



**ON THE NUMBER OF CERTAIN RELATIVELY PRIME SUBSETS
OF $\{1, 2, \dots, n\}$**

László Tóth

Department of Mathematics, University of Pécs, 7624 Pécs, Ifjúság u. 6, Hungary
ltoth@gamma.ttk.pte.hu

Received: 10/21/09, Revised: 4/9/10, Accepted: 4/11/10, Published: 9/8/10

Abstract

We give common generalizations of three formulae involving the number of relatively prime subsets of $\{1, 2, \dots, n\}$ with some additional constraints. We also generalize a fourth formula concerning the Euler-type function Φ_k , and investigate certain related divisor-type, sum-of-divisors-type and gcd-sum-type functions.

1. Introduction

For a nonempty subset A of $\{1, 2, \dots, n\}$ let $\gcd(A)$ denote the gcd of the elements of A . Then A is said to be relatively prime if $\gcd(A) = 1$, i.e., the elements of A are relatively prime. Let $f(n)$ denote the number of relatively prime subsets of $\{1, 2, \dots, n\}$. Then

$$f(n) = \sum_{d=1}^n \mu(d) \left(2^{\lfloor n/d \rfloor} - 1 \right), \quad n \in \mathbb{N}, \quad (1)$$

where μ is the Möbius function and $\mathbb{N} = \{1, 2, \dots\}$. A similar formula is valid for the number $f_k(n)$ of relatively prime k -subsets (subsets with k elements) of $\{1, 2, \dots, n\}$. These functions were investigated by M. B. Nathanson [6].

Let $R_k(n)$ denote the number of k -compositions of n such that the summands are relatively prime, i.e., $R_k(n)$ is the number of ordered k -tuples (a_1, a_2, \dots, a_k) of positive integers such that $a_1 + a_2 + \dots + a_k = n$ and $\gcd(a_1, a_2, \dots, a_k) = 1$. Then

$$R_k(n) = \sum_{d|n} \mu(d) \binom{n/d - 1}{k - 1}, \quad n, k \in \mathbb{N} \quad (2)$$

(see H. W. Gould [4], T. Shonhiwa [10]).

Furthermore, let $p_k(n)$ denote the number of partitions of n into k summands and consider the number $\bar{p}_k(n)$ of partitions of n into k parts such that the parts are relatively prime. Then

$$\bar{p}_k(n) = \sum_{d|n} \mu(d)p_k(n/d), \quad n, k \in \mathbb{N} \tag{3}$$

(see T. Shonhiwa [9]).

Also, consider the Euler-type functions $\Phi(n)$ and $\Phi_k(n)$, representing the number of nonempty subsets A of $\{1, 2, \dots, n\}$ and k -subsets A of $\{1, 2, \dots, n\}$, respectively, such that $\gcd(A)$ and n are coprime. Note that $\Phi_1(n) = \varphi(n)$ is Euler’s function. One has

$$\Phi_k(n) = \sum_{d|n} \mu(d) \binom{n/d}{k}, \quad n, k \in \mathbb{N}. \tag{4}$$

The functions Φ and Φ_k were defined and studied by M. B. Nathanson [6]. For further properties and generalizations see also M. Ayad, O. Kihel [1, 2] and their references.

It is the aim of the present paper to give common generalizations of formulae (1), (2), (3), (4). We also consider certain related divisor-type, sum-of-divisors-type and gcd-sum-type functions.

The definitions of our general arithmetical functions are given in Section 2. Assuming a natural condition we give arithmetical identities and formulas for certain formal series in Sections 3 and 4, respectively. In Section 5 we show that if two additional conditions are fulfilled, then we have asymptotic formulae for the values of these general functions. Finally, special cases and references to known results are given in Section 6.

We remark that the asymptotic formulae for the values $\Phi_k(n)$, given by M. B. Nathanson [6, Th. 4] are ambiguous. See Section 6, Case C of the present paper.

2. Arithmetical Functions

For $k, n \in \mathbb{N}$ let $\{1, \dots, n\}^k = \{(a_1, \dots, a_k) : a_1, \dots, a_k \in \mathbb{N}, 1 \leq a_1, \dots, a_k \leq n\}$ be the set of all ordered k -tuples of positive integers $\leq n$. Let $\mathcal{S} = (S(n))_{n \in \mathbb{N}}$ be a system of nonempty subsets $S(n)$ of the set $\bigcup_{\ell=1}^n \{1, \dots, n\}^\ell$. If $\underline{a} = (a_1, \dots, a_k)$ is a k -tuple in $S(n)$ (where $1 \leq k \leq n$) we denote by $\gcd(\underline{a})$ the gcd of the numbers a_1, \dots, a_k . Also, let $\gcd(\underline{a}, n)$ denote the gcd of $\gcd(\underline{a})$ and n (the gcd of a_1, \dots, a_k, n).

