MORE ON FINITE SUMS THAT INVOLVE RECIPROCALS OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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Abstract

In this paper we find closed forms, in terms of rational numbers, for certain finite sums. In each case the denominator of the summand consists of products of generalized Fibonacci numbers. Specifically, the denominator of each summand that we consider is a finite product $W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}\cdots$ in which $0 < m_1 < m_2 < \cdots$ are integers, $k \geq 1$ and $m \geq 0$ are integers, and $\{W_n\}$ is a sequence of generalized Fibonacci numbers. Here, therefore, we take a more general approach than that previously taken, where $(m_1, m_2, m_3, \dots)$ was taken to be $(1, 2, 3, \dots)$.

1. Introduction

We begin by establishing the notation for three pairs of integer sequences that are featured in this paper. Let $a \geq 0$ and $b \geq 0$ be integers with $(a, b) \neq (0, 0)$. For p a positive integer we define, for all integers n, the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b,$$

and

$$\overline{W}_n = W_{n-1} + W_{n+1}.$$

For $(a, b, p) = (0, 1, 1)$ we have $\{W_n\}=\{F_n\}$, and $\{\overline{W}_n\}=\{L_n\}$, which are the Fibonacci and Lucas numbers, respectively. Retaining the generality afforded by the parameter p, and taking $(a, b) = (0, 1)$, we write $\{W_n\}=\{U_n\}$, and $\{\overline{W}_n\}=\{V_n\}$, which are integer sequences that generalize the Fibonacci and Lucas numbers, respectively.

[1] I dedicate this paper to the memory of the late Samuel Melham. He was father, mentor, role model, and dear friend.

Let α and β denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then the closed forms (the Binet forms) for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

and

$$\overline{W}_n = A\alpha^n + B\beta^n.$$

Set $\Delta = p^2 + 4$. Then $\overline{U}_n = V_n$, and $\overline{V}_n = \Delta U_n$, so that $\overline{F}_n = L_n$, and $\overline{L}_n = 5F_n$. We require also the constant $e_W = AB = b^2 - pab - a^2$.

Throughout this paper we take $k \geq 1$, $m \geq 0$, and $n \geq 2$ to be integers. Also let $0 < m_1 < m_2 < m_3$ be integers. Define the following sums:

$$S_2(k, m, n, m_1) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}}{W_{ki+m}W_{k(i+m_1)+m}},$$

$$S_3^U(k, m, n, m_1, m_2) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}U_{k(i+m_1)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}},$$

$$S_3^V(k, m, n, m_1, m_2) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}V_{k(i+m_1)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}},$$

$$S_3^{\overline{W}}(k, m, n, m_1, m_2) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}_{k(i+m_1)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}},$$

and

$$S_4(k, m, n, m_1, m_2, m_3) = \sum_{i=1}^{n-1} \frac{1}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}W_{k(i+m_3)+m}}.$$

In this paper we evaluate each of the five sums above in terms of rational numbers. In so doing we take a more general approach than in [2], where we considered the same five sums with $(m_1, m_2, m_3) = (1, 2, 3)$. In [2] we also gave closed forms for similar sums that had longer products in the denominator of the summand. We do likewise here, but we work in a more general setting. Specifically, in [2] the denominators of the summands that we considered were finite products $W_{ki+m}W_{k(i+1)+m}W_{k(i+2)+m}\cdots$. In this paper the denominators of the summands that we consider are finite products $W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}\cdots$ in which $0 < m_1 < m_2 < \cdots$ are integers. For background and references we refer the interested reader to Section 2 in [2].

In Section 2 we present one result, the closed form for S_2, and this is the only result of its type that we have found. In each of Sections 3-8 we present a selection of the results that we have found. Our aim is to highlight the kinds of results that we have discovered, rather than to present long lists of results. In Section 9 we give a sample proof.

