INTEGER SUBSETS WITH HIGH VOLUME AND LOW PERIMETER

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Received: 7/14/11, Revised: 2/6/12, Accepted: 4/7/12, Published: 4/18/12

Abstract

We explore a variation of the isoperimetric problem in which integer subsets take the role of geometric figures. More specifically, we consider the sequence $P(n)$ introduced by Miller *et al.* and described in OEIS A186053. We provide the first exact formulas for $P(n)$ including recursive relations via auxiliary functions as well as concise and satisfying representations and even quasi-explicit formulas. We also discuss some of the intricate fractal-like symmetry of the sequence and the development of algorithms for computing $P(n)$. We conclude with open questions for further research.

1. Introduction

One of the most widely-known classical geometry problems is the so-called *isoperimetric problem*, one equivalent variation of which is:

> If a figure in the plane has area A, what is the smallest possible value for its perimeter?

In the Euclidean plane, the optimal configuration is a circle, implying that any figure with area A has perimeter at least $2\sqrt{A\pi}$, and this lower bound is obtained if and only if the figure is a circle.

In 2011, Miller et al. [2] extended the isoperimetric problem in a new direction, in which integer subsets took the role of geometric figures. For any integer subset A, they defined its *volume* as the sum over all its elements, and they defined its *perimeter* as the sum of all elements $x \in A$ such that $\{x-1, x+1\} \not\subset A$. Thus, the volume can be thought of as the sum of all the elements of A, and the perimeter can be thought of as the sum of all the elements on the "boundary" of A (that is to say, the elements of A whose successor and predecessor are not both in A).

The main focus of [2] was to examine the relationships between a set's perimeter and its volume. More specifically, the authors wanted to answer the corresponding "isoperimetric question"[1]:

> If a subset of $\{0, 1, \ldots\}$ has volume n, what is the smallest possible value for its perimeter?

Adopting their notation, we will let $P(n)$ denote this value throughout this paper[2].

Because their work is so recent, Miller et al. are the only ones who have published on this variation of the isoperimetric problem or on the function $P(n)$. Their work was to provide bounds for $P(n)$, by which they were able to determine its asymptotic behavior. Specifically, their main result was

Theorem 1. (Miller et al., 2011) *Let $P(n)$ be as defined. Then $P(n) \sim \sqrt{2}n^{1/2}$. Moreover, for all $n \geq 1$,*

$$\sqrt{2}n^{1/2} - 1/2 < P(n) < \sqrt{2}n^{1/2} + (2n^{1/4} + 8)\log_2 \log_2 n + 58. \tag{1}$$

Their proof of the lower bound will be reproduced in following sections. However, their proof of the upper bound is via a construction argument, which we will not reproduce here since we will analytically derive a tighter bound in Theorem 12.

Beyond the inequalities in (1) provided by Miller et al., nothing else has been published on $P(n)$ except for some values for small n. It should be noted that [2] provides very good bounds on a related function, in which the sets of interest are allowed to have both negative as well as positive elements. This result was also via a construction argument and is not relevant to this paper.

1.1. Outline of Results

In this paper, we focus on improving the few results known on $P(n)$, including deriving multiple exact formulas and developing an understanding of its interesting long-term behavior. Many of these results are stated in terms of a closely related function, $Q(n)$, which is briefly defined as

$$Q(n) := \min_{A \subseteq \{0,1,\ldots\}} \Big\{ per(A^c) : vol(A) = n \Big\}.$$

Since it proves to be intimately related to $P(n)$, we provide results on $Q(n)$ as well.

We begin in Section 3 with several prelimary lemmas including those used in [2]. Then in Section 4, we define auxilary functions, with which we combinatorially derive several recursive formulas for $P(n)$. We then introduce the function $Q(n)$ and derive similar formulas for it as well.

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[1]They focused on this question in particular because it turns out that all of the other related extremal questions relating a set's volume and perimeter are trivial.

[2]This is sequence A186053 in OEIS.

In Section 5, we relate the functions $P(n)$ and $Q(n)$ by providing yet more recurrences for both of them, from which we see that each function completely determines the other. With this in place, we move on to Section 6, in which we use these recurrences to determine several analytic results for $P(n)$ and $Q(n)$, including upper and lower bounds and derivations of their asymptotic behavior.

Our work then culminates in Section 7, in which we state and prove the strongest results of the paper. By appealing to our analytic bounds on $P(n)$ and $Q(n)$, we show that for all sufficiently large values of n, the recurrences of Section 5 admit certain drastic simplifications. By then combining this result with rigorous computer calculations, we arrive at the main theorem of the paper[3]:

Theorem 18. *Let $P(n)$ and $Q(n)$ be as given. Then if $n \geq 0$ is not one of the* 177 *known counterexamples tabulated in Table 1 of the appendix (in particular, for all $n > 149,894$), we have*

$$P(n) = f(n) + Q(g(n)) \quad \textit{and} \quad Q(n) = 1 + f(n) + P(g(n)),$$

where the functions $f(n)$ and $g(n)$, given by

$$f(n) := \left\lfloor \sqrt{2n} + 1/2 \right\rfloor = \left\lceil \sqrt{2n} \right\rceil, \quad \textit{and} \quad g(n) := \frac{f(n)[f(n)+1]}{2} - n,$$

are the smallest nonnegative integers satisfying $[1 + 2 + 3 + \cdots + f(n)] - g(n) = n$.

With this, we derive several other satisfying and revealing reccurence relations and quasi-explicit representations for $P(n)$ and $Q(n)$. We also briefly demonstate and discuss the intricate fractal-like symmetry of the graphs of these functions. We then conclude in Section 9 by noting applications in the design of algorithms related to this problem and with some open questions for future research.

For an earlier version of this paper with somewhat more detail, see [**?**].

2. Definitions and Notation

For the reader's possible convenience, a brief list of definitions used throughout the paper is given here. In each definition, A is assumed to be a subset of $\{0, 1, 2, \ldots\}$, and n and k are assumed to be nonnegative integers.

- The *boundary* of A, ∂A, is $\partial A := \{z \in A : \{z-1, z+1\} \not\subseteq A\}$. In words, it is the set of elements of A whose successor or predecessor is not in A.
- The *volume* and *perimeter* of A are defined as

$$vol(A) := \sum_{z \in A} z, \qquad \text{and} \qquad per(A) := \sum_{z \in \partial A} z,$$

[3] More adequate introductions of the functions f and g are given in Section 6.

respectively. For convention, $vol(\emptyset) = per(\emptyset) = 0$.