Consider the following functions:

(i) the function counting the number of all relatively prime k -tuples of $S(n)$, i.e.,

$$f_S(n) = \#\{\underline{a} \in S(n) : \gcd(\underline{a}) = 1\}, \tag{5}$$

(ii) the divisor-type function

$$\tau_S(n) = \#\{\underline{a} \in S(n) : \gcd(\underline{a}) \mid n\}, \tag{6}$$

(iii) the sum-of-divisors-type function

$$\sigma_S(n) = \sum_{\substack{\underline{a} \in S(n) \\ \gcd(\underline{a}) \mid n}} \gcd(\underline{a}), \tag{7}$$

(iv) the Euler-type function attached to S , given by

$$\phi_S(n) = \#\{\underline{a} \in S(n) : \gcd(\underline{a}, n) = 1\}, \tag{8}$$

(v) the gcd-sum-type function

$$P_S(n) = \sum_{\underline{a} \in S(n)} \gcd(\underline{a}, n). \tag{9}$$

3. Arithmetical Identities

If $\underline{a} = (a_1, \dots, a_k)$ is a k -tuple and $j \in \mathbb{N}$ let $j\underline{a}$ denote the k -tuple $j\underline{a} = (ja_1, \dots, ja_k)$. Assume in what follows that for every $n \in \mathbb{N}$, $S(n)$ is an arbitrary nonempty subset of $\bigcup_{\ell=1}^n \{1, \dots, n\}^\ell$ such that the following condition is valid:

(C₁) For any $j \in \mathbb{N}$, $1 \leq j \leq n$, $j\underline{a} \in S(n)$ holds if and only if $\underline{a} \in S(\lfloor n/j \rfloor)$.

Theorem 1. *Assuming condition (C₁), we have for any $n \in \mathbb{N}$,*

$$\sum_{j=1}^n f_S(\lfloor n/j \rfloor) = \#S(n), \tag{10}$$

$$f_S(n) = \sum_{j=1}^n \mu(j) \#S(\lfloor n/j \rfloor). \tag{11}$$

Proof. Grouping the k -tuples $\underline{a} = (a_1, \dots, a_k) \in S(n)$ according to the values $\gcd(\underline{a}) = j$, where $1 \leq j \leq n$, $a_1 = jb_1, \dots, a_k = jb_k$, $\gcd(\underline{b}) = \gcd(b_1, \dots, b_k) = 1$, and using the given condition (C₁),

$$\#S(n) = \sum_{j=1}^n \sum_{\substack{\underline{a} \in S(n) \\ \gcd(\underline{a})=j}} 1 = \sum_{j=1}^n \sum_{\substack{j\underline{b} \in S(n) \\ \gcd(\underline{b})=1}} 1 = \sum_{j=1}^n \sum_{\substack{\underline{b} \in S(\lfloor n/j \rfloor) \\ \gcd(\underline{b})=1}} 1 = \sum_{j=1}^n f_S(\lfloor n/j \rfloor),$$

which proves (10). Now (11) follows at once by Möbius inversion. Alternatively, for a direct proof of (11) we use the following property of the Möbius function: $\sum_{d|n} \mu(d) = \delta_{n,1}$ (Kronecker-delta) and obtain

$$\begin{aligned} f_S(n) &= \sum_{\underline{a} \in S(n)} \sum_{j|\gcd(\underline{a})} \mu(j) = \sum_{j=1}^n \mu(j) \sum_{\underline{b} \in S(n)} 1 \\ &= \sum_{j=1}^n \mu(j) \sum_{\underline{b} \in S(\lfloor n/j \rfloor)} 1 = \sum_{j=1}^n \mu(j) \#S(\lfloor n/j \rfloor). \end{aligned}$$

□

Theorem 2. *Let h be an arbitrary arithmetic function. Then, assuming condition (C_1) , we have for any $n \in \mathbb{N}$,*

$$\sum_{\substack{\underline{a} \in S(n) \\ \gcd(\underline{a})|n}} h(\gcd(\underline{a})) = \sum_{d|n} h(d) f_S(n/d), \tag{12}$$

$$\tau_S(n) = \sum_{d|n} f_S(d), \quad \sigma_S(n) = \sum_{d|n} d f_S(n/d). \tag{13}$$

Proof. If $\underline{a} \in S(n)$, $\gcd(\underline{a}) = d \mid n$, then $\underline{a} = d\underline{b}$, where $\underline{b} \in S(n/d)$ by condition (C_1) and $\gcd(\underline{b}) = 1$. Hence

$$\sum_{\substack{\underline{a} \in S(n) \\ \gcd(\underline{a})|n}} h(\gcd(\underline{a})) = \sum_{d|n} h(d) \sum_{\substack{\underline{b} \in S(n/d) \\ \gcd(\underline{b})=1}} 1 = \sum_{d|n} h(d) f_S(n/d),$$

which proves (12). For $h(n) = 1$ and $h(n) = n$, respectively, we have (13). □

Theorem 3. *Assuming (C_1) , we have for any $n \in \mathbb{N}$,*

$$\sum_{d|n} \phi_S(d) = \#S(n), \tag{14}$$

$$\phi_S(n) = \sum_{d|n} \mu(d) \#S(n/d). \tag{15}$$

Proof. Similar to the proof of Theorem 1. We give the direct proof of (15):

$$\begin{aligned} \phi_S(n) &= \sum_{\underline{a} \in S(n)} \sum_{d|(\gcd(\underline{a}),n)} \mu(d) = \sum_{d|n} \mu(d) \sum_{\underline{b} \in S(n)} 1 \\ &= \sum_{d|n} \mu(d) \sum_{\underline{b} \in S(n/d)} 1 = \sum_{d|n} \mu(d) \#S(n/d). \end{aligned}$$

