CONVOLUTION IDENTITIES FOR STIRLING NUMBERS OF THE FIRST KIND

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Abstract

We derive several new convolution identities for the Stirling numbers of the first kind. As a consequence we obtain a new linear recurrence relation which generalizes known relations.

1. Introduction

The *Stirling numbers* of the first and the second kind are fundamental objects in combinatorics, with important applications in other areas of mathematics. In this paper we will be mainly concerned with Stirling numbers *of the first kind*, $s(n,k)$. They can be defined by the generating function

$$x(x-1)\dots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k. \tag{1}$$

This means that they are the coefficients connecting the two most fundamental bases of the vector space of single-variable polynomials (while the inverse transformation between these two bases is given by the Stirling numbers of the second kind). The main combinatorial interpretation of $|s(n,k)|$ is the number of ways to arrange n objects into k cycles; see, e.g., [4, p. 259].

It is the purpose of this paper to derive some convolution identities, as well as an apparently new class of linear identities, all of which can also be seen as recurrence relations. The most basic recurrence is the "triangular" relation

$$s(n+1,k) = s(n,k-1) - n\,s(n,k), \qquad 1 \le k \le n. \tag{2}$$

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Other important types of recurrences are the linear relations

$$s(n,m) = \sum_{k=m}^{n} n^{k-m} s(n+1,k+1) \qquad (m \geq 1), \tag{3}$$

$$s(n+1,m+1) = \sum_{k=m}^{n} (-1)^{m-k} \binom{k}{m} s(n,k), \tag{4}$$

$$s(n,m) = \sum_{k=m}^{n} \binom{k}{m} s(n+1,k+1). \tag{5}$$

The best known convolution identity is

$$\binom{m}{r} s(n,m) = \sum_{k=m-r}^{n-r} \binom{n}{k} s(n-k,r) s(k,m-r), \tag{6}$$

valid for $0 \leq r \leq m$ (see, e.g., [1, p. 824]). The following identity is of a somewhat different type:

$$\frac{s(a+b,a+b-n)}{(a+b-1)\binom{a+b-2}{a-1}} = \sum_{k=0}^{n} \frac{s(a,a-k)s(b,b-(n-k))}{(a+b-n-1)\binom{a+b-n-2}{a-k-1}}, \tag{7}$$

It can be obtained from Equation (6.46) in [4, p. 272]. Among the more basic properties, the following will be used in this paper:

$$s(0,0) = s(n,n) = 1, \quad s(n,0) = 0 \quad \text{for} \quad n \geq 1, \tag{8}$$

$$s(n,1) = (-1)^{n-1}(n-1)!, \tag{9}$$

$$s(n,2) = (-1)^{n}(n-1)! H_{n-1}, \tag{10}$$

where $H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$ is the kth harmonic number, with $H_0 = 0$. By convention we also have

$$s(n,k) = 0 \qquad \text{for} \quad k < 0 \quad \text{and for} \quad k > n. \tag{11}$$

These and numerous other properties can be found, e.g., in the books [1, Ch. 24], [3], [4], or in the on-line resources [11] or [10, A008277]. Although there are some advantages to the bracket notation used in [4] (see also [7]) we use here the main competing notation $s(n,k)$.

In Section 2 we will prove two convolution identities, which will then be used in Section 3 to derive a linear relation. This will be shown to be a generalization of some known identities. We finish with some additional remarks in Section 4.

2. Convolution Identities

Our first convolution identity is related to the sum

$$\tilde{s}(n,k,m) := \sum_{r=0}^{m} \binom{m}{r} \frac{s(n-r,k+m)}{(n-r)!}, \tag{12}$$

which arose in connection with recurrences for Bernoulli numbers of the second kind [2]. No closed from for $\tilde{s}(n,k,m)$ seems to exist; however, we have the following relation.

Theorem 1. *For any nonnegative integers n, k, and m we have*

$$\sum_{r=0}^{m} \binom{m}{r} \frac{s(n-r,k+m)}{(n-r)!} = \frac{1}{n!} \sum_{j=0}^{m} (-1)^j s(n-m,k+j) s(m+1,m+1-j). \tag{13}$$

The identity (13) holds for all $n, k, m \geq 0$, but for $n < k+m$ it is trivially true and quite meaningless since in this case all summands on both sides of (13) vanish, according to (11). The proof of Theorem 1 is based on the following lemma which is of interest in its own right.