- $P(n) := \min_{A \subseteq \{0,1,\ldots\}} \Big\{ per(A) : vol(A) = n \Big\}$.
- The *complement* of A is $A^c := \{0, 1, \ldots\} \setminus A = \{z \in \{0, 1, \ldots\} : z \notin A\}$.
- $Q(n) := \min_{A \subseteq \{0,1,\ldots\}} \Big\{ per(A^c) : vol(A) = n \Big\}$.
- The helper functions $p(n; k)$ and $q(n; k)$ are defined as

$$\begin{aligned} p(n;k) &:= \min_{A \subseteq \{0,1,\ldots,k\}} \Big\{ per(A) : vol(A) = n \Big\}, \\ q(n;k) &:= \min_{A \subseteq \{0,1,\ldots,k\}} \Big\{ per(A^c) : vol(A) = n \Big\}. \end{aligned}$$

- The special helper function $\sigma(n; k)$ is

$$\sigma(n;k) := \min_{A \subseteq \{0,1,\ldots,k\}} \Big\{ per(A^c) : vol(A) = n, \quad \text{and} \quad k \in A \Big\}.$$

- The functions $f(n)$ and $g(n)$ are given by

$$f(n) = \left[\sqrt{2n}\right], \qquad g(n) = \frac{f(n)[f(n)+1]}{2} - n = \frac{\left[\sqrt{2n}\ \right]^2 + \left[\sqrt{2n}\ \right]}{2} - n,$$

  where $[x]$ denotes the nearest integer function. In **Proposition 8**, we show these are the smallest nonnegative integers satisfying $[1+\cdots+f(n)] - g(n) = n$.
- For $N \geq 1$ (e.g., $N = 149,894$), $\phi(n; N) = \phi(n) := \min\{i \geq 0 : g^i(n) \leq N\}$.
- The function $R(n)$ is recursively defined as $R(0) := 0$, and for all $n \geq 1$, we have $R(n) := 1/2 + f(n) + R(g(n))$.

3. Preliminary Results

The following is used throughout [2] particularly in their lower bound on $P(n)$.

Lemma 2. (Miller et al., 2011) *Assume A is a finite nonempty subset of* $\{0, 1, \ldots\}$, *and let m denote its maximum element. Then*

$$m \leq per(A) \leq vol(A) \leq \frac{m(m+1)}{2}.$$

Using this lemma, the following lower bound is immediately attained.

Proposition 3. *Assume $A \subseteq \{0, 1, \ldots\}$ is finite. Then we have*

$$\sqrt{2vol(A)} - 1/2 \leq \frac{-1 + \sqrt{1 + 8vol(A)}}{2} \leq per(A).$$

Moreover, for any positive integer n, this implies $\sqrt{2}n^{1/2} - 1/2 \leq P(n)$.

As stated before, except for the previously mentioned constructive upper bound on $P(n)$, these two results are all that has been published about $P(n)$. The remainder of the paper is devoted to new results.

3.1. Miscellaneous Lemmas

Lemma 4. *Let $\emptyset \neq A \subseteq \{0, 1, \ldots\}$ be finite with maximum element m. Then*

$$m + 1 \leq per(A^c)$$

with equality if and only if $\{1, \ldots, m\} \subseteq A$.

Proof. Let A be as given. Then $m \in A$, but we know $m + 1 \notin A$. Therefore, $m + 1 \in \partial A^c$ implying that $m + 1 \leq per(A^c)$. Now since $m + 1 \in \partial A^c$, we know that $m + 1 = per(A^c)$ if and only if ∂A^c is equal to either $\{m + 1\}$ or $\{0, m + 1\}$. But this happens if and only if $\{1, 2, \ldots, m\} \subseteq A$, as desired. □

Proposition 5. *Assume $A \subseteq \{0, 1, \ldots\}$ is finite. Then we have*

$$\sqrt{2vol(A)} + 1/2 \leq \frac{-1 + \sqrt{1 + 8vol(A)}}{2} + 1 \leq per(A^c).$$

Moreover, for any positive integer n, this implies

$$\sqrt{2}n^{1/2} + 1/2 \leq Q(n).$$

Proof. This follows from the previous lemma just as Proposition 3. □

4. Recurrence Relations using Auxilary Functions

We now derive our first set of recurrence relations for $P(n)$ and $Q(n)$. Although the relations derived in Section 5 are more revealing, the relations presented here follow naturally, and they motivate the introduction of important auxilary functions. Moreover, due to their convenient structure, these relations are used extensively in the design of algorithms for computing values, as we briefly discuss in Section 9.

4.1. First Recurrence for $P(n)$

As is often the case in analyzing discrete functions, we may obtain an exact recurrence relation for $P(n)$ in terms of a related auxillary function. In our case, recall that $P(n)$ is the minimum perimeter among all subsets of $\{0, 1, \ldots\}$ having volume n. This suggests defining an auxilary function, $p(n; k)$, as

$$p(n;k) = \min_{A \subseteq \{0,1,\ldots,k\}} \Big\{ per(A) : vol(A) = n \Big\}.$$

Then for all $n \geq 0$, we have

$$P(n) = \min_{k \in \{0,1,\ldots\}} \left\{ \min_{A \subseteq \{0,1,\ldots,k\}} \Big\{ per(A) : vol(A) = n \Big\} \right\} = \min_{k \in \{0,1,\ldots\}} \Big\{ p(n;k) \Big\}.$$

From its definition, it is clear that for all fixed n, the function $p(n; k)$ is monotonically decreasing with k. Moreover, for all $K \geq n$, we have $p(n; K) = p(n; n)$ since any subset of $\{0, 1, \ldots\}$ having volume n must necessarily be a subset of $\{0, 1, 2, \ldots , n\}$. Therefore the above equation simplifies to

$$P(n) = \min_{k \in \{0,1,\ldots\}} \left\{ p(n;k) \right\} = \lim_{k \to \infty} p(n;k) = p(n;n). \tag{2}$$

Thus, we now seek a recurrence for $p(n; k)$, which will provide us with $P(n)$ by calculating $p(n; n)$.

For notational convenience, let $S(n; k)$ denote the set of all subsets of $\{0, 1, \ldots , k\}$ having volume n. Then consider the following paritition of $S(n; k)$

$$S(n;k) = \bigcup_{l=0}^{k+1} \Big\{ A \in S(n;k) : \{l, \ldots, k\} \subseteq A \quad \text{and} \quad l-1 \notin A \Big\}.$$

From this partition, it follows that

$$p(n;k) = \min_{l \in \{0,\ldots,k+1\}} \left\{ \min_{A \in S(n;k)} \Big\{ per(A) : \{l,\ldots,k\} \subseteq A \quad \text{and} \quad l-1 \notin A \Big\} \right\}. \tag{3}$$

Now let $0 \leq l \leq k + 1$ be fixed. Then we have

$$\begin{aligned}
&\min_{A \in S(n;k)} \Big\{ per(A) : \{l,\ldots,k\} \subseteq A \quad \text{and} \quad l-1 \notin A \Big\} \\
&\quad = \min_{B \subseteq \{0,\ldots,l-2\}} \Big\{ per(B \cup \{l, l+1, \ldots, k\}) : vol(B \cup \{l, l+1, \ldots, k\}) = n \Big\} \\
&\quad = \begin{cases} p(n;k-1) & \text{if } l = k+1, \\ k + p(n-k;k-2) & \text{if } l = k, \\ k + l + p\big(n - [k(k+1) - l(l-1)]/2; l-2\big) & \text{if } 0 \leq l < k. \end{cases}
\end{aligned}$$

