G(\mathbb{Z}, D)$ where D is a distance set can be recast in the language of p-adic numbers. The reverse is also true as illustrated above and in Theorems 3, 5, 7. However, sometimes it may be more convenient to use one language over the other. For example, if D is the set of all positive multiples of the prime number 2 which are not divisible by 4, then the distance graph $G(\mathbb{Z}, D)$ is more neatly described as the distance graph $G(\mathbb{Z}, \mathcal{D})$ where $\mathcal{D} = \{1/2\}$ and x is adjacent to y iff $||x - y||_2 = 1/2$.

4. Distance Graphs With p-Adic Distance Sets in Finite Unions

In this section, we give exact chromatic numbers of several distance graphs arising from p-adic norms introduced in Section 3.

Theorem 2 *Let p be a prime, and let $\mathcal{D}(p)$ be a p-adic distance set of size k. Then $\chi(G(\mathbb{Z}, \mathcal{D}(p))) = p^k$.*

Proof. Suppose that $\mathcal{D}(p) = \{1/p^{n_i} : 1 \le i \le k\}$ for some integers $n_1 > n_2 > \ldots > n_k \ge 0$. Define $f : \mathbb{Z} \to \{0, 1, \ldots, p-1\}^k$ by

$$f(x) := (a_{n_k}, a_{n_{k-1}}, \ldots, a_{n_1})$$

where a_{n_i} is the coefficient of p^{n_i} in the p-adic representation of x for $i = 1, 2, \ldots, k$. We show that f does indeed give a coloring of $G(\mathbb{Z}, \mathcal{D})$. Let $x, y \in \mathbb{Z}$ and suppose that x has p-adic representation $\ldots a_2a_1a_0$ and that y has p-adic representation $\ldots b_2b_1b_0$. If x is adjacent to y then for some i we have that

$$a_{n_i} \ne b_{n_i}$$

yet

$$a_{n_i-1} \ldots a_1a_0 = b_{n_i-1} \ldots b_1b_0.$$

whence $f(x) \ne f(y)$. Thus $\chi(G(\mathbb{Z}, \mathcal{D})) \le p^k$. On the other hand, observe that the subgraph induced by $V := \{\sum_{i=1}^{k} a_{n_i}p^{n_i} : 0 \le a_i \le p-1 \; for \; i = 1, 2, \ldots, k\}$ forms a clique in $G(\mathbb{Z}, \mathcal{D})$ since the p-adic norm of the difference of two distinct elements of V is of the form $1/p^{n_i}$ for some $1 \le i \le k$. Thus $\chi(G(\mathbb{Z}, \mathcal{D})) \ge p^k$. This completes the proof. □

Corollary 3 *Suppose that $n_1 > n_2 > \ldots > n_k \geq 0$ are integers and*

$$D = \{ap^{n_i} : 1 \leq i \leq k \text{ and } a \in \mathbb{Z} \text{ with } p \nmid a\}.$$

Then $\chi(G(\mathbb{Z}, D)) = p^k$.

Proof. This follows from Theorem 2 and the discussion in Section 3. □

Theorem 4 *Let $p_1, p_2, \ldots, p_t$ be a collection of distinct prime numbers. Then*

$$\chi\big(G(\mathbb{Z}, \mathcal{D}(p_1) \sqcup \mathcal{D}(p_2) \sqcup \ldots \sqcup \mathcal{D}(p_t))\big) = p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t},$$

where $k_i = |\mathcal{D}(p_i)|$ for $1 \leq i \leq t$.

Proof. Put $G := G(\mathbb{Z}, \mathcal{D}(p_1) \sqcup \mathcal{D}(p_2) \sqcup \ldots \sqcup \mathcal{D}(p_t))$. By Theorem 2, we can choose a coloring $f_i : \mathbb{Z} \to \{0, 1, \ldots, p_i - 1\}^{k_i}$ of $G(\mathbb{Z}, \mathcal{D}(p_i))$. Define $f : \mathbb{Z} \to \prod_{i=1}^{t} \{0, 1, \ldots, p_i - 1\}^{k_i}$ by

$$f(x) = (f_1(x), f_2(x), \ldots, f_t(x)).$$

Then, by using similar arguments to those of Theorem 2, f is a coloring of G. Thus $\chi(G) \leq p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t}$.