□

Theorem 4. *Assuming (C_1) , we have for every $n \in \mathbb{N}$,*

$$P_S(n) = \sum_{d|n} d\phi_S(n/d), \tag{16}$$

$$P_S(n) = \sum_{d|n} \varphi(d)\#S(n/d), \tag{17}$$

(where φ is Euler’s function).

Proof. Grouping the k -tuples $\underline{a} \in S(n)$ according to the values $(\gcd(\underline{a}), n) = d$, where $\underline{a} = j\underline{b}$, $(\gcd(\underline{b}), n/d) = 1$, we obtain (16) using condition (C_1) :

$$P_S(n) = \sum_{d|n} d \sum_{\substack{\underline{b} \in S(n) \\ (\gcd(\underline{b}), n/d)=1}} 1 = \sum_{d|n} d\phi_S(n/d).$$

Now using (16) and (15),

$$\begin{aligned} P_S(n) &= n \sum_{d|n} \frac{1}{d} \phi_S(d) = n \sum_{d|n} \frac{1}{d} \sum_{e|d} \mu(e)\#S(d/e) = n \sum_{ejm=n} \frac{\mu(e)}{em} \#S(m) \\ &= n \sum_{m\ell=n} \frac{1}{m} \#S(m) \sum_{ej=\ell} \frac{\mu(e)}{e} = n \sum_{m\ell=n} \frac{1}{m} \#S(m) \frac{\varphi(\ell)}{\ell} = \sum_{m\ell=n} \#S(m)\varphi(\ell). \end{aligned}$$

□

Note, that if $\#S(n)$ is multiplicative in n , then the functions ϕ_S and P_S are also multiplicative, while f_S is, in general, not multiplicative, cf. Section 6, Case A.

4. Formal Series

Next we consider the formal power series of $(f_S(n))_{n \in \mathbb{N}}$. Let

$$F_S(x) = \sum_{n=1}^{\infty} \#S(n)x^n, \tag{18}$$

be the formal power series of $(\#S(n))_{n \in \mathbb{N}}$.

Theorem 5. *Assuming condition (C_1) , we have for any $n \in \mathbb{N}$,*

$$\sum_{n=1}^{\infty} f_S(n)x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} \mu(n)(1-x^n)F_S(x^n). \tag{19}$$

Proof. Using (11),

$$\sum_{n=1}^{\infty} f_S(n)x^n = \sum_{n=1}^{\infty} \sum_{j=1}^n \mu(j)\#S(\lfloor n/j \rfloor)x^n = \sum_{j=1}^{\infty} \mu(j) \sum_{n=j}^{\infty} \#S(\lfloor n/j \rfloor)x^n,$$

where the inner sum is

$$\begin{aligned} & \sum_{n=j}^{2j-1} \#S(1)x^n + \sum_{n=2j}^{3j-1} \#S(2)x^n + \sum_{n=3j}^{4j-1} \#S(3)x^n + \dots \\ &= \frac{1-x^j}{1-x} (\#S(1)x^j + \#S(2)x^{2j} + \#S(3)x^{3j} + \dots) = \frac{1-x^j}{1-x} F_S(x^j). \end{aligned}$$

□

Regarding the formal Lambert series of $(\phi_S(n))_{n \in \mathbb{N}}$ we have the following formula.

Theorem 6. *Assuming (C_1) , for every $n \in \mathbb{N}$ we have*

$$\sum_{n=1}^{\infty} \phi_S(n) \frac{x^n}{1-x^n} = F_S(x). \tag{20}$$

Proof. Apply (14) and the following well-known result: If A and B are any arithmetical functions such that $\sum_{d|n} A(d) = B(n)$ ($n \in \mathbb{N}$), then $\sum_{n=1}^{\infty} A(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} B(n)x^n$. □

5. Asymptotic Formulae

Assume in this section that, in addition to condition (C_1) , the following also hold:

(C_2) the sequence $(\#S(n))_{n \in \mathbb{N}}$ is increasing;

(C_3) $n\#S(n)/\#S(2n) \rightarrow 0$, as $n \rightarrow \infty$.

We show that under these conditions almost all sets in $S(n)$ are relatively prime. The proof of the next result is along the same lines as that of [6, Th. 2].