There is a finite sum that features throughout. For integers $0 \le l_1 < l_2$ this finite sum is

$$\Omega(k,m,n,l_1,l_2) = \sum_{i=l_1}^{l_2-1} \frac{(-1)^{ki}}{W_{k(i+2)+m}W_{k(i+n)+m}}.$$

2. A Closed Form for S_2

Theorem 1. *With S_2 as defined in Section 1, we have*

$$U_{m_1k}\left(S_2(k,m,n,m_1) - S_2(k,m,2,m_1)\right) = U_{k(n-2)}\Omega(k,m,n,0,m_1).$$

3. Closed Forms for S_3^U, S_3^V, and $S_3^{\overline{W}}$

For integers $k \ge 1$ and $0 < m_1 < m_2$ define

$$\begin{aligned}
c_1(k,m_1,m_2) &= W_{-m_1k}U_{(m_2-m_1)k},\\
c_2(k,m_1,m_2) &= U_{m_1k}W_{(m_2-m_1)k},\\
c_3(k,m_1,m_2) &= \overline{W}_{-m_1k}U_{(m_2-m_1)k},\\
c_4(k,m_1,m_2) &= U_{m_1k}\overline{W}_{(m_2-m_1)k},\\
c_5(k,m_1,m_2) &= V_{m_1k}U_{(m_2-m_1)k},\\
c_6(k,m_1,m_2) &= U_{m_1k}V_{(m_2-m_1)k},\\
c_7(k,m_1,m_2) &= U_{m_1k}U_{m_2k}U_{(m_2-m_1)k}.
\end{aligned}$$

To prevent the presentation from becoming too unwieldy we suppress certain arguments from quantities when there is no danger of confusion. For instance, c_1 denotes $c_1(k,m_1,m_2)$ when all the parameters are fixed; $S_3^U(k,m,n,m_1,m_2)$ is replaced by $S_3^U(n)$ when n varies and the other parameters are fixed; $\Omega(l_1,l_2)$ denotes $\Omega(k,m,n,l_1,l_2)$ when l_1 and l_2 vary and the other parameters are fixed.

Theorem 2. *Let S_3^U be as defined in Section 1. Then*

$$e_W c_7\left(S_3^U(n) - S_3^U(2)\right) = (-1)^{km_1+1}U_{k(n-2)}\left(c_1\Omega(0,m_1) - (-1)^{km_1}c_2\Omega(m_1,m_2)\right).$$

Theorem 3. *Let S_3^V be as defined in Section 1. Then*

$$e_W c_7\left(S_3^V(n) - S_3^V(2)\right) = (-1)^{km_1}U_{k(n-2)}\left(c_3\Omega(0,m_1) - (-1)^{km_1}c_4\Omega(m_1,m_2)\right).$$

Theorem 4. *Let $S_3^{\overline{W}}$ be as defined in Section 1. Then*

$$c_7\left(S_3^{\overline{W}}(n)-S_3^{\overline{W}}(2)\right)=U_{k(n-2)}\left(c_5\Omega(0,m_1)-c_6\Omega(m_1,m_2)\right).$$

4. Where the Summand has Four Factors in the Denominator

In this section we give the closed form for S_4. We also give the closed form for another finite sum in which the denominator of the summand has four factors. Furthermore, we indicate two similar finite sums whose closed forms we were able to find.

Let $0<m_1<m_2<m_3$ be integers. For $8\le i\le 10$ define $c_i=c_i(k,m_1,m_2,m_3)$ as

$$\begin{aligned}c_8&=U_{m_1k}U_{m_2k}U_{m_3k},\\ c_9&=U_{(m_2-m_1)k}U_{(m_3-m_1)k}U_{(m_3-m_2)k},\\ c_{10}&=U_{m_1k}U_{(m_3-m_2)k}U_{(m_3+m_2-m_1)k}.\end{aligned}$$

With c_7 as defined in the previous section we then have the following theorem which gives the closed form for S_4.

Theorem 5. *With S_4 as defined in Section 1, we have*

$$\begin{aligned}e_Wc_8c_9\left(S_4(n)-S_4(2)\right)&=(-1)^mU_{k(n-2)}\Big(c_9\Omega(0,m_1)-(-1)^{km_1}c_{10}\Omega(m_1,m_2)\\&\quad+(-1)^{k(m_1+m_2+m_3)}c_7\Omega(m_2,m_3)\Big).\end{aligned}$$

Next we give the closed form for a finite sum where the summand has four factors in the denominator and two factors in the numerator. Define

$$T_4(k,m,n,m_1,m_2,m_3)=\sum_{i=1}^{n-1}\frac{(-1)^{ki}\overline{W}_{k(i+m_1)+m}\overline{W}_{k(i+m_2)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}W_{k(i+m_3)+m}}.$$