For each $1 \leq i \leq t$, let A_i' be the set of all prime powers p_i^m such that $1/p_i^m \in \mathcal{D}(p_i)$ and let A_i be the set of all linear combinations of elements of A_i' with coefficients from $0, 1, 2, \ldots, p - 1$. Note that $|A_i| = p_i^{k_i}$ and put $M_i = p_i^{k_i+1}$. For each $(a_1, a_2, \ldots, a_t) \in A_1 \times A_2 \times \ldots \times A_t$, it follows from the Chinese remainder theorem that there exists an $a \geq 0$, unique modulo $M = \prod_i M_i$, such that $a \equiv a_i \bmod M_i$ for $1 \leq i \leq t$. Let A be the set of all such a's. Then $|A| = |A_1||A_2| \ldots |A_t|$. Next we show that the vertices in A form a clique. Let $a, a' \in A$ and suppose they correspond to $(a_1, a_2, \ldots, a_t)$ and $(a_1', a_2', \ldots, a_t')$. Then $a - a' \equiv (a_i - a_i') \bmod M_i$ so $||a - a'||_{p_i} = ||a_i - a_i'||_{p_i} \in \mathcal{D}(p_i)$. Thus a is joined to a' and the chromatic number of G is at least $|A| = p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t}$. □

Corollary 5 *Let $p_1, p_2, \ldots, p_t$ be a collection of distinct prime numbers. Corresponding to each prime p_i, $1 \leq i \leq t$, let D_i be a finite set of distinct non-negative powers of p_i of size $k_i := |D_i|$. Let*

$$D := \{ax : for\ some\ 1 \leq i \leq t, x \in D_i \text{ and } a \in \mathbb{Z} \text{ with } p_i \nmid a\}.$$

Then $\chi(G(\mathbb{Z}, D)) = p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t}$.

Proof. By the product formula (4), the graph $G(\mathbb{Z}, D)$ is isomorphic to the distance graph $G(\mathbb{Z}, \mathcal{D}(p_1) \sqcup \mathcal{D}(p_2) \sqcup \ldots \sqcup \mathcal{D}(p_t))$ of Theorem 4. □

Theorem 6 *Let $p_1, p_2, \ldots, p_t$ be a collection of distinct prime numbers. Then*

$$\chi\big(G(\mathbb{Z}, \mathcal{D}(p_1) \sqcap \mathcal{D}(p_2) \sqcap \ldots \sqcap \mathcal{D}(p_t))\big) = \min\{p_i^{k_i} : 1 \le i \le t\},$$

where $k_i = |\mathcal{D}(p_i)|$ for $1 \le i \le t$.

Proof. Put $G := G(\mathbb{Z}, \mathcal{D}(p_1) \sqcap \mathcal{D}(p_2) \sqcap \ldots \sqcap \mathcal{D}(p_t))\big)$. The upper bound follows from the observation that G is a subgraph of $G(\mathbb{Z}, \mathcal{D}(p_i))$ for each i and Theorem 2. For the reverse inequality, we may assume that