Theorem 7. *Assuming conditions (C_1) , (C_2) and (C_3) , for every $n \in \mathbb{N}$, $n \geq 3$, we have*

$$f_S(n) = \#S(n) - \#S(\lfloor n/2 \rfloor) + R(f_S)(n), \tag{21}$$

where

$$-(n-2)\#S(\lfloor n/3 \rfloor) \leq R(f_S)(n) \leq 0. \tag{22}$$

Furthermore, $f_S(n) \sim \#S(n)$, as $n \rightarrow \infty$.

Proof. By (10) we have using condition (C_2) ,

$$\#S(n) = f_S(n) + f_S(\lfloor n/2 \rfloor) + \sum_{j=3}^n f_S(\lfloor n/j \rfloor) \leq f_S(n) + \#S(\lfloor n/2 \rfloor) + (n-2)\#S(\lfloor n/3 \rfloor).$$

The numbers $2a_1, \dots, 2a_k$ are not relatively prime for any k -tuple $\underline{a} = (a_1, \dots, a_k)$, and $2\underline{a} \in S(n)$ holds if and only if $\underline{a} \in S(\lfloor n/2 \rfloor)$ by (C_1) . Hence

$$f_S(n) \leq \#S(n) - \#S(\lfloor n/2 \rfloor).$$

Here $\#S(\lfloor n/2 \rfloor)/\#S(n) \rightarrow 0$ and $n\#S(\lfloor n/3 \rfloor)/\#S(n) \rightarrow 0$, as $n \rightarrow \infty$. This follows from

$$\frac{n\#S(\lfloor n/2 \rfloor)}{\#S(n)} \leq 3 \frac{m\#S(m)}{\#S(2m)} \rightarrow 0, \quad n \rightarrow \infty,$$

where $m = \lfloor n/2 \rfloor$, using conditions (C_2) and (C_3) . Therefore, $f_S(n)/\#S(n) \rightarrow 1$, as $n \rightarrow \infty$. \square

Let $q(n)$ denote the least prime divisor of n . The next two results are inspired by [6, Th. 4].

Theorem 8. *Assuming (C_1) , (C_2) and (C_3) , we have for every $n \in \mathbb{N}$, $n > 1$,*

$$\phi_S(n) = \#S(n) - \#S\left(\frac{n}{q(n)}\right) + R(\phi_S)(n), \tag{23}$$

where

$$|R(\phi_S)(n)| \leq \tau(n)\#S\left(\left\lfloor \frac{n}{q(n)+1} \right\rfloor\right) \ll n^\varepsilon \#S\left(\left\lfloor \frac{n}{q(n)+1} \right\rfloor\right), \tag{24}$$

for every $\varepsilon > 0$, where $\tau(n)$ is the number of divisors of n . Furthermore, $\phi_S(n) \sim \#S(n)$, as $n \rightarrow \infty$.

Proof. By (15) we have

$$\begin{aligned} \phi_S(n) &= \#S(n) + \mu(q(n))\#S(n/q(n)) + \sum_{\substack{d|n \\ d > q(n)}} \mu(d)\#S(n/d) \\ &= \#S(n) - \#S(n/q(n)) + R(\phi_S)(n), \end{aligned}$$

where by condition (C_2) ,

$$|R(\phi_S)(n)| \leq \sum_{\substack{d|n \\ d > q(n)}} \#S(n/d) \leq \tau(n)\#S\left(\left\lfloor \frac{n}{q(n)+1} \right\rfloor\right).$$

Now, $\#S(n/q(n))/\#S(n) \rightarrow 0$ and $\tau(n)\#S(\lfloor n/(q(n)+1) \rfloor)/\#S(n) \rightarrow 0$, as $n \rightarrow \infty$. This is obtained by

$$\frac{n\#S(n/q(n))}{\#S(n)} \leq 3 \frac{m\#S(m)}{\#S(2m)} \rightarrow 0, \quad n \rightarrow \infty,$$

where $m = \lfloor n/2 \rfloor$, by conditions (C_2) and (C_3) . Consequently, $\phi_S(n)/\#S(n) \rightarrow 1$, as $n \rightarrow \infty$. □

Theorem 9. *Assuming (C_1) , (C_2) and (C_3) , we have for every $n \in \mathbb{N}$, $n > 1$,*

$$P_S(n) = \#S(n) + (q(n) - 1)\#S\left(\frac{n}{q(n)}\right) + R(P_S)(n), \tag{25}$$

where

$$0 \leq R(P_S)(n) \leq n\#S\left(\left\lfloor \frac{n}{q(n)+1} \right\rfloor\right), \tag{26}$$

and $P_S(n) \sim \#S(n)$, as $n \rightarrow \infty$.

