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ON BOUNDS FOR TWO DAVENPORT-TYPE CONSTANTS

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Abstract

Let G be an additive abelian group of finite order n and let A be a non-empty set of integers. The Davenport constant of G with weight A, $D_A(G)$, is the smallest $k \in \mathbb{Z}^+$ such that for any sequence $x_1, \ldots, x_k$ of elements in G, there exists a non-empty subsequence $x_{j_1}, \ldots, x_{j_r}$ and corresponding weights $a_1, \ldots, a_r \in A$ such that $\sum_{i=1}^{r} a_i x_{j_i} = 0$. Similarly, $E_A(G)$ is the smallest positive integer k such that for any sequence $x_1, \ldots, x_k$ of elements in G there exists a non-empty subsequence of exactly n terms, $x_{j_1}, \ldots, x_{j_n}$, and corresponding weights $a_1, \ldots, a_n \in A$ such that $\sum_{i=1}^{n} a_i x_{j_i} = 0$. We consider these constants when $G = \mathbb{Z}_n$ and $A = \{b^2 | b \in \mathbb{Z}_n^*\}$, proving lower bounds for each.

1. Introduction

Let G be an additive abelian group of finite order n. The Davenport constant of G, $D(G)$, is the smallest $k \in \mathbb{Z}^+$ such that for any sequence $x_1, \ldots, x_k$ of elements in G, there exists a non-empty subsequence $x_{j_1}, \ldots, x_{j_r}$ such that $\sum_{i=1}^{r} x_{j_i} = 0$. Let A be a non-empty set of integers. The Davenport constant of G with weight A, $D_A(G)$, is the smallest $k \in \mathbb{Z}^+$ such that for any sequence $x_1, \ldots, x_k$ of elements in G, there exists a non-empty subsequence $x_{j_1}, \ldots, x_{j_r}$ and corresponding weights $a_1, \ldots, a_r \in A$ such that $\sum_{i=1}^{r} a_i x_{j_i} = 0$. Similarly, $E_A(G)$ is the smallest positive integer k such that for any sequence $x_1, \ldots, x_k$ of elements in G there exists a non-empty subsequence of exactly n terms, $x_{j_1}, \ldots, x_{j_n}$, and corresponding weights $a_1, \ldots, a_n \in A$ such that $\sum_{i=1}^{n} a_i x_{j_i} = 0$.

In 2008, Adhikari, David, and Urroz [1] considered the case where G is $\mathbb{Z}_n$, the cyclic group of order n, and A is the set of quadratic residues modulo n,

$$A = A_n = \{b^2 | b \in \mathbb{Z}_n^*\}, \tag{1}$$

proving a collection of bounds for each of these constants. Unfortunately, the first theorem in that paper holds only for odd integers. In this work, we provide some counter-examples in the even case, then state and prove a corrected version of the theorem, explaining the error made in the original proof.

Fix $n \geq 2$, let $G = \mathbb{Z}_n$, and let $A = A_n$, as defined in (1). Let $\Omega(n)$ denote the number of prime factors of n counting multiplicity and let $\Omega_o(n)$ denote the number of odd prime factors of n counting multiplicity.

In [1, Theorem 1], it is claimed that

$$D_A(\mathbb{Z}_n) \geq 2\Omega(n) + 1 \quad \text{and} \quad E_A(\mathbb{Z}_n) \geq 2\Omega(n) + n. \tag{2}$$

Theorem 1. *The bounds in (2) are incorrect for even n. For example,*

$$\begin{array}{ll} D_A(\mathbb{Z}_2) = 2 < 3 = 2\Omega(2) + 1, & E_A(\mathbb{Z}_2) = 3 < 4 = 2\Omega(2) + n, \\ D_A(\mathbb{Z}_4) = 4 < 5 = 2\Omega(2) + 1, & E_A(\mathbb{Z}_4) = 7 < 8 = 2\Omega(2) + n, \\ D_A(\mathbb{Z}_{10}) = 4 < 5 = 2\Omega(2) + 1, & E_A(\mathbb{Z}_{10}) = 13 < 14 = 2\Omega(2) + n. \end{array}$$

Proof. First note that $A_2 = A_4 = \{1\}$ and so, for $n = 2$ or 4, $D_A(\mathbb{Z}_n) = D(\mathbb{Z}_n) = n$, from which the first two examples follow. For $n = 10$, we have $A = A_{10} = \{1, -1\}$, and so, from [2, Lemma 2.1], it follows that $D_A(10) \leq \lfloor \log_2 10 \rfloor + 1 = 4 < 5$. The remaining results follow, for $A = \{1\}$, from $E_A(G) = D_A(G) + n - 1$, which was proved in [3] and, for $A = \{-1, 1\}$, from $E_A(G) = n + \lfloor \log_2 n \rfloor$, which was proved in [2]. □

2. Corrected Version of the Theorem

We now state and prove our corrected version of the theorem. The proof follows closely the proof of the original theorem in [1].

Theorem 2. *For $n \geq 2$, $D_A(\mathbb{Z}_n) \geq 2\Omega_o(n) + 1$ and $E_A(\mathbb{Z}_n) \geq 2\Omega_o(n) + n$.*

Proof. Given $n \geq 2$, let $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, with $\alpha_0 \geq 0$ and $\alpha_i \geq 1$ for $i \geq 1$. To prove the first inequality, it suffices to produce a sequence of $2\Omega_o(n) = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_r)$ terms with no non-zero weighted zero-sum subsequence.