$$p_1^{k_1} < p_2^{k_2} < \cdots < p_t^{k_t}.$$

For each i, let A_i' be the set of all numbers p_i^k such that $1/p_i^k \in \mathcal{D}(p_i)$. Let $A_i := \{a(i,1) < a(i,2) < \ldots\}$ be the set of all non-zero linear combinations of elements of A_i' with coefficients from $0, 1, \ldots, p_i - 1$. Note that $|A_i| = p_i^{k_i} - 1$ and the set A_1 is the smallest. Put $M_i := p_i^{k_i+1}$. From the Chinese remainder theorem, for each $1 \le j \le p^{k_1}$, there exists an x_j, unique modulo $M_1 M_2 \ldots M_t$, such that $x_j \equiv a(\ell, j) \bmod M_\ell$ for $1 \le \ell \le t$. Now, for $1 \le \ell \le p^{k_1}$, $x_i - x_j \equiv a(\ell, i) - a(\ell, j) \bmod M_\ell$ so $||x_i - x_j||_{p_\ell} = ||a(\ell, i) - a(\ell, j)||_{p_\ell} \in \mathcal{D}(p_\ell)$. Thus x_i is joined to x_j. It follows that the $p_1^{k_1}$ numbers x_i form a clique in G. Thus $\chi(G) \ge p_1^{k_1}$. □

Corollary 7 *Let $p_1, p_2, \ldots, p_t$ be a collection of distinct prime numbers. Corresponding to each prime p_i, $1 \le i \le t$, let D_i be a finite set of non-negative powers of p_i of size $k_i := |D_i|$. Let*

$$D := \{a x_1 x_2 \ldots x_t : x_i \in D_i \text{ and } a \in \mathbb{Z} \text{ with } p_i \nmid a \text{ for } 1 \le i \le t\}.$$

Then $\chi(G(\mathbb{Z}, D)) = \min\{p_i^{k_i} : 1 \le i \le t\}$.

Proof. By the product formula (4), the graph $G(\mathbb{Z}, D)$ is isomorphic to the distance graph $G(\mathbb{Z}, \mathcal{D}(p_1) \sqcap \mathcal{D}(p_2) \sqcap \ldots \sqcap \mathcal{D}(p_t))$ of Theorem 6. □

Theorem 8 *Let $p_{i,j}$ $1 \le i \le t$, $1 \le j \le k_i$ be a collection of distinct prime numbers, and let $\mathcal{D}(p_{i,j})$ be a finite set of $p_{i,j}$-adic distances. If $\mathcal{D}_i := \sqcap_{1 \le j \le k_i} \mathcal{D}(p_{i,j})$ for each i, and $\mathcal{D} := \bigsqcup_{1 \le i \le t} \mathcal{D}_i$, then*

$$\chi(G(\mathbb{Z}, \mathcal{D})) = \prod_{i=1}^{t} \chi(G(\mathbb{Z}, \mathcal{D}_i)).$$

Proof. Let G denote the graph $G(\mathbb{Z}, \sqcup_{1\le i\le t}\mathcal{D}_i)$. By Theorem 6, for each $i = 1, 2, \ldots, t$, we can choose a coloring $f_i : \mathbb{Z} \to K_i$ of $G(\mathbb{Z}, \mathcal{D}_i)$ with $|K_i| = \chi(G(\mathbb{Z}, \mathcal{D}_i))$. Define $f : \mathbb{Z} \to \prod_{i=1}^{t} K_i$ by

$$f(x) = (f_1(x), f_2(x), \ldots, f_t(x)).$$

Then one easily checks that f is a coloring of G. Thus $\chi(G) \le \prod_{i=1}^{t} \chi(G(\mathbb{Z}, \mathcal{D}_i))$. To prove the reverse inequality, we use induction on t. More specifically, we will show by induction on t that the graph G contains a clique of size $\prod_{i=1}^{t} \chi(G(\mathbb{Z}, \mathcal{D}_i))$. For $t = 1$, this result follows from the proof of Theorem 6. Assume this result is true for $t \ge 1$, and let $a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ ($\ell = \prod_{i=1}^{t} \chi(G(\mathbb{Z}, \mathcal{D}_i))$ be vertices that form a clique in G. Let $b_1, b_2, \ldots, b_m$ ($m = \chi(G(\mathbb{Z}, \mathcal{D}_{t+1}))$) be vertices that form a clique in $G(\mathbb{Z}, \mathcal{D}_{t+1})$. Let $C_1 := \prod_{i=1}^{t} \prod_{j=1}^{k_i} p_{i,j}$ be the product of all prime numbers corresponding to the distance sets in $\mathcal{D}_1$, $\mathcal{D}_2$, ..., $\mathcal{D}_t$; and let $C_2 := \prod_{j=1}^{k_i} p_{t+1,j}$ be the product of all prime numbers corresponding to the distance sets in $\mathcal{D}_{t+1}$. Then for sufficiently large M the numbers