To discover the closed form for T_4 we needed to consider the specialized case where $m_3=m_1+m_2$. Accordingly, for $1\le i\le 3$ define $d_i=d_i(k,m_1,m_2)$ by

$$\begin{aligned}d_1&=U_{m_1k}^2U_{m_2k}^2U_{(m_2-m_1)k}U_{(m_1+m_2)k},\\ d_2&=U_{2m_1k}U_{2m_2k}U_{(m_2-m_1)k},\\ d_3&=-U_{m_1k}^2U_{m_2k}\left(V_{(2m_2-m_1)k}+3(-1)^{k(m_1+m_2)}V_{m_1k}\right).\end{aligned}$$

We then have

Theorem 6. *Let $0<m_1<m_2$ be integers, and let $m_3=m_1+m_2$. Then*

$$d_1\left(T_4(n)-T_4(2)\right)=U_{k(n-2)}\left(d_2\Omega(0,m_1)+d_3\Omega(m_1,m_2)+d_2\Omega(m_2,m_3)\right).$$

Notice that the assumption $m_3 = m_1 + m_2$ includes all instances where 0, m_1, m_2, and m_3 are in arithmetic progression. With the same assumptions as for Theorem 6, we have found closed forms for two other sums that we do not present here. These are

$$\sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}_{ki+m}\overline{W}_{k(i+m_3)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}W_{k(i+m_3)+m}},$$

and

$$\sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}_{k(i+m_2)+m}\overline{W}_{k(i+m_3)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}W_{k(i+m_3)+m}}.$$

Indeed, there are several other similar sums that can be considered.

At this point we give an example. Let $k = 2$, $m = 0$, and $(m_1, m_2) = (1, 2)$. Then for $W_n = F_n$ the result given in Theorem 6 becomes

$$8\sum_{i=1}^{n-1} \frac{L_{2(i+1)}L_{2(i+2)}}{F_{2i}F_{2(i+1)}F_{2(i+2)}F_{2(i+3)}} = 2 + F_{2(n-2)}\left(\frac{7}{3F_{2n}} - \frac{9}{8F_{2(n+1)}} + \frac{1}{3F_{2(n+2)}}\right).$$

5. Where the Summand has Five Factors in the Denominator

In this section, and in the sections that follow, we require quantities analogous to the c_i and d_i defined in the previous section. For these quantities we also use the pro-numerals c and d. In this section we take $0 < m_1 < m_2 < m_3 < m_4$ to be integers. Define the sum

$$S_5(k, m, n, m_1, \ldots, m_4) = \sum_{i=1}^{n-1} \frac{\overline{W}_{k(i+m_2)+m}}{W_{ki+m}W_{k(i+m_1)+m}\cdots W_{k(i+m_4)+m}}.$$

We now define quantities $c_i = c_i(k, m, m_1, m_2, m_3, m_4)$, for $1 \le i \le 5$, that will help us to succinctly give the closed form for this sum. Define

$$\begin{aligned} c_1 &= U_{m_1k}U_{m_2k}U_{m_3k}U_{m_4k}U^2_{(m_2-m_1)k}, \\ c_2 &= (-1)^m U^2_{(m_2-m_1)k}V_{m_2k}, \\ c_3 &= (-1)^{km_1+m+1}U_{m_1k}U_{m_3k}V_{m_2k}. \end{aligned}$$

We then have

Theorem 7. *Let $0 < m_1 < m_2$ be integers, and let $m_3 = 2m_2 - m_1$ and $m_4 = 2m_2$. Then*

$$\begin{aligned} e_W c_1 \left(S_5(n) - S_2(2)\right) &= U_{k(n-2)}\left(c_2\Omega(0, m_1) + c_3\Omega(m_1, m_2)\right. \\ &\quad \left. - c_3\Omega(m_2, m_3) - c_2\Omega(m_3, m_4)\right). \end{aligned}$$

We were unable to find the closed form for S_5 without the simplifying assumptions $m_3 = 2m_2 - m_1$ and $m_4 = 2m_2$. These assumptions include all instances where 0, m_1, m_2, m_3, and m_4 are in arithmetic progression, and they also bring a nice symmetry to the subscripts in the denominator of the summand. Indeed, under these assumptions the successive differences between the subscripts in the denominator of the summand are km_1, $k(m_2 - m_1)$, $k(m_2 - m_1)$, and km_1.