Therefore, by substituting into (3), we are able to obtain the recurrence

$$\begin{aligned} p(n;k) &= \min\Bigg\{ p(n;k-1), k + p(n-k;k-2), \\ &\qquad k + \min_{l \in \{0,\ldots,k-1\}} \Big\{ l + p\big(n - [k(k+1) - l(l-1)]/2; l-2\big) \Big\} \Bigg\}, \end{aligned} \tag{4}$$

which is valid for all $n \geq 1$ and for all $k \geq 1$. Moreover, as boundary conditions, which are clear from its definition, we have that $p(n;k)$ satisfies

$$p(n;k) = \begin{cases} 0 & \text{if } n = 0, \\ \infty & \text{if } n < 0 \text{ or } k \leq 0 < n. \end{cases}$$

Thus, this gives the following compact representation for $P(n)$ for all $n \geq 0$:

$$P(n) = \min\Big\{ p(n;n-1), n \Big\}. \tag{5}$$

4.2. Introduction of $Q(n)$ and Derivation of First Recurrences

Because of its intimate connections with the function $P(n)$ that will be explored in subsequent sections, we now introduce the function $Q(n)$, which is defined as

$$Q(n) = \min_{A \subseteq \{0,1,\ldots\}} \Big\{ per(A^c) : vol(A) = n \Big\}.$$

The difference between this function and the function $P(n)$ is subtle, and based on how similarly the two functions are defined, one would expect their behavior to be very close. As we will see, this is indeed the case, and the connections between $P(n)$ and $Q(n)$ are actually of fundamental importance. However, it is important for the reader to keep in mind the difference in how these functions are defined.

As with the function $P(n)$, we define the auxilary function $q(n;k)$ as

$$q(n;k) = \min_{A \subseteq \{0,1,\ldots,k\}} \Big\{ per(A^c) : vol(A) = n \Big\},$$

and just as before, for all $n \geq 0$, we have that

$$Q(n) = q(n;n). \tag{6}$$

Because of the difference between how the functions $P(n)$ and $Q(n)$ are defined, we now need to define a special auxilary function, $\sigma(n;k)$, in order to obtain a compact recurrence for $q(n)$. This function is defined by

$$\sigma(n;k) = \min_{A \subseteq \{0,1,\ldots,k\}} \Big\{ per(A^c) : vol(A) = n \quad \text{and} \quad k \in A \Big\}.$$

Note the similarities between $\sigma(n;k)$ and $q(n;k)$. In fact, for all $n \geq 1$ and $k \geq 0$,

$$q(n;k) = \min_{l \in \{1,2,\ldots,k\}} \Big\{ \sigma(n;l) \Big\}. \tag{7}$$

Using this equation and (6), we obtain that $Q(0) = 0$, and for all $n \geq 1$

$$Q(n) = \min_{l \in \{1,2,\ldots,n\}} \Big\{ \sigma(n;l) \Big\}. \tag{8}$$

Just as was the case for $P(n)$, in order to obtain a useful recurrence relation for $Q(n)$, it now only remains to find a recurrence for $\sigma(n;k)$. As before, we accomplish this by a simple partition yielding

$$\sigma(n;k) = k + 1 + \min \Big\{ \sigma(n-k;k-1) - k, \sigma(n-k;k-2), k-1+q(n-k;k-3) \Big\},$$

which we obtain by partitioning the subsets of interest into the three groups (I) sets containing $k-1$, (II) sets containing $k-2$ but not $k-1$, and (III) sets containing neither $k-2$ nor $k-1$.

At this point, we should note that some care must be given to the interpretation of the above equation, which depends on how we define $\sigma(0;0)$. However, if we note and state as a boundary condition that $\sigma(n,n) = 2n$ for all $n \geq 1$, then these concerns are effectively removed.

We then have a recurrence for σ. As boundary conditions for $\sigma(n;k)$, we have

$$\sigma(n;k) = \begin{cases} 2n & \text{if } n = k \geq 0, \\ \infty & \text{if } n < 0 \text{ or if } k \in \{0,1\} \text{ and } n > k, \\ \infty & \text{if } 0 \leq k > n \geq 0, \end{cases}$$

and for all $n \geq 2$, and $2 \leq k < n$, we have

$$\sigma(n;k) = k + 1 + \min \Big\{ k - 1 + q(n-k;t-3), \sigma(n-k;k-2), \sigma(n-k;k-1) - k \Big\}.$$

Thus, by using (8) we have a recurrence for $Q(n)$ as well.

5. More Direct Recurrence Relations

Using different partitions of the sets of interest, we derive the following recurrences, from which we see the first connections between the functions $P(n)$ and $Q(n)$.

5.1. Recurrence for $P(n)$ Involving $q(n;k)$ and $\sigma(n;k)$

We may calculate $P(n)$ by a "more direct" recurrence relation, which is found by partitioning all sets of volume n first according to their maximum element, m, and then according to the largest integer smaller than m not contained in each set.

Let A be a set of volume n, let m be its maximum element, and let l be the largest element of $\{-1, 0, \ldots, m\}$ not contained in A. Then A may be written uniquely as $A = \{0, 1, 2, \ldots, m\} \setminus B$ for some set $B \subseteq \{0, 1, \ldots, l\}$, where the volume of B is equal to $(1 + 2 + \cdots + m) - n$ and $l \in B$. If $l = m - 1$, then $per(A) = per(B^c)$. Else, we have $per(A) = m + per(B^c)$.

From this observation, we obtain that for all $n \geq 2$

$$P(n) = \min_{m \geq 1} \Big\{ m + q([1+2+\cdots+m] - n; m-2), \sigma([1+2+\cdots+m] - n; m-1) \Big\}, \quad (9)$$

where $q(n; k)$ and $\sigma(n; k)$ are defined as earlier.

5.2. Recurrence for $Q(n)$ Involving $p(n; k)$

As before, we also have a simple recurrence that can be used to calculate $Q(n)$ "more directly." Let A be a set of volume n and maximum element m. Then the set A may be written uniquely in the form $A = \{0, 1, 2, \ldots, m\} \setminus B$ for some set $B \subseteq \{0, 1, \ldots, m-1\}$, where the volume of B is equal to $(1+2+\cdots+m)-n$. Now we know that for all such sets A and B, we have $per(A^c) = per(B) + (m + 1)$.

This observation leads to the simple and beautiful recurrence that for all $n \geq 2$,

$$Q(n) = 1 + \min_{m \geq 1} \Big\{ m + p([1 + 2 + \cdots + m] - n; m - 1) \Big\}, \quad (10)$$

where $p(n; k)$ is as defined earlier.

6. Analysis of Recurrences

Although equations (9) and (10) appear somewhat intractible (and they offer little or no computational advantage over the first recurrences of Section 4), they turn out to be crucial in understanding the behavior of $P(n)$ (and of $Q(n)$ as well). In Section 7, we are able to greatly simplify these recurrences, but in order to do so, we must first derive some analytic bounds on $P(n)$ and $Q(n)$.

6.1. Relevant Lemmas and Notions

Lemma 6. *For all $n \in \{1, 2, \ldots\}$, there are unique integers $f(n)$ and $g(n)$ satisfying*

$$n = [0 + 1 + \cdots + f(n)] - g(n),$$

such that $0 \leq g(n) < f(n)$. Moreover, $f(n)$ and $g(n)$ are given by[4]

$$f(n) = \left\lceil \frac{-1 + \sqrt{1 + 8n}}{2} \right\rceil, \qquad \textit{and} \qquad g(n) = \frac{f(n)[f(n) + 1]}{2} - n.$$

[4]We will use these representations for $f(n)$ and $g(n)$ so that $f(0) = g(0) = 0$ is well-defined.