$$C_2^M a_i + C_1^M b_j \ for\ 1 \le \ell\ and\ 1 \le j \le m$$

form a clique of the desired size. □

Example 9 Suppose that $\mathcal{D}(2) = \{1/2\}$, $\mathcal{D}(3) = \{1\}$ and $\mathcal{D}(5) = \{1/5, 1/25\}$. Consider the graph $G = G(\mathbb{Z}, (\mathcal{D}(2) \sqcap \mathcal{D}(3)) \sqcup \mathcal{D}(5))$. Then by Theorem 8 we have that $\chi(G) = \min(2, 3) \cdot 25 = 50$. Let D be the set of all numbers of the form $2a$ where a is relatively prime to 6 together with the set of numbers of the form $5b$ and $25c$ where b and c are not divisible by 5. Then $G(\mathbb{Z}, D)$ is isomorphic to the graph G.

Theorem 10 *Let $p_{i,j}$ $1 \le i \le t$, $1 \le j \le k_i$ be a collection of distinct prime numbers, and let $\mathcal{D}(p_{i,j})$ be a finite set of $p_{i,j}$-adic distances. If $\mathcal{D}_i := \bigsqcup_{1\le j\le k_i} \mathcal{D}(p_{i,j})$ for each i, and $\mathcal{D} := \sqcap_{1\le i\le t} \mathcal{D}_i$, then*

$$\chi(G(\mathbb{Z}, \mathcal{D})) = \min\{\chi(G(\mathbb{Z}, \mathcal{D}_i)) : 1 \le i \le t\}.$$

Proof. Put $\rho := \min\{(\chi(G(\mathbb{Z}, \mathcal{D}_i)) : 1 \le i \le t\}$. Let G denote the graph $G(\mathbb{Z}, \mathcal{D}_1 \sqcap \mathcal{D}_2 \sqcap \ldots \sqcap \mathcal{D}_t)$. Clearly G is a subgraph of $G(\mathbb{Z}, \mathcal{D}_i)$ for any i. This implies that $\chi(G) \le \rho$.

To prove the reverse inequality, as in the proof of Theorem 8, we show by induction on t that the graph G contains a clique of size ρ. Put $\kappa_i := \chi(G(\mathbb{Z}, \mathcal{D}_i))$

for $i = 1, 2, \ldots, t+1$. After rearranging the $\mathcal{D}_i$'s we can assume that $\kappa_1 \le \kappa_2 \le \ldots \le \kappa_{t+1}$.

Suppose that $a_1, a_2, \ldots, a_{\kappa_1} \in \mathbb{Z}$ form a clique in G and that $b_1, b_2, \ldots, b_{\kappa_{t+1}}$ form a clique in $G(\mathbb{Z}, \mathcal{D}_{t+1})$. Put $C_1 := \prod_{i=1}^{t} \prod_{j=1}^{k_i} p_{i,j}$ and $C_2 := \prod_{j=1}^{k_i} p_{t+1,j}$. Then for sufficiently large M, the numbers

$$C_2^M a_i + C_1^M b_i \ for\ 1 \le i \le \kappa_1$$

form a clique of the desired size. $\square$

Example 11 Suppose that $\mathcal{D}(2) = \{1/2\}$, $\mathcal{D}(3) = \{1\}$ and $\mathcal{D}(5) = \{1/5,\ 1/25\}$. Consider the graph $G = G(\mathbb{Z}, (\mathcal{D}(2) \sqcup \mathcal{D}(3)) \sqcap \mathcal{D}(5))$. Then by Theorem 10, we have that

$$\chi(G) = \min(\chi(G(\mathbb{Z}, \mathcal{D}(2) \sqcup \mathcal{D}(3))), \chi(G(\mathbb{Z}, \mathcal{D}(5)))).$$

By Theorem 4 we have that $\chi(G(\mathbb{Z}, \mathcal{D}(2) \sqcup \mathcal{D}(3))) = 6$ and by Theorem 2 we have that $\chi(G(\mathbb{Z}, \mathcal{D}(5))) = 25$. Thus $\chi(G) = 6$.