Proof. Similar to the above, by (17) we have

$$\begin{aligned} P_S(n) &= \#S(n) + \varphi(q(n))\#S(n/q(n)) + \sum_{\substack{d|n \\ d > q(n)}} \varphi(d)\#S(n/d) \\ &= \#S(n) + (q(n) - 1)\#S(n/q(n)) + R(P_S)(n), \end{aligned}$$

where by (C_2) ,

$$0 \leq R(P_S)(n) \leq \#S\left(\left\lfloor \frac{n}{q(n)+1} \right\rfloor\right) \sum_{d|n} \varphi(d) = n\#S\left(\left\lfloor \frac{n}{q(n)+1} \right\rfloor\right).$$

□

6. Special Cases

We now consider special cases for which condition (C_1) is verified and the results of Sections 3 and 4 can be applied. Furthermore, we indicate the cases for which (C_2) and (C_3) also hold and the asymptotic results of Section 5 apply.

Case A. $\mathcal{S} = \mathcal{S}(k\text{-tuples})$

Let $k \in \mathbb{N}$ be fixed and let \mathcal{S} be the set of all ordered k -tuples, i.e., $S(n) = S(k\text{-tuples})(n) := \{1, \dots, n\}^k$ for all $n \in \mathbb{N}$. Then $\#S(k\text{-tuples})(n) = n^k$, which is multiplicative in n . Conditions (C_1) and (C_2) hold true, (C_3) fails. Here

$$f_{S(k\text{-tuples})}(n) = \sum_{j=1}^n \mu(j) \lfloor n/j \rfloor^k, \quad n \in \mathbb{N}, \tag{27}$$

which is well-known (see [8, Th. 2], [12]). Note that for $k = 1$, Equation (27) gives $\sum_{j=1}^n \mu(j) \lfloor n/j \rfloor = 1$ ($n \in \mathbb{N}$). Furthermore, $J_k(n) := \phi_{S(k\text{-tuples})}(n) = \sum_{d|n} \mu(d) (n/d)^k$ is the Jordan function, and $P_k(n) := P_{S(k\text{-tuples})}(n) = \sum_{d|n} \varphi(d) (n/d)^k$ is the generalized Pillai function (see [11, 13]). Note that the functions J_k and P_k are multiplicative, while $f_{S(k\text{-tuples})}$ is not multiplicative.

We point out that

$$F_{S(k\text{-tuples})}(x) = \sum_{n=1}^{\infty} n^k x^n = \frac{1}{(1-x)^{k+1}} \sum_{j=0}^k a(k, j) x^{k-j}, \tag{28}$$

where $a(k, j)$ are the Eulerian numbers, representing the number of permutations of $\{1, \dots, k\}$ having j rises (see, e.g., [3, Section 6.5]). Therefore, by Theorems 5 and 6 we obtain

$$\sum_{n=1}^{\infty} f_{S(k\text{-tuples})}(n) x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{\mu(n)(1-x^n)}{(1-x^n)^{k+1}} \sum_{j=0}^k a(k, j) x^{n(k-j)}, \tag{29}$$

$$\sum_{n=1}^{\infty} J_k(n) \frac{x^n}{1-x^n} = \frac{1}{(1-x)^{k+1}} \sum_{j=0}^k a(k, j) x^{k-j}. \tag{30}$$

Formula (30) is given in [3, p. 199], and for $k = 1$ we reobtain the familiar formula $\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1-x^n} = \frac{x}{(1-x)^2}$.

Case B. $\mathcal{S} = \mathcal{S}(\text{all } k\text{-tuples})$

Let \mathcal{S} be the system of all ordered k -tuples, where $1 \leq k \leq n$; that is, $S(n) = S(\text{all } k\text{-tuples})(n) := \bigcup_{k=1}^n \{1, \dots, n\}^k$ for all $n \in \mathbb{N}$. Then $\#S(\text{all } k\text{-tuples})(n) = n + n^2 + \dots + n^k$. Conditions (C_1) and (C_2) hold true and (C_3) fails.

Case C. $\mathcal{S} = \mathcal{S}(k\text{-sets})$

Let $k \in \mathbb{N}$ be fixed and let \mathcal{S} be the system of all ordered k -tuples $\underline{a} = (a_1, \dots, a_k)$ such that $a_1, \dots, a_k \in \mathbb{N}, 1 \leq a_1 < a_2 < \dots < a_k \leq n$; that is, $S(n)$ is the family of all sets with k elements from $\{1, \dots, n\}$, notation $S(k\text{-sets})(n)$. Then $\#S(k\text{-sets})(n) = \binom{n}{k}$. (C_1) and (C_2) are valid, (C_3) fails.

This is a special case investigated by M. B. Nathanson [6] (see Introduction). Note that Theorems 7, 8, and 9, concerning asymptotic estimates, are not valid now. It is trivial that $\Phi_k(n) \leq \binom{n}{k}$. In [6, Th. 4] it is stated that for n odd, $\Phi_k(n) = \binom{n}{k} + \mathcal{O}(n^{\lfloor n/3 \rfloor})$ (for n even another similar result is also given). But here $n^{\lfloor n/3 \rfloor} \sim n^{k+1}$, while $\binom{n}{k} \sim n^k$, so this does not give any new information on the size of $\Phi_k(n)$.