Having defined these functions, we may now restate previous lemmas involving $P(n)$ and $Q(n)$ in these terms. The most important result we will use combines Propositions 3 and 5 as follows:

Corollary 7. *Restating earlier results in new notation, for all $n \geq 1$, we have that*

$$P(n) \geq f(n), \qquad \textit{and} \qquad Q(n) \geq f(n) + 1.$$

Finally, before moving on, we must present two more results on $f(n)$ and $g(n)$.

Proposition 8. *Let $f(n)$ be as before. Then for all integers $n \geq 0$, we have*

$$f(n) = \left\lceil \frac{-1 + \sqrt{1+8n}}{2} \right\rceil = \left\lceil \sqrt{2n} - 1/2 \right\rceil = \left[\sqrt{2n} \right],$$

where $[x]$ is the nearest integer function.

Proof. It suffices to show the first part of the stated equation holds, and the fact that $\sqrt{2n}$ is never a half-integer will complete the proof. Now by way of contradiction, suppose that the first two representations are not equal. Then this would imply that there exist integers $p \in \mathbb{Z}$ and $n \in \{0, 1, \ldots\}$ such that

$$\sqrt{2n} - 1/2 \leq p < \frac{\sqrt{1+8n} - 1}{2},$$

which implies $8n \leq (2p+1)^2 < 8n + 1$. But since n and p are integers, this forces $8n = (2p+1)^2$, which taken modulo 2 yields a contradiction. $\square$

Proposition 9. *Let $g(n)$ be as defined. Then for all integers $L \geq 0$ and $n \geq 0$,*

$$g^L(n) \leq 2 \cdot (n/2)^{1/2^L}.$$

Proof. The proof is by induction on L. If $L = 0$, then the claim is trivially true, establishing the base case. Suppose the claim holds for $L = m$. Then for all $n \geq 0$,

$$g(n) \leq f(n) - 1 < \sqrt{2n} - 1/2 < \sqrt{2n},$$

which implies $g^{m+1}(n) = g(g^m(n)) < \sqrt{2 \cdot g^m(n)}$. Then using the induction hypothesis and that the square root function is increasing completes the proof. $\square$

6.2. Upper Bounds and Asymptotics for $P(n)$ and $Q(n)$

Using the recurrences of Section 5, we now obtain simple upper bounds on $P(n)$ and $Q(n)$, which taken with the last few lemmas, yield good absolute bounds in n.

Theorem 10. *Let $f(n)$ and $g(n)$ be defined as before. Then for all $n \geq 0$, we have*

$$P(n) \leq f(n) + Q(g(n)), \quad \textit{and} \quad Q(n) \leq 1 + f(n) + P(g(n)).$$

Proof. For $n = 0$ and $n = 1$, the two inequalities hold. Then for all $n \geq 2$, we may appeal to (9) to obtain

$$\begin{aligned} P(n) &= \min_{m\geq 1}\Big\{m + q([1+\cdots+m]-n; m-2), \sigma([1+\cdots+m]-n; m-1)\Big\} \\ &\leq f(n) + \min\Big\{q(g(n); f(n)-2), \sigma(g(n); f(n)-1)\Big\} \\ &= f(n) + q(g(n); f(n)-1) = f(n) + Q(g(n)), \end{aligned}$$

and the corresponding inequality for $Q(n)$ is proven analogously. □

Corollary 11. *For all nonnegative integers n and L, we have that*

$$\begin{aligned} P(n) &\leq L + P(g^{2L}(n)) + \sum_{i=0}^{2L-1} f(g^i(n)), \qquad \textit{and} \\ Q(n) &\leq L + Q(g^{2L}(n)) + \sum_{i=0}^{2L-1} f(g^i(n)), \end{aligned}$$

where $g^i(n)$ is the i-fold composition of g evaluated at n.

Theorem 12. *Let $P(n)$ and $Q(n)$ be as given. Then $P(n) \sim Q(n) \sim \sqrt{2}n^{1/2}$. Moreover, for all $n > 2$,*

$$\begin{aligned} \sqrt{2}n^{1/2} - 1/2 < \quad P(n) &\leq \sqrt{2}n^{1/2} + (2^{3/4}\cdot n^{1/4}+1)[\log_2(\log_2(n/2)) - 1] + 7, \\ \sqrt{2}n^{1/2} + 1/2 < \quad Q(n) &\leq \sqrt{2}n^{1/2} + (2^{3/4}\cdot n^{1/4}+1)[\log_2(\log_2(n/2)) - 1] + 7. \end{aligned}$$

Proof. The lower bounds in the asserted inequalities have already been proven. To prove the upper bounds, we merely combine the results in the last corollary with the past few bounds on $f(n)$ and $g(n)$. More specifically, assuming $n > 2$, we know from Proposition 9 that if $L \geq (\log_2(\log_2(n/2)) - 1)/2$, then

$$g^{2L}(n) \leq 2\cdot (n/2)^{1/2^{(\log_2(\log_2(n/2))-1)}} = \cdots = 8.$$

By considering values of $P(n)$ and $Q(n)$ for $n \leq 8$, we see that $g^{2L}(n) \leq 8$ implies $P(g^{2L}(n)) \leq 7$ and $Q(g^{2L}(n)) \leq 7$. Now by the last few results, we have

$$\begin{aligned} P(n) &\leq L + P(g^{2L}(n)) + \sum_{i=0}^{2L-1} f(g^i(n)) \leq L + P(g^{2L}(n)) + \sum_{i=0}^{2L-1} \sqrt{2g^i(n)} + 1/2 \\ &\leq 2L + P(g^{2L}(n)) + \sqrt{2n} + 2\sum_{i=1}^{2L-1}\sqrt{(n/2)^{1/2^i}} \\ &\leq 2L + P(g^{2L}(n)) + \sqrt{2n} + 4L(n/2)^{1/4}. \end{aligned}$$

Then taking $L = (\log_2(\log_2(n/2)) - 1)/2$ proves the bound. The inequality for $Q(n)$ is proven analogously. □

Note that these bounds on $P(n)$ are slightly better than those of [2] stated in **Theorem 1**. Also note that the upper bound on the summation is very crude. However, these bounds are sufficient for our purposes.

7. Obtaining *Good* Recurrences for $P(n)$ and $Q(n)$

Although the bounds in Theorem 12 are rather good, they reveal nothing about the actual fluctuations of $P(n)$ and $Q(n)$. And although we have already obtained multiple recurrence relations for finding exact values, these relations all involve auxilary helper functions, multiple variables, and unweildy minimum functions. In this section, we combine our analytic bounds and combinatorial results to obtain surprisingly simple and satisfying recurrence relations for $P(n)$ and $Q(n)$ and even quasi-explicit formulae.