5. Distance Graphs With p-Adic Distance Sets in Infinite Unions

As discussed in Section 3, an Euclidean distance set can be transformed into a p-adic distance set in infinite unions $\bigsqcup_{i=1}^{\infty} \mathcal{D}_i$ where $\mathcal{D}_i$ is the intersection of $\{1/p^e\}$, p primes, $e \in \mathbb{N}_0$. In the light of this, we would like to know the chromatic number of the p-adic distance graph $G(\mathbb{Z}, \mathcal{D})$ with distance set $\mathcal{D} = \bigsqcup_{i=1}^{\infty} \mathcal{D}_i$ where $\mathcal{D}_i$ has the form $\{1/p_1^{k_1}\} \sqcap \{1/p_2^{k_2}\} \sqcap \ldots \sqcap \{1/p_t^{k_t}\}$ for $t \in \mathbb{N}_0$ and distinct primes $p_1, \ldots, p_t$.

In this section, we give a characterization of p-adic distance graphs with finite chromatic number when all the $\mathcal{D}_i$ are described in terms of fixed primes $p_1, \ldots, p_t$.

Lemma 12 *Let $p_1 < p_2 < \ldots < p_t$ be t prime numbers, and let $\ell_1, \ell_2, \ldots, \ell_t$ be fixed non-negative integers. Let $\Lambda := \Lambda(\ell_1, \ell_2, \ldots, \ell_t)$ be the collection of all t-tuples $(k_1, k_2, \ldots, k_t)$ such that $k_i \le \ell_i$ for some $i = 1, 2, \ldots, t$. Put*

$$\mathcal{D} := \bigsqcup_{(k_1, k_2, \ldots, k_t) \in \Lambda} \left(\{1/p_1^{k_1}\} \sqcap \{1/p_2^{k_2}\} \sqcap \ldots \sqcap \{1/p_t^{k_t}\} \right).$$

Then the distance graph $G(\mathbb{Z}, \mathcal{D})$ has finite chromatic number.

Proof. First, we consider a special type Λ' of Λ, where $\Lambda' := \Lambda'(\ell_1, \ell_2, \ldots, \ell_t)$ is any collection of t-tuples $(k_1, k_2, \ldots, k_t)$ such that $k_i \geq \ell_i$ for each $i = 1, 2, \ldots, t$, with equality for at least one i.

Claim: $\chi(G(\mathbb{Z}, \mathcal{D})) \leq p_1 p_2 \ldots p_\ell$ when $\Lambda = \Lambda'$.

Proof of Claim: Let $x \in \mathbb{Z}$ and let $x := \ldots a_2^{(i)} a_1^{(i)} a_0^{(i)}$ be the p_i-adic representation. Then we will show that

$$x \to (a_{\ell_1}^{(1)}, a_{\ell_2}^{(2)}, \ldots, a_{\ell_t}^{(t)})$$

is a proper coloring of the graph $G(\mathbb{Z}, \mathcal{D})$. If two integers x, y are adjacent and their p_i-adic representations are $x := \ldots a_2^{(i)} a_1^{(i)} a_0^{(i)}$ and $y := \ldots b_2^{(i)} b_1^{(i)} b_0^{(i)}$ for each i, there exists $(k_1, k_2, \ldots, k_t) \in \Lambda'$ such that $\|a_{k_i}^{(i)} - b_{k_i}^{(i)}\|_{p_i} = 1/p_i^{k_i}$ for all $1 \leq i \leq t$. At those i with $k_i = \ell_i$, which is guaranteed to occur in Λ', we have $a_{\ell_i}^{(i)} \neq b_{\ell_i}^{(i)}$. The Claim follows from that $0 \leq a_{\ell_i}^{(i)} \leq p_i - 1$ for each $1 \leq i \leq t$.