Next define the sum

$$T_5(k, m, n, m_1, \ldots, m_4) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}_{k(i+m_1)+m} \cdots \overline{W}_{k(i+m_3)+m}}{W_{ki+m}W_{k(i+m_1)+m} \cdots W_{k(i+m_4)+m}},$$

in which the numerator of the summand contains a product of three terms from the sequence $\{\overline{W}_n\}$. For $1 \le i \le 4$ define the quantities $d_i = d_i(k, m_1, m_2, m_3, m_4)$ as

$$\begin{aligned} d_1 &= U^2_{(m_2-m_1)k} V_{m_1k} V_{m_2k} V_{m_3k}, \\ d_2 &= -V^2_{(m_2-m_1)k} U_{m_1k} V_{m_2k} U_{m_3k}. \end{aligned}$$

Then with c_1 as defined at the beginning of this section we have

Theorem 8. *Let $0 < m_1 < m_2$ be integers, and let $m_3 = 2m_2 - m_1$ and $m_4 = 2m_2$. Then*

$$\begin{aligned} c_1 \left(T_5(n) - T_2(2)\right) &= U_{k(n-2)} \left(d_1\Omega(0, m_1) + d_2\Omega(m_1, m_2)\right. \\ &\quad \left. - d_2\Omega(m_2, m_3) - d_1\Omega(m_3, m_4)\right). \end{aligned}$$

We have found closed forms for several similar sums that we do not present here. The most notable of these is

$$\sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}_{ki+m}\overline{W}_{k(i+m_2)+m}\overline{W}_{k(i+m_4)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}W_{k(i+m_3)+m}W_{k(i+m_4)+m}},$$

where the assumptions are the same as for Theorems 7 and 8.

At this point we give a second example. Let $k = 1$, $m = 0$, and $(m_1, m_2) = (1, 2)$. Then for $W_n = L_n$ the result given in Theorem 8 becomes

$$\sum_{i=1}^{n-1} \frac{(-1)^i F_{i+1}F_{i+2}F_{i+3}}{L_iL_{i+1}L_{i+2}L_{i+3}L_{i+4}} + \frac{1}{154} = \frac{F_{n-2}}{125}\left(\frac{2}{3L_n} + \frac{1}{4L_{n+1}} + \frac{1}{7L_{n+2}} + \frac{2}{11L_{n+3}}\right).$$

6. Where the Summand has Six Factors in the Denominator

In this section we take $0 < m_1 < m_2 < m_3 < m_4 < m_5$ to be integers. Furthermore, here, and in the sequel, we take $g \ge 1$ to be an integer.

Define the sum

$$S_6(k,m,n,m_1,\ldots,m_5)=\sum_{i=1}^{n-1}\frac{(-1)^{ki}}{W_{ki+m}W_{k(i+m_1)+m}\cdots W_{k(i+m_5)+m}}.$$

For $1\le i\le 4$ define the quantities $c_i=c_i(g,k)$ as

$$\begin{aligned}c_1 &= U_{gk}U_{2gk}U_{3gk}U_{4gk}U_{5gk},\\ c_2 &= -V_{gk}V_{3gk},\\ c_3 &= (-1)^{gk}\left(V_{6gk}+V_{2gk}+2(-1)^{gk}\right).\end{aligned}$$

In the theorem that follows we take the m_i to be certain multiples of g.

Theorem 9. *Let* $(m_1,m_2,m_3,m_4,m_5)=(g,2g,3g,4g,5g)$. *Then*

$$\begin{aligned}e_W^2c_1\left(S_6(n)-S_6(2)\right) &= U_{k(n-2)}\left(\Omega(0,g)+c_2\Omega(g,2g)+c_3\Omega(2g,3g)\right.\\ &\quad\left.+\,c_2\Omega(3g,4g)+\Omega(4g,5g)\right).\end{aligned}$$

For a second result where the summand has six factors in the denominator define the sum

$$T_6(k,m,n,m_1,\ldots,m_5)=\sum_{i=1}^{n-1}\frac{\overline{W}_{k(i+m_1)+m}\overline{W}_{k(i+m_5)+m}}{W_{ki+m}W_{k(i+m_1)+m}\cdots W_{k(i+m_5)+m}}.$$