Using the formula $x(x-1)\cdots(x-n+1) = \sum_{k=1}^n s(n, k)x^k$, where $s(n, k)$ are the Stirling numbers of the first kind, we obtain from (4),

$$\Phi_k(n) = \frac{1}{k!} \sum_{m=1}^k s(k, m)J_m(n), \tag{31}$$

where J_m is the Jordan function (see Case A). It is well-known that $\sum_{n \leq x} J_m(n) = \frac{1}{(m+1)\zeta(m+1)}x^{m+1} + \mathcal{O}(x^m)$ for $m \geq 2$, and we obtain

$$\sum_{n \leq x} \Phi_k(n) = \frac{1}{(k+1)!\zeta(k+1)}x^{k+1} + \mathcal{O}(x^k), \quad k \geq 2, \tag{32}$$

giving the average order of $\Phi_k(n)$.

As new results we also give the following ones:

$$\tau_{S(k\text{-sets})}(n) = \sum_{d|n} f_{S(k\text{-sets})}(d) = \sum_{d|n} \sum_{j=1}^d \mu(j) \binom{\lfloor d/j \rfloor}{k}, \quad n \in \mathbb{N}, \tag{33}$$

where for $k = 1$ this is the usual divisor function $\tau(n) := \sum_{d|n} 1$,

$$P_{S(k\text{-sets})}(n) = \sum_{d|n} d\Phi_k(n/d) = \sum_{d|n} \varphi(d) \binom{n/d}{k}, \quad n \in \mathbb{N}, \tag{34}$$

$$\sum_{n=1}^{\infty} f_{S(k\text{-sets})}(n)x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{\mu(n)x^{kn}}{(1-x^n)^k}, \tag{35}$$

$$\sum_{n=1}^{\infty} \Phi_k(n) \frac{x^n}{1-x^n} = \frac{x^k}{(1-x)^{k+1}}, \tag{36}$$

where (35) and (36) follow by the familiar formula $\sum_{n=1}^{\infty} \binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}$.

Also, by (34) and (32) we obtain

$$\sum_{n \leq x} P_{\mathcal{S}(k\text{-sets})}(n) = \frac{\zeta(k)}{(k+1)! \zeta(k+1)} x^{k+1} + \mathcal{O}(\psi_k(x)), \quad k \geq 2, \tag{37}$$

where $\psi_k(x) = x^k$ for $k \geq 3$ and $\psi_2(x) = x^2 \log x$.

Case D. $\mathcal{S} = \mathcal{S}(\text{sets})$

Let \mathcal{S} be the system of all ordered k -tuples (a_1, \dots, a_k) such that $a_1, \dots, a_k \in \mathbb{N}, 1 \leq a_1 < a_2 < \dots < a_k \leq n$, where $1 \leq k \leq n$; that is, $\mathcal{S}(n)$ is the family of all nonempty subsets of $\{1, \dots, n\}$, notation $\mathcal{S}(\text{sets})(n)$. Then $\#\mathcal{S}(\text{sets})(n) = 2^n - 1$. All of (C_1) , (C_2) and (C_3) hold true. Hence the asymptotic estimates are valid.

This is the another special case studied by Nathanson [6]. We have the following additional results:

$$P_{\mathcal{S}(\text{sets})}(n) = \sum_{d|n} \varphi(d) 2^{n/d} - n, \quad n \in \mathbb{N}, \tag{38}$$

$$\sum_{n=1}^{\infty} f_{\mathcal{S}(\text{sets})}(n) x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{\mu(n) x^n}{1-2x^n}, \tag{39}$$

$$\sum_{n=1}^{\infty} \phi_{\mathcal{S}(\text{sets})}(n) \frac{x^n}{1-x^n} = \frac{x}{(1-x)(1-2x)}, \tag{40}$$

where (39) and (40) are obtained by $\sum_{n=1}^{\infty} (2^n - 1)x^n = \frac{x}{(1-x)(1-2x)}$. Note that formula (39) is given, without proof, in [7, Item A085945]. Note also that $1 + \frac{1}{n} P_{\mathcal{S}(\text{sets})}(n) = \frac{1}{n} \sum_{d|n} \varphi(d) 2^{n/d}$ is exactly the number of circular permutations of two distinct elements taken n at a time (repetitions allowed). As a consequence we obtain that $P_{\mathcal{S}(\text{sets})}(n) \equiv 0 \pmod{n}$ for any $n \in \mathbb{N}$.

Furthermore, $\phi_{\mathcal{S}(\text{sets})}(n) = \sum_{d|n} \mu(d) 2^{n/d}$ ($n \in \mathbb{N}, n > 1$), given in [6, Th. 3]. It follows that $\phi_{\mathcal{S}(\text{sets})}(n) \equiv 0 \pmod{n}$ for any $n \in \mathbb{N}, n > 1$, by a well-known result ($\frac{1}{n} \sum_{d|n} \mu(d) 2^{n/d}$ represents the number of irreducible polynomials of degree n over the field $\mathbb{Z}/2\mathbb{Z}$).