Now, result of Theorem follows from the two observations: (1) Λ is a finite union of translates of sets of the form of Λ', and (2) if $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_r$ are p-adic distance sets, then $\chi(G(\mathbb{Z}, \sqcup_{1\leq i\leq r}\mathcal{D}_i)) \leq \prod_{1\leq i\leq r} \chi(G(\mathbb{Z}, \mathcal{D}_i))$. □

Lemma 13 *Let $p_1 < p_2 < \ldots < p_t$ be t prime numbers. Suppose that $\mathcal{D}_1, \mathcal{D}_2, \ldots, D_r$ are distance sets of the form*

$$\{1/p_1^{k_1}\} \sqcap \{1/p_2^{k_2}\} \sqcap \ldots \sqcap \{1/p_t^{k_t}\} \tag{5}$$

with the property that: whenever $i < j$ and $1/p_\alpha^k$ appears in $\mathcal{D}_i$ for some α, $1 \leq \alpha \leq t$, then $1/p_\alpha^{k'}$ appears in $\mathcal{D}_j$ for some $k' > k$. Then the graph $G(\mathbb{Z}, \bigsqcup_{i=1}^r \mathcal{D}_i)$ has a clique of size at least 2^r.

Proof. For a distance set $\mathcal{D}_i$ of the form in (5), we define $P(\mathcal{D}_i) := p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t}$ to be the product of the relevant prime powers involved in the definition of $\mathcal{D}$. Let K be the set of linear combinations for the form $\sum_{i=1}^r a_i P(\mathcal{D}_i)$, where the a_i's take on the values 0 or 1. Then since the subgraph induced by K is the complete graph on 2^r vertices, the desired result follows. □

Theorem 14 *Let $p_1 < p_2 < \ldots < p_t$ be prime numbers and let Λ be a nonempty subset of $\mathbb{N}_0^t$. Put*

$$\mathcal{D} := \bigsqcup_{(k_1, k_2, \ldots, k_t) \in \Lambda} \left(\{1/p_1^{k_1}\} \sqcap \{1/p_2^{k_2}\} \sqcap \ldots \sqcap \{1/p_t^{k_t}\} \right).$$

Then the following are equivalent:

(i) *The chromatic number of the distance graph $G(\mathbb{Z}, \mathcal{D})$ is finite.*

(ii) *There exists $(\ell_1, \ell_2, \ldots, \ell_t) \in \mathbb{N}_0^t$ such that for all $(k_1, k_2, \ldots, k_t) \in \Lambda$ we have that $k_i \leq \ell_i$ for some $1 \leq i \leq t$.*

Proof. The fact that (14) implies (14) follows from Lemma 12

For non-negative integers $k_1, k_2, \ldots, k_t$, let $\Lambda(k_1, k_2, \ldots, k_t)$ be the collection of all t-tuples $(m_1, m_2, \ldots, m_t)$ such that $m_i \leq k_i$ for some $i = 1, 2, \ldots, t$. Assume that $G(\mathbb{Z}, \mathcal{D})$ has finite chromatic number but that (14) is false. Let $u_1 := (k_1, k_2, \ldots, k_t) \in \Lambda$.