For each $1 \leq i \leq r$, fix $v_i \in \mathbb{Z}_n$ such that, modulo p_i, $v_i \notin A_{p_i} \cup \{0\}$. (Note that, since $p_i > 2$, $A_{p_i} \cup \{0\} \subsetneq \mathbb{Z}_{p_i}$, while $A_2 \cup \{0\} = \mathbb{Z}_2$. This is precisely the problem invalidating the proof giving in [1]: it was not possible for a v_2 to exist satisfying the given conditions.)

For $1 \leq i \leq r$ and $0 \leq j_i \leq \alpha_i - 1$, define $x_{i,j_i} = np_i^{j_i - \alpha_i}$ and $y_{i,j_i} = -v_i x_{i,j_i}$. Let S be the $2\Omega_o(n)$-term sequence:

$$x_{1,0}, y_{1,0}, x_{1,1}, y_{1,1}, \ldots, x_{1,\alpha_1-1}, y_{1,\alpha_1-1}, x_{2,0}, \ldots, y_{2,\alpha_2-1}, \ldots, x_{r,\alpha_r-1}, y_{r,\alpha_r-1}.$$

Suppose that S has a non-empty weighted zero-sum subsequence. Then there exist $s_{i,j_i}, t_{i,j_i} \in A_n \cup \{0\}$, not all zero, such that

$$\sum_{i,j_i} (s_{i,j_i} x_{i,j_i} + t_{i,j_i} y_{i,j_i}) = 0. \tag{3}$$

Fix an arbitrary k, $1 \le k \le r$ and notice that for $(i, j_i) \neq (k, 0)$, $p_k | x_{i,j_i}$ and $p_k | y_{i,j_i}$. So reducing equation (3) modulo p_k yields

$$s_{k,0}x_{k,0} + t_{k,0}y_{k,0} \equiv 0 \pmod{p_k}. \tag{4}$$

Since $x_{k,0}$ is a unit modulo p_k, the congruence simplifies to

$$s_{k,0} \equiv v_k t_{k,0} \pmod{p_k}. \tag{5}$$

Suppose that $s_{k,0} \neq 0$. Then, recalling that $s_{k,0}$, $t_{k,0} \in A_n \cup \{0\}$, it follows that $s_{k,0} \not\equiv 0 \pmod{p_k}$, and so $t_{k,0} \neq 0$. Thus, there exist units, u_1, $u_2 \in \mathbb{Z}_n$ such that ${u_1}^2 = s_{k,0}$ and $u_2^2 = t_{k,0}$. But then, by (5), $v_k \equiv (u_1 u_2^{-1})^2 \pmod{p_k}$, which is a contradiction, since $v_k \notin A_{p_k}$. Thus $s_{k,0} = 0$ and so $v_k t_{k,0} \equiv 0 \pmod{p_k}$. Since v_k is defined to be non-zero modulo p_k, $t_{k,0} \equiv 0 \pmod{p_k}$, and thus $t_{k,0} = 0$.

Now, fix ℓ, $0 < \ell < \alpha_k$, and assume by induction that for all $j_k < \ell$, $s_{k,j_k} = t_{k,j_k} = 0$. Reducing equation (3) modulo $p_k^{\ell+1}$, yields

$$s_{k,j_k}x_{k,j_k} + t_{k,j_k}y_{k,j_k} \equiv 0 \pmod{p_k^{\ell+1}}.$$

Dividing through by p_k^ℓ, we find that

$$s_{k,\ell}\frac{x_{k,\ell}}{p_k^\ell} + t_{k^\ell}\frac{y_{k,\ell}}{p_k^\ell} \equiv 0 \pmod{p_k}.$$

So $\dfrac{x_{k,\ell}}{p_k^\ell}(s_{k,\ell} - v_k t_{k,\ell}) \equiv 0 \pmod{p_k}$. Since $\dfrac{x_{k,\ell}}{{p_k}^\ell}$ is a unit modulo p_k, $s_{k,\ell} \equiv v_k t_{k,\ell} \pmod{p_k}$. Using the same arguments as above, $s_{k,\ell} = t_{k,\ell} = 0$. Hence by induction, for all j_k, $s_{k,j_k} = t_{k,j_k} = 0$. Since k was arbitrary, we have that for all i, j_i, $s_{i,j_i} = t_{i,j_i} = 0$, which is a contradiction.

Hence, S is a sequence of length $2\Omega_o(n)$ that does not have a non-empty weighted zero-sum subsequence. Therefore, $D_A(n) \ge 2\Omega_o(n) + 1$, as desired.

Finally, to prove the bound on $E_A(n)$, let T be a sequence of length $D_A(n)-1$ with no weighted zero-sum subsequence. Let T' be the sequence obtained by appending $n-1$ zeros to T. Then T' is a sequence of $D_A(n) + n - 2$ terms with no zero-sum subsequence of exactly n terms. Thus, $E_A(n) > D_A(n) + n - 2$, and so $E_A(n) \ge 2\Omega_o(n) + n$. □

References

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