Case E. $\mathcal{S} = \mathcal{S}(k\text{-multisets})$

Let $k \in \mathbb{N}$ be fixed and let \mathcal{S} be the system of all ordered k -tuples $\underline{a} = (a_1, \dots, a_k)$ such that $a_1, \dots, a_k \in \mathbb{N}, 1 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n$; that is, $\mathcal{S}(n)$ is the family

of all multisets with k elements from $\{1, \dots, n\}$, denoted by $S(k\text{-multisets})(n)$. Now $\#S(k\text{-multisets})(n) = \binom{n+k-1}{k}$. (C_1) and (C_2) are valid and (C_3) fails.

This is a special case investigated by Shonhiwa [8]. We have for $n \in \mathbb{N}$,

$$f_{S(k\text{-multisets})}(n) = \sum_{j=1}^n \mu(j) \binom{\lfloor n/j \rfloor + k - 1}{k}, \tag{41}$$

$$\phi_{S(k\text{-multisets})}(n) = \sum_{d|n} \mu(d) \binom{n/d + k - 1}{k}, \tag{42}$$

$$P_{S(k\text{-multisets})}(n) = \sum_{d|n} \varphi(d) \binom{n/d + k - 1}{k}, \tag{43}$$

where (42) is given in [8, p. 70].

Here $F_{S(k\text{-multisets})}(x) = \sum_{n=1}^{\infty} \binom{n+k-1}{k} x^n = \frac{x}{(1-x)^{k+1}}$ and obtain

$$\sum_{n=1}^{\infty} f_{S(k\text{-multisets})}(n) x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{\mu(n)(1-x^n)x^n}{(1-x^n)^{k+1}}. \tag{44}$$

Case F. $\mathcal{S} = \mathcal{S}(\text{all multisets})$

Take all nonempty multisubsets of $\{1, \dots, n\}$. Then we have $\#S(\text{all multisets})(n) = \sum_{k=1}^n \binom{n+k-1}{k} = \binom{2n}{n} - 1$. All of (C_1) , (C_2) , and (C_3) hold true. Hence the asymptotic results are also valid:

$$f_{S(\text{multisets})}(n) \sim \phi_{S(\text{multisets})}(n) \sim P_{S(\text{multisets})}(n) \sim \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}, \quad n \rightarrow \infty. \tag{45}$$

Case G. $\mathcal{S} = \mathcal{S}(k\text{-compositions})$

Let $k \in \mathbb{N}$ be fixed and let $S(n) = S(k\text{-compositions})(n) := \{(a_1, \dots, a_k) : a_1, \dots, a_k \in \mathbb{N}, a_1 + \dots + a_k = n\}$. Then $\#S(k\text{-compositions})(n) = \binom{n-1}{k-1}$. Conditions (C_1) and (C_2) hold, while (C_3) does not hold.

If $\underline{a} \in S(k\text{-compositions})(n)$, then $\gcd(\underline{a}) \mid n$, and hence $\gcd(\underline{a}, n) = \gcd(\underline{a})$ and we obtain that $f_{S(k\text{-compositions})}(n) := R_k(n) = \phi_{S(k\text{-compositions})}(n)$ for any $n \in \mathbb{N}$, where $R_k(n)$ was given in the Introduction. Also, $\tau_{S(k\text{-compositions})}(n) =$

$\binom{n-1}{k-1}$. In this case we have, from Theorems 4 and 6,

$$P_{\mathcal{S}(k\text{-compositions})}(n) = \sum_{d|n} \varphi(d) \binom{n/d-1}{k-1}, \quad n \in \mathbb{N}, \tag{46}$$

$$\sum_{n=1}^{\infty} R_k(n) \frac{x^n}{1-x^n} = \frac{x^k}{(1-x)^k} \tag{47}$$

(for (47), see [4, 10]).

Case H. $\mathcal{S} = \mathcal{S}(\text{all compositions})$

Let $S(n) = S(\text{all compositions})(n) := \bigcup_{1 \leq k \leq n} \{(a_1, \dots, a_k) : a_1, \dots, a_k \in \mathbb{N}, a_1 + \dots + a_k = n\}$. Then $\#S(\text{all compositions})(n) = 2^{n-1}$. All of (C_1) , (C_2) , (C_3) are valid. Hence, among others,

$$f_{S(\text{all compositions})}(n) = \phi_{S(\text{all compositions})}(n) \sim P_{S(\text{all compositions})}(n) \sim 2^{n-1}, \quad n \rightarrow \infty. \tag{48}$$

Case I. $\mathcal{S} = \mathcal{S}(k\text{-partitions})$

Let $k \in \mathbb{N}, k \geq 2$, be fixed and let $S(n) = S(k\text{-partitions}) := \{(a_1, \dots, a_k) : 1 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n, a_1 + \dots + a_k = n\}$, where $\#S(k\text{-partitions}) = p_k(n)$ is the number of k -partitions of n .