Lemma 2. (*a*) *The sum $\tilde{s}(n,k,m)$ defined in* (12) *satisfies the recurrence relation*

$$(n+1)\,\tilde{s}(n+1,k-1,m+1) = \tilde{s}(n,k-1,m) + (m+1)\,\tilde{s}(n,k,m). \tag{14}$$

(*b*) *Let $t(n,k,m)$ denote the sum on the right-hand side of* (13)*. Then we have*

$$t(n+1,k-1,m+1) = t(n,k-1,m) + (m+1)\,t(n,k,m). \tag{15}$$

Proof. (a) We begin by introducing the closely related expressions

$$\tilde{t}(n,k,m) := n!\,\tilde{s}(n,k,m) = \sum_{r=0}^{m} \binom{m}{r} \binom{n}{r} r!\, s(n-r,k+m). \tag{16}$$

Then it is clear that (14) holds if and only if (15) holds for $\tilde{t}(n,k,m)$. We first apply the identity $(m+1)\binom{m}{r} = \binom{m+1}{r+1}(r+1)$ to (16) and then use the binomial identity $\binom{n}{r-1} = \binom{n+1}{r} - \binom{n}{r}$ to obtain

$$
\begin{aligned}
(m+1)\,\tilde{t}(n,k,m) &= \sum_{r=0}^{m} \binom{m+1}{r+1}\binom{n}{r}(r+1)!\,s(n-r,k+m)\\
&= \sum_{r=1}^{m+1} \binom{m+1}{r}\binom{n}{r-1} r!\,s(n-r+1,k+m)\\
&= \sum_{r=0}^{m+1} \binom{m+1}{r}\binom{n+1}{r} r!\,s(n-r+1,k+m)\\
&\quad - \sum_{r=0}^{m+1} \binom{m+1}{r}\binom{n}{r} r!\,s(n-r+1,k+m),
\end{aligned}
$$

where we extended the last two sums to include the case $r=0$; note that these two terms cancel each other. Since the second-last sum is simply $\tilde{t}(n+1,k-1,m+1)$, we are done if we can show that the last sum is $\tilde{t}(n,k-1,m)$. To do this, using the convention $\binom{m}{-1}=0$ we write the sum in question as

$$
\begin{aligned}
&\sum_{r=0}^{m+1} \left(\binom{m}{r}+\binom{m}{r-1}\right)\binom{n}{r} r!\,s(n-r+1,k+m)\\
&= \sum_{r=0}^{m+1} \binom{m}{r}\binom{n}{r} r!\,s(n-r+1,k+m)\\
&\quad + \sum_{r=0}^{m} \binom{m}{r}\binom{n}{r+1}(r+1)!\,s(n-r,k+m)\\
&= \sum_{r=0}^{m} \binom{m}{r}\binom{n}{r} r!\,\big[s(n-r+1,k+m)+(n-r)\,s(n-r,k+m)\big]\\
&= \tilde{t}(n,k-1,m),
\end{aligned}
$$

since by (2) the term in square brackets is $s(n-r,k+m-1)$. Note that in the second-last step we used the simple identity $\binom{n}{r+1}(r+1)=\binom{n}{r}(n-r)$. This proves (14).

(b) For (15) we use again (2) to obtain

$$
\begin{aligned}
&(m+1)\, t(n,k,m) \\
&\quad = \sum_{j=0}^{m} (-1)^j s(n-m,k+j)\big[(m+1)\, s(m+1,m+1-j)\big] \\
&\quad = \sum_{j=0}^{m} (-1)^j s(n-m,k+j)\big[s(m+1,m-j) - s(m+2,m+1-j)\big] \\
&\quad = \sum_{j=1}^{m+1} (-1)^{j-1} s(n-m,k-1+j) \\
&\qquad\qquad\qquad \times \big[s(m+1,m+1-j) - s(m+2,m+2-j)\big] \\
&\quad = -\sum_{j=0}^{m} (-1)^j s(n-m,k-1+j) s(m+1,m+1-j) + s(n-m,k-1) \\
&\quad + \sum_{j=0}^{m+1} (-1)^j s(n+1-(m+1),k-1+j)\, s(m+2,m+2-j) - s(n-m,k-1) \\
&\quad = -t(n,k-1,m) + t(n+1,k-1,m+1),
\end{aligned}
$$

where we have also used (8) to deal with the initial and final terms in the above sums. $\square$

Proof of Theorem 1. The sums $t(n,k,m)$ and $\tilde{t}(n,k,m)$ satisfy the same recurrence, and we have $t(n,k,0) = \tilde{t}(n,k,0) = s(n,k)$ for all k and n. Hence they must be identical, i.e., (13) must hold. $\square$

A result similar in nature to (13) can be derived with a different method.

Theorem 3. *For positive integers n and $r \le n+1$ and for integers $0 \le m \le n$ we have*

$$
\begin{aligned}
&\sum_{j=0}^{n-r+1} \binom{r-1+j}{j} m^j s(n,r-1+j) \\
&\qquad\qquad = \sum_{j=0}^{r} (-1)^{m+1-r+j} s(m+1,r-j) s(n-m,j). \qquad (17)
\end{aligned}
$$

