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AN EXPLICIT EVALUATION OF THE GOSPER SUM

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Abstract

In this paper, we give a new method to derive a binomial series identity discovered by J.M. Borwein and R. Girgensohn.

– Dedicado a la memoria de Julia Villacorta

1. Introduction

The Gosper sum is defined by J.M. Borwein and R. Girgensohn as

$$b_3(k) = \sum_{n=1}^{\infty} \frac{n^k}{\binom{3n}{n} 2^n}.$$

In a recent paper [3], Borwein and Girgensohn indicated that

$$b_3(-2) = \frac{\pi^2}{24} - \frac{1}{2}\ln^2 2. \tag{1}$$

This identity was later proved by N. Batir [2]. Batir showed using integrals that for $|x| < 27/4$, and integer $n \geq 2$,

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2 \binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\phi(x)-1}\right) - \frac{1}{2}\ln^2\left(\frac{\phi^3(x)+1}{(\phi(x)+1)^3}\right), \tag{2}$$

where

$$\phi(x) = \left[\frac{27 - 2x + 3\sqrt{81-12x}}{2x}\right]^{1/3}.$$

By substituting $x = 1/2$ in the above and the identities $\arctan\left(\frac{\pi}{\sqrt{144}}\right) = \frac{\sqrt{3}}{2\varphi - 1}$ and $\frac{\varphi^3+1}{(\varphi+1)^3} = \frac{1}{2}$, where $\varphi = \phi(1/2) = \sqrt[3]{26+15\sqrt{3}}$, Batir deduced (1).

In the next section, we will present a generalization of (1). Our identity involves computations that seem to be less complicated than that of Batir.

2. Main Theorem

Theorem 1. *For* $-\frac{1}{2} \le t \le 1$*, we have*

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{3n}{n}} \frac{t^{3n}}{(1+t)^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{t}{1+t}\right)^n + 2\sum_{n=1}^{\infty} \frac{t^n}{n^2} \cos\left(n \arctan\left(\frac{\sqrt{(3-t)(1+t)}}{1-t}\right)\right). \tag{3}$$

Proof. We begin with the identity

$$(1-xf)(1-xg)(1-xh) = 1 - x(f+g+h) + x^2(fg+gh+fh) - x^3 fgh, \tag{4}$$

which is valid for all complex numbers x, f, g, h. Suppose f, g and h satisfy the relations

$$f + g + h = fgh, \quad fg + gh + fh = 0.$$

Then

$$f = \frac{h\left(-1 + i\sqrt{3+4h^2}\right)}{2(1+h^2)} \quad \text{and} \quad g = \frac{h\left(-1 - i\sqrt{3+4h^2}\right)}{2(1+h^2)}. \tag{5}$$

Substituting (5) into (4), we find that

$$(1-xh)\left(1 - x\left(\frac{h\left(-1+i\sqrt{3+4h^2}\right)}{2(1+h^2)}\right)\right)\left(1 - x\left(\frac{h\left(-1-i\sqrt{3+4h^2}\right)}{2(1+h^2)}\right)\right) = 1 - \frac{h^3}{1+h^2}x(1+x^2). \tag{6}$$

We next replace x in (6) by $-x$ and deduce that

$$(1+xh)\left(1 + x\left(\frac{h\left(-1+i\sqrt{3+4h^2}\right)}{2(1+h^2)}\right)\right)\left(1 + x\left(\frac{h\left(-1-i\sqrt{3+4h^2}\right)}{2(1+h^2)}\right)\right) = 1 + \frac{h^3}{1+h^2}x(1+x^2). \tag{7}$$

Multiplying (6) and (7), we obtain the identity

$$(1-x^2h^2)\left(1-x^2\left(\frac{h\left(-1+i\sqrt{3+4h^2}\right)}{2(1+h^2)}\right)^2\right)\left(1-x^2\left(\frac{h\left(-1-i\sqrt{3+4h^2}\right)}{2(1+h^2)}\right)^2\right)$$
$$= 1-\frac{h^6}{(1+h^2)^2}x^2(1+x^2)^2. \tag{8}$$

Next, we replace h^2 by h and x^2 by $-x$ in (8) and rewrite (8) as

$$(1+xh)\left(1+\frac{xh}{(1+h)}\left(\frac{-1+i\sqrt{3+4h}}{2\sqrt{1+h}}\right)^2\right)\left(1+\frac{xh}{(1+h)}\left(\frac{-1-i\sqrt{3+4h}}{2\sqrt{1+h}}\right)^2\right)$$
$$= 1+\frac{h^3}{(1+h)^2}x(1-x)^2. \tag{9}$$

Writing

$$\left(\frac{-1+i\sqrt{3+4h}}{2\sqrt{1+h}}\right)^2 = e^{i\arctan\left\{\sqrt{3+4h}/(1+2h)\right\}},$$

we deduce that

$$(1+xh)\left(1+\frac{xh}{1+h}e^{i\arctan\left(\sqrt{4h+3}/(1+2h)\right)}\right)\left(1+\frac{xh}{1+h}e^{-i\arctan\left(\sqrt{4h+3}/(1+2h)\right)}\right)$$
$$= 1+\frac{h^3}{(1+h)^2}x(1-x)^2. \tag{10}$$