Conditions (C_1) and (C_2) are valid. Condition (C_3) does not hold, since $p_k(n) \sim \frac{n^{k-1}}{(k-1)!k!}$, as $n \rightarrow \infty$ (see, e.g., [5, Chapter 4]).

Here $f_{S(k\text{-partitions})}(n) := \bar{p}_k(n) = \phi_{S(k\text{-partitions})}(n)$, like in Case H, where $\bar{p}_k(n)$ is the function studied in [9], see Introduction. Note that $p_3(n) = (n^2 - 1)/12$ for any $n \equiv \pm 1 \pmod{6}$ (see [5, Chapter 4]). We obtain from (3) that for any $n \in \mathbb{N}, n > 1$, such that $6 \nmid n$,

$$\bar{p}_3(n) = \sum_{d|n} \mu(d) \bar{p}_3(n/d) = \frac{1}{12} \sum_{d|n} \mu(d) (n^2/d^2 - 1) = \frac{1}{12} J_2(n), \tag{49}$$

where $J_2(n) = n^2 \prod_{p|n} (1 - 1/p^2)$ is the Jordan function of order 2 (see Case A).

Similarly, from (17) we have for any $n \in \mathbb{N}$, such that $6 \nmid n$,

$$P_{S(3\text{-partitions})}(n) = \frac{1}{12} (P_2(n) - n), \tag{50}$$

where $P_2(n) = \sum_{d|n} \varphi(d) (n/d)^2$ (see Case A).

Case J. $S = S(\text{all partitions})$

Let $S(n) = S(\text{all partitions}) := \bigcup_{k=1}^n \{(a_1, \dots, a_k) : 1 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n, a_1 + \dots + a_k = n\}$. Here $\#S(\text{all partitions}) = p(n)$ is the number of unrestricted partitions of n . Each of (C_1) , (C_2) , and (C_3) hold.

We have

$$\begin{aligned} f_{S(\text{all partitions})}(n) &= \phi_{S(\text{all partitions})}(n) \\ &\sim P_{S(\text{all partitions})}(n) \sim p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}}, \quad n \rightarrow \infty, \end{aligned} \quad (51)$$

where $K = \pi\sqrt{2/3}$, by the result of Hardy and Ramanujan.

Finally, we note that one can consider other special cases too. Let, for example, $S(n) = \bigcup_{k=1}^m \{(a_1, \dots, a_k) : 1 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n, a_1 + \dots + a_k = n\}$, where m is fixed, $1 \leq m \leq n$. Then $\#S(n)$ is the number of partitions of n with at most m summands. Conditions (C_1) and (C_2) hold, while (C_3) does not hold.

References

- [1] M. Ayad, O. Kihel, The number of relatively prime subsets of $\{1, 2, \dots, n\}$, *Integers*, **9** (2009), A14, 4 pp. (electronic).
- [2] M. Ayad, O. Kihel, On relatively prime sets, *Integers*, **9** (2009), A28, 10 pp. (electronic).
- [3] L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions*, Reidel, Dordrecht–Boston, 1974.
- [4] H. W. Gould, Binomial coefficients, the bracket function, and compositions with relatively prime summands, *Fibonacci Quart.*, **2** (1964), 241–260.
- [5] M. Hall, *Combinatorial Theory*, 2nd ed, John Wiley & Sons, 1998.
- [6] M. B. Nathanson, Affine invariants, relatively prime sets, and a phi function for subsets of $\{1, 2, \dots, n\}$, *Integers*, **7** (2007), A1, 7 pp. (electronic).
- [7] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, see <http://www.research.att.com/~njas/sequences>
- [8] T. Shonhiwa, A generalization of the Euler and Jordan totient functions, *Fibonacci Quart.*, **37** (1999), 67–76.
- [9] T. Shonhiwa, On relatively prime decompositions and related results, *Quaest. Math.*, **24** (2001), no. 4, 565–573.
- [10] T. Shonhiwa, Compositions with pairwise relatively prime summands within a restricted setting, *Fibonacci Quart.*, **44** (2006), 316–323.

- [11] L. Tóth, A generalization of Pillai's arithmetical function involving regular convolutions, *Acta Math. Inform. Univ. Ostraviensis*, **6** (1998), 203–217, see [http://www.ttk.pte.hu/matek/ltoth/TothGeneralizationPillai\(1998\).pdf](http://www.ttk.pte.hu/matek/ltoth/TothGeneralizationPillai(1998).pdf)
- [12] L. Tóth, The probability that k positive integers are pairwise relatively prime, *Fibonacci Quart.*, **40** (2002), 13–18.
- [13] L. Tóth, A gcd-sum function over regular integers modulo n , *J. Integer Seq.*, **12** (2009), Article 09.2.5, see <http://www.cs.uwaterloo.ca/journals/JIS/VOL12/Toth/toth3.html>.