7.1. New Lower Bounds on $P(n)$ and $Q(n)$

Lemma 13. *Let n and k be positive integers with $k < f(n)$. Then $p(n;k)$, $q(n;k)$, and $\sigma(n;k)$ are all infinite.*

Proof. If $k < f(n)$, there are no subsets of $\{0, 1, \ldots, k\}$ with volume n. □

Lemma 14. *Let n and m be positive integers with $m > f(n)$. Then we have*

$$\begin{aligned} m + p([1 + \cdots + m] - n; m - 1) &\geq f(n) + \sqrt{2(g(n) + f(n) + 1)} + 1/2 \\ m + q([1 + \cdots + m] - n; m - 2) &\geq f(n) + \sqrt{2(g(n) + f(n) + 1)} + 3/2. \end{aligned}$$

Proof. Using the simple lower bound in **Theorem 12**, we obtain

$$\begin{aligned} p([1 + \cdots + m] - n; m - 1) &\geq P([1 + \cdots + m] - n) \\ &\geq \sqrt{2([1 + \cdots + m] - n)} - 1/2 \\ &\geq \sqrt{2(g(n) + [f(n) + 1] + \cdots + m)} - 1/2 \\ &\geq \sqrt{2(g(n) + f(n) + 1)} - 1/2, \end{aligned}$$

which proves the first inequality. The second is proven in the same way. □

Lemma 15. *Let n and m be positive integers with $m \geq f(n)$. Then we have*

$$\sigma([1 + 2 + \cdots + m] - n; m - 1) \geq 2f(n) - 2.$$

Proof. First, we may assume $f(n) \geq 2$. Let $A \subseteq \{0, 1, \ldots, m - 1\}$ be such that $vol(A) = [1 + 2 + \cdots + m] - n$ and $m - 1 \in A$. By way of contradiction, suppose that $per(A^c) < 2f(n) - 2$. Now if $m \geq 2f(n) - 2$, then since $m - 1 \in \partial A$, this would imply that $per(A^c) \geq m \geq 2f(n)-2$. Therefore, we may assume that $m \leq 2f(n)-3$. Now since $m \geq f(n)$, the volume of A may be written as

$$vol(A) = [1+2+\cdots+m] - n < f(n) + [f(n)+1] + \cdots + m,$$

and because $m \leq 2f(n)-3 = [f(n)-2]+[f(n)-1]$, we also have

$$vol(A) < [f(n)] + [f(n)+1] + \cdots + [m-1] + [f(n)-2] + [f(n)-1] = \sum_{i=f(n)-2}^{m-1} i.$$

From this, we know there is at least one element of $\{f(n)-2, f(n)-1, \ldots, m-2\}$ that is not contained in A, since otherwise $vol(A)$ would be too large. Let $l \in A^c$ be the largest integer satisfying $f(n)-2 \leq l \leq m-2$. Then since $m-1 \in A$, we know that $l \in \partial A^c$, which implies

$$per(A^c) \geq l + m \geq f(n) - 2 + m \geq f(n) - 2 + f(n) = 2f(n) - 2.$$

But this contradicts the assumption that $per(A^c) < 2f(n)-2$. $\square$

With these lemmas, we are now able to prove the following lower bounds.

Theorem 16. *Let $P(n)$ and $Q(n)$ be as given. Then for all $n \geq 2$, we have*

$$\begin{aligned} P(n) &\geq f(n) + \min\left\{Q(g(n)), \sqrt{2(g(n)+f(n)+1)} + 3/2, f(n)-2\right\} \\ Q(n) &\geq 1 + f(n) + \min\left\{P(g(n)), \sqrt{2(g(n)+f(n)+1)} + 1/2\right\}. \end{aligned}$$

Proof. Starting with (9) and applying Lemmas 13, 14, and **15**, we obtain

$$\begin{aligned} P(n) &= \min_{m > f(n)} \Big\{ f(n) + q(g(n); f(n)-2), m + q([1+2+\cdots+m]-n; m-2), \\ &\qquad \sigma(g(n); f(n)-1), \sigma([1+2+\cdots+m]-n; m-1) \Big\} \\ &\geq f(n) + \min\left\{Q(g(n)), \sqrt{2(g(n)+f(n)+1)} + 3/2, f(n)-2\right\}. \end{aligned}$$

The second inequality is proven analogously by starting with (10). $\square$

7.2. Squeezing an Equation from Inequalities (Eventually)

At this point, we have simple upper bounds on $P(n)$ and $Q(n)$ provided by Theorem 10 and nearly simple lower bounds from Theorem 16, which are complicated by the "min" operators. Suppose we could show that *eventually* $P(g(n))$ and $Q(g(n))$ happen to be the smallest terms in each minimum. Then our lower bounds would simplify drastically and our lower and upper bounds would squeeze together, yielding a simple pair of mutual recurrences valid for all sufficiently large n.

As it turns out, we can in fact prove this claim, as follows:

Proposition 17. *Let $P(n)$ and $Q(n)$ be as given. Then there exists an $N \in \mathbb{Z}$ such that for all $n \geq N$*

$$\begin{aligned} P(g(n)) &= \min\left\{P(g(n)), \sqrt{2(g(n)+f(n)+1)}+1/2\right\} \qquad \textit{and} \\ Q(g(n)) &= \min\left\{Q(g(n)), \sqrt{2(g(n)+f(n)+1)}+3/2, f(n)-2\right\}. \end{aligned}$$

Moreover, these claims hold if we take N to be $2,500,000$.

Proof. We will first prove there is such an $N \in \mathbb{Z}$. Then we will discuss why we may take N to be $2,500,000$. We need to show that for sufficiently large n, $P(g(n)) \leq \sqrt{2(g(n)+f(n)+1)}+1/2$. From **Theorem 12**, we know

$$P(r) \leq \sqrt{2r} + o(\sqrt{r}).$$

Therefore, there exists a constant G such that for all $r \geq G$, we have

$$P(r) \leq \sqrt{2r} + o(\sqrt{r}) \leq \sqrt{4r}.$$

From this, it follows that for all n, if $g(n) \geq G$, then we have

$$P(g(n)) \leq \sqrt{4g(n)} \leq \sqrt{2(g(n)+f(n)+1)}+1/2.$$

Let $M := \max_{0 \leq k \leq G} P(k)$, and let $n \geq M^2(M^2+1)/2$ be arbitrary. Now if $g(n) \geq G$, then we know the claim holds. Therefore, we can assume $g(n) < G$. But if this is the case, then we know $P(g(n)) \leq M$, which implies

$$P(g(n)) \leq M \leq \sqrt{f(n)} \leq \sqrt{2(g(n)+f(n)+1)}+1/2.$$

Therefore, for all $n \geq M^2(M^2+1)/2 =: N_P$, the first equation holds. In the same way, we may find a constant N_Q after which the second inequality holds. Thus, taking $N := \max\{N_P, N_Q\}$ proves the existence of such an integer N.

Now proving that we may in fact take N to be $2,500,000$, follows from somewhat lengthy but routine refinements of the previous argument. In the above notation, the main idea is to first obtain any analytic upper bound on G, which is then refined by using computer calculated data to compare $P(r)$ with $\sqrt{4r}$ to make G as small as possible. Using this technique for both N_P and N_Q then proves the claim. $\square$

With this proposition, we are able to prove our main result.