Since we are assuming that (14) is false, it follows that $\Lambda \not\subseteq \Lambda(k_1, k_2, \ldots, k_t)$. Thus we can find and element $u_2 \in \Lambda \setminus \Lambda(k_1, k_2, \ldots, k_t)$. Then u_2 has the form $u_2 := (k_1', k_2', \ldots, k_t')$ of Λ where $k_1' > k_1$, $k_2' > k_2$, ..., $k_t' > k_t$. Continuing in this way, we can find a sequence $u_1, u_2, \ldots \in \Lambda$ such that their respective coordinates form strictly increasing sequences. Now put

$$\mathcal{D}' := \bigsqcup_{(k_1,k_2,\ldots,k_t)\in\{u_i \mid i=1,2,\ldots\}} \left(\{1/p_1^{k_1}\} \sqcap \{1/p_2^{k_2}\} \sqcap \ldots \sqcap \{1/p_t^{k_t}\}\right).$$

Then $G(\mathbb{Z}, \mathcal{D}')$ is a subgraph of $G(\mathbb{Z}, \mathcal{D})$ with infinite chromatic number by Lemma 13. This implies that the chromatic number of $G(\mathbb{Z}, \mathcal{D})$ is infinite, a contradiction. $\square$

Corollary 15 *Suppose that $p_1 < p_2 < \ldots < p_t$ are prime numbers, and $\Lambda \subseteq \mathbb{N}_0^t$. Let D be the distance set of all positive numbers of the form $ap_1^{k_1}p_2^{k_2}\ldots p_t^{k_t}$ with $(k_1, k_2, \ldots, k_t) \in \Lambda$ and $a \in \mathbb{Z}$ such that $\gcd(a, p_1p_2\ldots p_t) = 1$. Then the chromatic number of $G(\mathbb{Z}, D)$ is finite iff there exist nonnegative integers $\ell_1, \ell_2, \ldots, \ell_t$ such that no number in D is divisible by $p_1^{\ell_1}p_2^{\ell_2}\ldots p_t^{\ell_t}$.*

6. Applications: p-Adic Method for Euclidean Graphs

We show simple applications of p-adic results to bound the chromatic numbers on Euclidean distance graphs. First, we give a general bound on the chromatic number of a distance graph $G(\mathbb{Z}, D)$.

Theorem 16 *Let $D := \{d_1 < d_2 < \ldots\}$ be a (possibly finite) set of positive integers. For each prime number p, let*

$$\mathcal{D}(p) := \{||d_i||_p : i = 1, 2, \ldots\}.$$

Then

$$\chi(G(\mathbb{Z}, D)) \leq \min\{p^{|\mathcal{D}(p)|} : p \text{ is prime}\}.$$

Proof. The the graph $G(\mathbb{Z}, D)$ is a subgraph of the graph $G(\mathbb{Z}, \mathcal{D}(p))$ which, from Theorem 4, has chromatic number $p^{|\mathcal{D}(p)|}$. $\square$

It follows from this theorem that if the chromatic number of $G(\mathbb{Z}, D)$ is infinite, then arbitrarily high powers of every prime number appear among the divisors of the numbers in the set D.

The following example compares two different p-adic results in applications.

Example 17 Let P be the set of primes and let $c > 1$ be an integer. We obtain upper bounds on the chromatic number of the distance set of the translate $P + c$. Suppose that p is the smallest integer that divides c. Then the elements of $P + c$ are of the form either $p + c$ or $q + c$, $q \in P \setminus \{p\}$.

To apply Theorem 16, note that $p | \, p + c$ and $p \nmid q + c$. The p-adic distance set $\mathcal{D}$ corresponding to D satisfies the inclusion $\mathcal{D} \subset \mathcal{D}(p) = \{1/p^0, 1/p^e\}$ where $||p + c||_p = 1/p^e$ for some $e \geq 1$. Hence $\chi(G(\mathbb{Z}, P + c)) \leq p^2$.

We can obtain a better upper bound by applying Theorem 4. Observe that $2 | \, p+c$ (say $||p + c||_2 = 1/2^\ell$) and $p \nmid q + c$. Putting $\mathcal{D}(2) = \{1/2^\ell\}$ and $\mathcal{D}(p) = \{1/p^0\}$, we have $\mathcal{D} \subset \mathcal{D}(2) \sqcup \mathcal{D}(p)$ hence $\chi(G(\mathbb{Z}, P + c)) \leq 2p$.