By taking the logarithm of both sides of (10), dividing by x, and then integrating over $0 \le x \le 1$, we have

$$\begin{aligned}\int_0^1 \frac{\ln(1+xh)}{x}dx &+ \int_0^1 \frac{1}{x}\ln\left(1+\frac{xh}{1+h}e^{i\arctan\left(\sqrt{4h+3}/(1+2h)\right)}\right)dx \\ &+ \int_0^1 \frac{1}{x}\ln\left(1+\frac{xh}{1+h}e^{-i\arctan\left(\sqrt{4h+3}/(1+2h)\right)}\right)dx \\ &= \int_0^1 \frac{1}{x}\ln\left(1+\frac{h^3}{(1+h)^2}x(1-x)^2\right)dx \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{h^{3n}}{(1+h)^{2n}}\int_0^1 x^{n-1}(1-x)^{2n}dx. \end{aligned} \tag{11}$$

In the left side of the last identity, the integration is valid for $-\frac{1}{2} \le h \le 1$; in the right side the interval is $\frac{3}{2}[(\sqrt{2}-1)^{\frac{1}{3}} - (\sqrt{2}+1)^{\frac{1}{3}}] \le h \le 3$.

Using the result

$$\int_0^1 x^{n-1}(1-x)^{2n}dx = \frac{(n-1)!(2n)!}{(3n)!} = \frac{1}{n\binom{3n}{n}},$$

and observing that

$$\begin{aligned}\int_0^1 \frac{1}{x}\ln\left(1+\frac{xh}{1+h}e^{i\arctan\left(\frac{\sqrt{4h+3}}{1+2h}\right)}\right)dx \\ = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left(\frac{h}{1+h}\right)^n \\ \times\left(\cos\left(n\arctan\left(\frac{\sqrt{4h+3}}{1+2h}\right)\right)+i\sin\left(n\arctan\left(\frac{\sqrt{4h+3}}{1+2h}\right)\right)\right),\end{aligned}$$

we deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}h^n + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left(\frac{h}{1+h}\right)^n\cos\left(n\arctan\left(\frac{\sqrt{4h+3}}{1+2h}\right)\right) \\ = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\binom{3n}{n}}\frac{h^{3n}}{(1+h)^{2n}}.\end{aligned}$$

If we set $h/(1+h) = -t$ in (11), then (3) follow. The left side of identity (3) can be write as:

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{3n}{n}}\frac{t^{3n}}{(1+t)^n} = \frac{t^3}{3(1+t)}\cdot{}_4F_3\left(1,1,1,\frac{3}{2};\frac{4}{3},\frac{5}{3},2;\frac{4t^3}{27(1+t)}\right)$$

where ${}_4F_3(a,b,c,d;e,f,g;z)$ is a hypergeometric function (see [1, Chapter 5] for the definition and properties). That implies the inequality $|\frac{t^3}{1+t}| < \frac{27}{4}$ or the more explicit result

$$\frac{3}{2}[(\sqrt{2}-1)^{\frac{1}{3}} - (\sqrt{2}+1)^{\frac{1}{3}}] < t < 3,$$

and in the case of the function $\sum_{n=1}^{\infty} \frac{x^n \cos nu}{n^2}$, where u is a constant or a variable, the radius of convergence is at least $|x| \leq 1$. □

Corollary 2 *Let $t = 1$. Then the series in* (3) *converges and we have*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{2}\right)^n + 2\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(n\frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{3n}{n}} \frac{1}{2^n}. \tag{12}$$

Using Euler's identity [4, pp. 39-41]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{2}\right)^n = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2 ,$$

and the identity

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(n\frac{\pi}{2}\right) = -\frac{\pi^2}{24},$$

we complete the proof of (1).

We can derive, using (3) and Batir's evaluation, the next identity:

Corollary 3 *We have*

$$6 \arctan^2\left(\frac{\sqrt{3}}{2\phi(u) - 1}\right) - \frac{1}{2}\ln^2\left(\frac{\phi^3(u) + 1}{(\phi(u) + 1)^3}\right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1 - 2\cos u}{2(1 - \cos u)}\right)^n + 2\sum_{n=1}^{\infty} \frac{(1 - 2\cos u)^n}{n^2} \cos nu, \tag{13}$$

where

$$\phi(u) = \frac{[27(1-\cos u)-(1-2\cos u)^3+3\sqrt{81(1-\cos u)^2-6(1-\cos u)(1-2\cos u)^3}]^{\frac{1}{3}}}{(1-2\cos u)},$$

with the restriction $0 \leq \cos u \leq \frac{3}{4}$.

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References

[1] M. Abramowitz and I.A. Stegun (Editors), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1972.

[2] N. Batir, *On the series* $\sum_{k=1}^{\infty} \binom{3k}{k}^{-1} k^{-n} x^k$, Proc. Indian Acad. Sci. (Math. Sci.) **115** (2005), 371-381.

[3] J.M. Borwein and R. Girgensohn, *Evaluations of binomial series*, Aequationes Math., **70** (2005) 25-36.

[4] L. Euler Euler, *Opera Omnia*, Ser. I, Vol. 14, Berlin, 1924.