Theorem 18. *Let $P(n)$ and $Q(n)$ be as given. Then if $n \geq 0$ is not one of the 177 known counterexamples tabulated in Table 1 of the appendix (in particular, for all $n > 149,894$), we have*

$$P(n) = f(n) + Q(g(n)) \quad \textit{and} \quad Q(n) = 1 + f(n) + P(g(n)),$$

where as before, the functions $f(n)$ and $g(n)$, given by

$$f(n) := \left\lfloor \sqrt{2n} + 1/2 \right\rfloor = \left[\sqrt{2n} \right] \quad \text{and} \quad g(n) := \frac{f(n)[f(n)+1]}{2} - n,$$

are also the smallest nonnegative integers satisfying $[1 + 2 + \cdots + f(n)] - g(n) = n$.

Proof. If $n \geq 2,500,000$, then the result follows by using the previous proposition to simplify the lower bounds of Theorem 16 and comparing these to the upper bounds in Theorem 10.

On the other hand, if $0 \leq n < 2,500,000$, then the result holds by performing an exhaustive computer seach for counterexamples[5]. There are only 177 counterexamples in this range, as tabulated in Table 1 of the appendix. In particular, if $n > 149,894$, then the claim holds since $149,894$ is the largest counterexample. □

8. Corollaries and Remarks

There are many interesting implications of Theorem 18; from this result, many things can be discovered about the behavior of $P(n)$ and $Q(n)$, and the intimate connection between these two functions is made evident. Although these results can be formulated simply as algebraic statements about the recurrence relations, the corresponding geometric statements about the graphs of these functions is perhaps more enlightening.

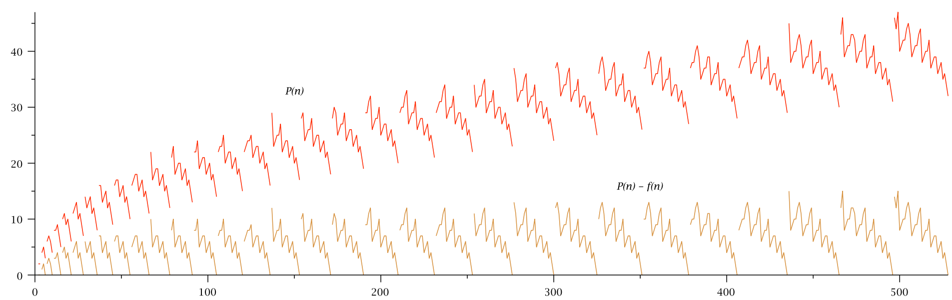

Figure 1: Graph of $P(n)$ (*higher*) and $P(n) - f(n) = P(n) - [\sqrt{2n}]$ (*lower*)

Examining Figures 1 and 2 suggests several apparent patterns of the graphs of these functions. For example, we see that the graphs $P(n)$ and $Q(n)$ are each "drifting" upwards by a translation of $f(n)$. After compensating for this drift, the patterns in the graphs become more apparent.

[5]A brief discussion of the algorithms used for this search is provided in Section 9. Code is available on request.

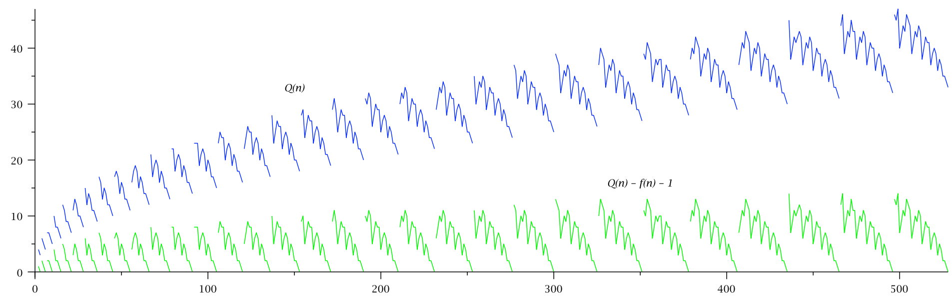

Figure 2: Graph of $Q(n)$ (*higher*) and $Q(n) - f(n) - 1 = Q(n) - [\sqrt{2n}] - 1$ (*lower*)

Now the curves $P(n) - f(n)$ and $Q(n) - f(n) - 1$ (shown as the 'lower' curves in the previous figures) appear to be almost "periodic" in a sense, with zeroes at $0, 1, 3, 6, 10, \ldots$. This apparent behavior is even more pronounced when the values of these functions are laid out in the following triangular array

$$\begin{array}{ccccc} & & \{a_n\}_{n=0}^{\infty} & & \\ & & & & a_0 \\ & & & & a_1 \\ & & & a_2 & a_3 \\ & & a_4 & a_5 & a_6 \\ & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}, \text{ yielding } \begin{array}{ccccc} & & \{(f(n), g(n))\}_{n=0}^{\infty} & & \\ & & & & (0,0) \\ & & & & (1,0) \\ & & & (2,1) & (2,0) \\ & & (3,2) & (3,1) & (3,0) \\ & (4,3) & (4,2) & (4,1) & (4,0) \\ (5,4) & (5,3) & (5,2) & (5,1) & (5,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}.$$

Then arranging values in this triangular manner, we have

$$\begin{array}{cccccccccc} & & \multicolumn{6}{c}{\{P(n) - f(n)\}_{n=0}^{\infty}} & & \\ & & & & & & & & & 0 \\ & & & & & & & & & 0 \\ & & & & & & & & 0 & 0 \\ & & & & & & & 1 & 2 & 0 \\ & & & & & & 2 & 3 & 2 & 0 \\ & & & & & 3 & 3 & 4 & 2 & 0 \\ & & & & 4 & 5 & 3 & 4 & 2 & 0 \\ & & & 4 & 5 & 6 & 3 & 4 & 2 & 0 \\ & & 6 & 4 & 5 & 6 & 3 & 4 & 2 & 0 \\ & 7 & 7 & 4 & 5 & 6 & 3 & 4 & 2 & 0 \\ 6 & 7 & 7 & 4 & 5 & 6 & 3 & 4 & 2 & 0 \end{array} \qquad \begin{array}{cccccccccc} & & \multicolumn{6}{c}{\{Q(n) - f(n) - 1\}_{n=0}^{\infty}} & & \\ & & & & & & & & & \text{-1} \\ & & & & & & & & & 0 \\ & & & & & & & & 1 & 0 \\ & & & & & & & 2 & 1 & 0 \\ & & & & & & 2 & 2 & 1 & 0 \\ & & & & & 4 & 2 & 2 & 1 & 0 \\ & & & & 5 & 4 & 2 & 2 & 1 & 0 \\ & & & 3 & 5 & 4 & 2 & 2 & 1 & 0 \\ & & 6 & 3 & 5 & 4 & 2 & 2 & 1 & 0 \\ & 7 & 6 & 3 & 5 & 4 & 2 & 2 & 1 & 0 \\ 6 & 7 & 6 & 3 & 5 & 4 & 2 & 2 & 1 & 0 \end{array}.$$