7. Future Research

Characterization of Distance Sets. It is easy to see that if the chromatic number of $G(\mathbb{Z}, D)$ is infinite, then multiples of every positive integer appear in the the set D. If there is no multiple of m, then the m-coloring assigning colors to integers modulo m would be a proper coloring of $\mathbb{Z}$. On the other hand, it is interesting to see that Theorem 14, obtained from a generalized form of p-adic distance sets, induces the same necessity condition as follows. Let $D = \{d_1, d_2, \ldots\}$ be a distance set and suppose that the graph $G(\mathbb{Z}, D)$ has infinite chromatic number. Let m be a positive integer and suppose that m has prime factorization $p_1^{\ell_1} p_2^{\ell_2} \ldots p_t^{\ell_t}$. Let Λ be the set of all t-tuples $(k_1, k_2, \ldots, k_t) \in \mathbb{N}_0^t$ such that some d_i has factorization of the form $a p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t}$ where $\gcd(a, m) = 1$. Form the distance set

$$\mathcal{D} := \bigsqcup_{(k_1, k_2, \ldots, k_t) \in \Lambda} \left(\{1/p_1^{k_1}\} \sqcap \{1/p_2^{k_2}\} \sqcap \ldots \sqcap \{1/p_t^{k_t}\} \right).$$

The distance graph $G_1 := G(\mathbb{Z}, D)$ is a subgraph of $G_2 := G(\mathbb{Z}, \mathcal{D})$. Since G_1 has infinite chromatic number, the same must be true for G_2. Theorem 14 implies that D contains multiples of m.

Based on our methods, we conjecture on the characterization of Euclidean distance sets having infinite chromatic number.

Conjecture 18 Suppose that D is a given distance set. The chromatic number of Euclidean distance graph $G(\mathbb{Z}, D)$ is infinite iff for every finite partition of

$D = \cup_{1 \le j \le k} D_j$, there exists j, $1 \le j \le k$, such that some multiples of every integer appear in the set $D_j = \{d_1 < d_2 < \ldots\}$ and $\inf_{d_i \in D_j} d_{i+1}/d_i = 1$.

In the above conjecture, considering finite partitions is important as can be shown as follows. If $D := \{n!, n!+1 : n \in \mathbb{Z}\}$, then D contains multiples of every integer and $\inf_{d_i \in D} d_{i+1}/d_i = 1$. Partition $D = D_1 \cup D_2$ where $D_1 = \{n! : n \in \mathbb{Z}\}$ and $D_2 = \{n! + 1 : n \in \mathbb{Z}\}$. Then $\inf_{d_i \in D_j} d_{i+1}/d_i > 1$ for $j = 1, 2$, and Theorem 1 of Ruzsa–Tuza–Voigt implies that the chromatic numbers of both graphs $G(\mathbb{Z}, D_j)$ are finite. Their finite union graph $G(\mathbb{Z}, D)$ also has finite chromatic number.

Generalization to Rings of Integers. In this paper we considered p-adic norm distance graphs in addition to the usual Euclidean distance graphs. From a number theoretic point of view, this approach is quite natural because both classes of norms go hand in hand by the product formula (4). In future work the authors will consider distance graphs where $\mathbb{Z}$ is replaced by other rings R with suitable norms. Specifically we will consider univariate polynomial rings and rings of integers in number fields. It would be of interest to consider the distance graph $G(\mathbb{F}_q[x], D)$ where $\mathbb{F}_q$ is a finite field with q elements and D is a (non-archimedian) distance set because for such polynomial rings, *all* norms are non-archemedian. If R is a ring of integers in a number field, then it is possible for R to have several inequivalent archemedian norms. It would interesting to study how the structure of the distance graphs would change when one transitions from $\mathbb{Z}$ to general rings of integers. It is hoped that by studying such generalisations, light would be shed on the standard distance graph problem on $\mathbb{Z}$.

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