Proof. In the definition (1) we substitute n by $m+1$, replace x by $-x$, and multiply both sides by $(-1)^{m+1}$, to obtain

$$\prod_{\nu=0}^{m}(x+\nu) = \sum_{j=0}^{m+1}(-1)^{m+1-j}s(m+1,j)x^j. \tag{18}$$

Next, we substitute n by $n-m$ in (1), to get

$$\prod_{\nu=0}^{n-m-1}(x-\nu) = \sum_{j=0}^{n-m}s(n-m,j)x^j. \tag{19}$$

The result will be obtained by evaluating the product of the expressions (18) and (19) in two different ways. The product of the left-hand sides is easily seen to be

$$x\prod_{\nu=0}^{n-1}\big((x+m)-\nu\big) = x\sum_{j=0}^{n}s(n,j)(x+m)^j = x\sum_{j=0}^{n}s(n,j)\sum_{t=0}^{j}\binom{j}{t}m^{j-t}x^t,$$

where we have used (1) again. Changing the order of summation, we see that the expression becomes

$$\begin{aligned} x\sum_{t=0}^{n}\sum_{j=0}^{n-t}&\binom{t+j}{j}m^j s(n,t+j)x^t \\ &= \sum_{r=1}^{n+1}\left(\sum_{j=0}^{n-r+1}\binom{r-1+j}{j}m^j s(n,r-1+j)\right)x^r. \end{aligned}$$

The product of the right-hand sides of (18) and (19) is

$$\sum_{r=0}^{n+1}\left(\sum_{j=0}^{r}(-1)^{m+1-r+j}s(m+1,r-j)s(n-m,j)\right)x^r.$$

Equating the coefficients of x^r, for $1 \le r \le n+1$, in the last two expressions, we obtain the result. $\square$

3. A Linear Recurrence

By comparing the right-hand sides of (17) and (13), we obtain another interesting identity. Indeed, set $r = k + m + 1$ in (17); then we have

$$\sum_{j=0}^{n-k-m} \binom{k+m+j}{j} m^j s(n, k+m+j)$$
$$= \sum_{j=0}^{k+m+1} (-1)^{k+j} s(m+1, k+m+1-j) s(n-m, j). \quad (20)$$

Since $s(m+1, 0) = s(m+1, x) = 0$ for $x > m+1$, the sum on the right actually ranges only from $j = k$ to $j = k+m$. If we now replace j by $j-k$ in this sum, we see that it is identical with the sum on the right-hand side of (13). Changing the summation on the left-hand side of (20), we now get the following apparently new identity from (13).

Theorem 4. *For integers $n \geq 1$ and $0 \leq m \leq n$, $0 \leq k \leq n-m$ we have*

$$\sum_{j=k+m}^{n} \binom{j}{k+m} m^{j-k-m} s(n, j) = n! \sum_{r=0}^{m} \binom{m}{r} \frac{s(n-r, k+m)}{(n-r)!}. \quad (21)$$

We consider some special cases. For $m = 0$ or for $k = n-m$ the identity is trivial. However, for $m = 1$ we get the following more meaningful summation.

Corollary 5. *For integers $n \geq 1$ and $0 \leq k \leq n-1$ we have*

$$\sum_{j=0}^{n-k-1} \binom{n-j}{k+1} s(n, n-j) = s(n, k+1) + ns(n-1, k+1). \quad (22)$$

This identity is in fact known; see [5], (52.2.19), or [6], p. 185, for equivalent formulations. For related identities, see [4], p. 265. If we take $k = 0$ in (22) and use (9), we get the identity

$$\sum_{j=1}^{n} js(n, j) = (-1)^{n-2}(n-2)!,$$

valid for $n \geq 2$. It can be found in [6], p. 186; it is also (incorrectly) quoted in [5], identity (52.2.6).

Similarly, if we let $m = 2$ in (21) and make a small change in the summation, then we get

Corollary 6. *For integers $n \geq 2$ and $0 \leq k \leq n-2$ we have*

$$\sum_{j=0}^{n-k-2} \binom{k+2+j}{k+2} 2^j s(n, k+2+j) \tag{23}$$
$$= s(n, k+2) + 2n\, s(n-1, k+2) + n(n-1)\, s(n-2, k+2),$$

and in particular, for $n \geq 3$,

$$\sum_{j=2}^{n} \binom{j}{2} 2^{j-2} s(n, j) = (-1)^n (n-3)!\, (2H_{n-3} - 3)\,. \tag{24}$$

The identity (24) follows easily from (23) by taking $k = 0$ and using (8)–(10).

4. Further Remarks

1. A result on Bernoulli numbers of the second kind in the authors' recent paper [2] can also be seen as a class of convolution identities for Stirling numbers of the first kind. Indeed, it was pointed out in [2] how we can obtain the identity

$$\sum_{j=0}^{m-1} (-1)^j \frac{s(m, j) s(m, m-j)}{\binom{m}{j}} = m!(m+1)! \sum_{r=0}^{m} \binom{m}{r} \frac{s(2m+1-r, m+1)}{(2m+1-r)!}.$$

2. Two rather general convolution identities for Stirling *polynomials* are given in [4, p. 272]; they can be rewritten in terms of identities for Stirling numbers of both kinds. Without going into details we just mention that $s(n, n-k)$ can be shown to be a polynomial in n of degree $2k$. After dividing out $k+1$ obvious linear factors, one obtains the kth Stirling polynomial. These polynomials were apparently first introduced by Nielsen in his treatise on the Gamma function [8], then studied in great detail in his rare memoir [9]. Stirling polynomials are also treated to some extent in [6] and in [4], as already cited. As usual, the reader should be aware of differences in notation which in this case also involves differences in the signs of the coefficients.

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