Then it appears that the rows (read from right to left) of the triangle for $\{P(n) - f(n)\}$ 'approach' $0, 2, 4, 3, 6, 5, 4, 7, 7, 6, \ldots$, and the rows of $\{Q(n) - f(n) - 1\}$ 'approach' $0, 1, 2, 2, 4, 5, 3, 6, 7, 6, \ldots$. Moreover, these two sequences seem to be just

$\{Q(t)\}$ and $\{P(t)\}$, respectively. In fact, this follows as our first corollary of Theorem 18:

Corollary 19. *Let $\{P(n)-f(n)\}_{n=0}^{\infty}$ and $\{Q(n)-f(n)-1\}_{n=0}^{\infty}$ be arranged in the triangular manner previously discussed. Then unless n is one of the 177 counterexamples in Table 1 of the appendix, reading the rows of $\{P(n)-f(n)\}$ from to right to left exactly agrees with $Q(t)$, and reading the rows of $\{Q(n)-f(n)-1\}$ exactly agrees with $P(t)$.*

Proof. This follows immediately by how the triangular array was constructed. □

Formulating this as a geometric statement is to say that except for 177 particular points, each "lump" in the graphs of $P(n)-f(n)$ and $Q(n)-f(n)-1$ is simply a reflection of a partial copy of $Q(n)$ or $P(n)$, respectively. Thus, the graph of $P(n)$ eventually consists solely of "shifted" and reflected partial copies of $Q(n)$, and similarly the graph of $Q(n)$ eventually consists solely of "shifted" and reflected partial copies of $P(n)$. This mutual similarity of the two functions also induces self-similarity as shown in the following results.

Corollary 20. *If $g(n) < f(n)-1$, and if n and $n-f(n)$ are not one of the 177 values in* ***Table 1****,*

$$P(n) = 1 + P(n-f(n)) \quad \textit{and} \quad Q(n) = 1 + Q(n-f(n)).$$

Proof. Use Theorem 18 and note if $g(n) \neq f(n)-1$, then $g(n) = g(n-f(n))$. □

This corollary states that with a finite number of exceptions, unless n is one of the values at the far left of a row, the value for n in the triangle for $\{P(n)\}_{n=0}^{\infty}$ (or $\{Q(n)\}_{n=0}^{\infty}$) is one more than the value directly above that entry in the triangle.

Corollary 21. *If n and $g(n)$ are not one of the 177 values listed in Table 1 of the appendix (and in particular, if $g(n) > 149,894$), then we have*

$$\begin{aligned} P(n) &= 1 + f(n) + f(g(n)) + P(g^2(n)) \qquad \textit{and} \\ Q(n) &= 1 + f(n) + f(g(n)) + Q(g^2(n)). \end{aligned}$$

Proof. This follows immediately by applying Theorem 18 twice. □

This last recurrence is readily 'solved' yielding the quasi-explicit equations:

Proposition 22. *For all $n \geq 0$, let $\phi(n; 149,894) = \phi(n)$ denote the smallest nonnegative integer satisfying $g^{\phi(n)}(n) \leq 149,894$. Then for all $n \geq 0$, we have*

$$P(n) = \begin{cases} P(g^{\phi(n)}(n)) + \sum_{i=1}^{\phi(n)} f(g^{i-1}(n)) + \phi(n)/2 & \textit{if } \phi(n) \textit{ is even} \\ Q(g^{\phi(n)}(n)) + \sum_{i=1}^{\phi(n)} f(g^{i-1}(n)) + [\phi(n)-1]/2 & \textit{if } \phi(n) \textit{ is odd,} \end{cases}$$

$$Q(n) = \begin{cases} Q(g^{\phi(n)}(n)) + \sum_{i=1}^{\phi(n)} f(g^{i-1}(n)) + \phi(n)/2 & \textit{if } \phi(n) \textit{ is even} \\ P(g^{\phi(n)}(n)) + \sum_{i=1}^{\phi(n)} f(g^{i-1}(n)) + [\phi(n)+1]/2 & \textit{if } \phi(n) \textit{ is odd.} \end{cases}$$

Proof. This follows easily from the previous corollary. Although the function $\phi(n)$ is much too elusive for most honest mathematicians to call these equations truly "explicit", they ought not be considered recursive. This is because even though P and Q are referenced on the right-hand side, their arguments are bounded; therefore, by appealing to Table 1, those terms are effectively known. □

This gives rise to the following, perhaps surprising fact:

Corollary 23. *Let $P(n)$ and $Q(n)$ be as given. Then for all $n \geq 0$, we have*

$$-1 \leq Q(n) - P(n) \leq 2.$$

Proof. For all $n \geq 0$, we can appeal to Proposition 22 to obtain that

$$Q(n) - P(n) = \begin{cases} Q(g^{\phi(n)}(n)) - P(g^{\phi(n)}(n)), & \text{if } \phi(n) \text{ is even,} \\ P(g^{\phi(n)}(n)) - Q(g^{\phi(n)}(n)) + 1, & \text{if } \phi(n) \text{ is odd.} \end{cases}$$

Moreover, for our purposes, we can assume that $g^{\phi(n)}(n)$ is one of the 177 counterexamples tabulated in Table 1 or else we could continue to appeal to Theorem 18 until this is the case. But looking at a table of these 177 values, we see that if k is one of those exceptions, then $0 \leq Q(k) - P(k) \leq 2$. □

Corollary 24. *Recursively define the function $R(n)$ by $R(0) := 0$, and for $n \geq 1$, $R(n) := 1/2 + f(n) + R(g(n))$. Then for all $n \geq 0$, we have*

$$0 \leq R(n) - \frac{P(n) + Q(n)}{2} \leq 9,$$

$$-1/2 \leq R(n) - P(n) \leq 9, \qquad \textit{and} \qquad -1 \leq R(n) - Q(n) \leq 8 + 1/2.$$

Proof. This is proven just as the last corollary. □

9. Conclusion

We conclude by discussing applications for computing $P(n)$ and $Q(n)$ and by listing some open questions.

9.1. "Sufficiently Large" and Computer Algorithms

In Proposition 17, we state results that hold for all sufficiently large values of n (in particular, for all $n \geq 2,500,000$). We then use this result to prove Theorem 18, and we use a computer aided search to completely classify all counterexamples, which brings up a brief discussion of algorithms.

The most naïve approach to compute $P(n)$ would be simply to list all sets of volume n and find which has the smallest perimeter. This would require roughly $\mathcal{O}(2^n)$ time and $\mathcal{O}(n)$ memory, which is much too slow for large n, and a different approach is needed.

Using the recurrence relations in Section 4, dynamic programming enables us to design algorithms for computing $P(n)$ and $Q(n)$ taking $\mathcal{O}\left(n^2 f(n)\right) = \mathcal{O}\left(n^{2.5}\right)$ time and using $\mathcal{O}\left(n^2\right)$ memory. We can reduce this memory requirement to roughly $\mathcal{O}(n)$ by employing a custom data structure, which benefits from the fact that for fixed n, functions such as $p(n;k)$ seem to take very few distinct values. Using these algorithms, the author was able to check all values of $P(n)$ and $Q(n)$ for $n \leq 3,500,000$, which is more than enough to obtain the results of Theorem 18.

These computations were verified on multiple machines, which collectively took several days of running time and used a few megabytes of memory. However, now that we have used these results to rigorously prove Theorem 18 and Proposition 22, we may use these to compute $P(n)$ or $Q(n)$ in only $\mathcal{O}(\phi(n)) \leq \mathcal{O}(\log_2 \log_2(n/2))$ time using no additional memory, which is a *vast* improvement. Moreover, we can compute a list of $P(0), P(1), \ldots, P(n)$ [or $Q(0), Q(1), \ldots, Q(n)$] in $\mathcal{O}(n)$ time using only the required $\mathcal{O}(n)$ memory.

Thus, one can now simply use Theorem 18 and the 177 values in Table 1 to compute $P(n)$ and $Q(n)$ extremely quickly, and $P(n)$ and $Q(n)$ can be tabulated essentially as far out as desired. The author is more than willing to provide anyone interested with code and calculated results.

10. Open Questions

There are several possible areas of future research. Because the function $P(n)$ was first introduced so recently, this paper serves as a comprehensive overview of all that is known.

- Little is known about the behavior of the functions $p(n;k)$, $q(n;k)$, and $\sigma(n;k)$.

- It appears that for any fixed $n \leq 100,000$ the function $p(n;k)$ takes at most two finite values as k varies. This may be interesting and might be proveable by focusing on Proposition 17.

- Very little or nothing whatsoever is known about $\phi(n;N)$ from Proposition 22.

- Nothing is known about the function $R(n)$ of Corollary 24.

- Characterizing sets for which $P(n)$ is obtained may be interesting. It seems likely that the partitions used and the code developed in this paper would help with that. Moreover, the result of Theorem 18 seems likely to help with this.

– Providing more direct (i.e., less analytic) proofs for these results would likely be quite enlightening.

– There seems to be no pattern or unifying properties for the 177 counterexamples tabulated in Table 1. Alternate proofs of the main results may shed light on these seemingly sporadic values.

– This paper considered the function $\min_{A \subseteq X}\{per(A) : vol(A) = n\}$ for $X = \{0, 1, 2, \ldots\}$, and [2] also considered this function for the set $X = \mathbb{Z}$. It may be interesting to consider the corresponding function for different ambient sets X. For instance, $X = \{1, 2, 3, \ldots\}$ or $X = \{a_1, a_2, a_3, \ldots\}$, where the boundary of $A \subseteq X$ is defined as $\partial A := \{a_i : \{a_{i-1}, a_{i+1}\} \nsubseteq A\}$ may be interesting.

References

[1] Devlin, Patrick, Sets with High Volume and Low Perimeter, see `arXiv:1107.2954v1 [math.CO]`.

[2] Miller, Steven J.; Morgan, Frank; Newkirk, Edward; Pedersen, Lori; and Seferis, Deividas *Isoperimetric Sets of Integers* , Mathematics Magazine **84** (2011), 37–42.

Appendix

The 177 counterexamples to Theorem 18 are tabulated below. Entries of the form (*123*) are not actually counterexamples to the theorem, and they are included here only for completeness.

n	P(n)	Q(n)	n	P(n)	Q(n)
(*0*)	0	0	2508	(*88*)	88
2	2	(*4*)	2581	89	89
4	4	(*6*)	2867	(*94*)	94
7	6	(*7*)	2945	95	95
8	7	(*7*)	3250	(*100*)	100
11	8	(*10*)	3333	101	101
16	10	(*12*)	3336	(*103*)	104
17	11	(*11*)	3503	104	105
29	14	(*15*)	3588	104	104
92	(*22*)	23	3657	(*106*)	106
125	25	(*25*)	3745	107	107
154	28	28	3748	(*109*)	110
155	29	(*29*)	3925	110	111
174	29	29	4015	110	110
361	(*38*)	38	4016	111	(*112*)
390	39	(*39*)	4107	111	111
441	(*42*)	42	4466	116	116
473	43	43	4467	117	(*118*)
529	(*46*)	46	4563	117	117
564	47	47	4564	118	(*119*)
601	49	(*50*)	4661	118	118
637	49	49	5186	(*123*)	124
704	54	(*55*)	5289	123	123
742	53	53	5806	(*130*)	131
743	54	55	5915	130	130
783	54	54	6026	131	131
837	(*53*)	54	6461	(*137*)	138
1003	(*58*)	59	6576	137	137
1147	62	62	6693	138	138
1184	(*63*)	64	6811	139	139
1340	67	67	7151	(*144*)	145
1341	68	(*69*)	7272	144	144
1380	(*68*)	69	7395	145	145
1394	68	68	7396	146	(*146*)
1548	72	72	7436	(*143*)	143
1549	73	(*74*)	7519	146	146
1606	73	73	8003	151	151
1665	74	74	8132	152	152
1771	77	77	8133	153	(*153*)
1772	78	(*79*)	8262	153	153
1833	78	78	8305	(*151*)	151
1896	79	79	9222	(*159*)	159
2173	(*82*)	82	9454	163	163
2241	83	83	10086	(*163*)	164
2279	86	86	10187	(*167*)	167

(Continued on the next page)

n	**P(n)**	**Q(n)**
10478	169	(*169*)
11200	(*175*)	175
11245	(*172*)	173
11505	177	(*177*)
12261	(*183*)	183
12467	(*181*)	182
12580	185	(*185*)
12583	(*187*)	188
12904	188	189
13066	188	188
13370	(*191*)	191
13703	193	(*193*)
13752	(*190*)	191
14041	196	197
14210	196	196
14381	197	197
15052	204	(*205*)
15227	205	(*206*)
15402	204	204
15403	205	206
15580	205	205
15759	206	206
16511	(*208*)	209
17254	(*214*)	215
17441	214	214
17985	(*217*)	218
18955	223	223
19152	(*224*)	224
19522	(*226*)	227
20532	(*232*)	232
20533	233	(*234*)
20737	233	233
21122	(*235*)	236
21961	(*241*)	242
22172	241	241
22173	242	(*243*)
22385	242	242
22654	(*241*)	241
22814	244	(*244*)
23656	(*250*)	251
23875	250	250
23876	251	(*252*)
24096	251	251
24541	253	(*253*)
24598	(*251*)	251
26855	262	262
28726	(*271*)	271
28783	(*268*)	269
28968	272	272
30910	(*281*)	281
31161	282	282
33174	(*291*)	291
33434	292	292
35518	(*301*)	301
35787	302	302
36391	(*301*)	302
37147	307	307
39125	(*312*)	313
39625	317	317
39626	318	319
39909	318	318
41958	(*323*)	324
44890	(*334*)	335
47921	(*345*)	346
50126	353	353
51051	(*356*)	357
53326	364	364
53327	365	(*365*)
53655	365	365
56625	375	375
56626	376	(*376*)
56964	376	376
61851	(*389*)	389
65764	(*401*)	401
66129	402	(*402*)
69797	(*413*)	413
70173	414	(*414*)
73950	(*425*)	425
74337	426	(*426*)
78223	(*437*)	437
78621	438	(*438*)
108375	510	510
114014	523	523
129359	(*554*)	554
136036	(*568*)	568
142881	(*582*)	582
149894	(*596*)	596

Table 1: Comprehensive list of exceptions to Theorem 18.