For $1\le i\le 6$ define the quantities $d_i=d_i(g,k,m)$ by

$$\begin{aligned}d_1 &= U_{gk}U_{2gk}U_{4gk}U_{5gk}U_{6gk},\\ d_2 &= (-1)^mV_{gk}V_{6gk},\\ d_3 &= (-1)^{m+1}V_{gk}\left(2(-1)^{gk}V_{8gk}-V_{6gk}+2V_{2gk}\right),\\ d_4 &= (-1)^mV_{gk}\left(V_{10gk}-2(-1)^{gk}V_{8gk}+2V_{6gk}+(-1)^{gk}V_{4gk}\right),\\ d_5 &= (-1)^{m+1}V_{gk}\left((-1)^{gk}V_{8gk}+V_{6gk}-2(-1)^{gk}V_{4gk}+3V_{2gk}\right),\\ d_6 &= 2(-1)^{gk+m}V_{5gk}.\end{aligned}$$

Once again, in the theorem that follows we take the m_i to be multiples of g. These multiples are different from those considered in Theorem 9.

Theorem 10. *Let* $(m_1,m_2,m_3,m_4,m_5)=(g,2g,4g,5g,6g)$. *Then*

$$\begin{aligned}e_Wd_1\left(T_6(n)-T_6(2)\right) &= U_{k(n-2)}\left(d_2\Omega(0,g)+d_3\Omega(g,2g)+d_4\Omega(2g,4g)\right.\\ &\quad\left.+\,d_5\Omega(4g,5g)+d_6\Omega(5g,6g)\right).\end{aligned}$$

At this point we give a third numerical example. In theorem 10 let $k=1$, $m=0$, and $g=2$ and take $W_n=F_n$. In order to lucidly state this example we set $a_0=161$,

$a_1 = -\frac{161}{2}$, $a_2 = -\frac{2053}{3}$, $a_3 = \frac{2053}{5}$, $a_4 = \frac{2851}{4}$, $a_5 = -\frac{5702}{13}$, $a_6 = \frac{5702}{21}$, $a_7 = -\frac{2851}{17}$, $a_8 = -\frac{1228}{55}$, $a_9 = \frac{1228}{89}$, $a_{10} = \frac{41}{144}$, $a_{11} = -\frac{41}{233}$. Then

$$83160 \sum_{i=1}^{n-1} \frac{L_{i+2}L_{i+12}}{F_i F_{i+2} F_{i+4} F_{i+8} F_{i+10} F_{i+12}} - \frac{8665272}{352529} = F_{n-2} \sum_{i=0}^{11} \frac{a_i}{F_{n+i}}.$$

Recall that in Theorems 7 and 8 we defined m_3 and m_4 in terms of the two parameters m_1 and m_2, and this enabled us to find the closed forms in question. Indeed, as the number of factors in the denominator of the summand increases we need to express the m_i in terms of fewer parameters in order for us to be able to find the associated closed form. Theorems 9 and 10 deal with finite sums in which the denominator of the summand consists of six factors. In these cases we need to express the m_i in terms of a single parameter g. The same is true when the denominator of the summand consists of seven of more factors, as we see in the sections that follow. We stress that for the results contained in Theorems 9 to 12 we aimed for more generality by trying to express the m_i in terms of m_1 and m_2, but our efforts proved fruitless. Out of necessity, therefore, we resorted to expressing the m_i in terms of the single parameter g. This enabled us to obtain many more results analogous to those presented in Theorems 9 to 12.

7. Where the Summand Has Seven Factors in the Denominator

In this section we take $0 < m_1 < m_2 < m_3 < m_4 < m_5 < m_6$ to be integers. In this section, and the next, we state only one theorem although we have discovered many more.

Define the sum

$$S_7(k, m, n, m_1, \ldots, m_6) = \sum_{i=1}^{n-1} \frac{\overline{W}_{k(i+m_1)+m} \overline{W}_{k(i+m_3)+m} \overline{W}_{k(i+m_5)+m}}{W_{ki+m} W_{k(i+m_1)+m} W_{k(i+m_2)+m} \cdots W_{k(i+m_6)+m}}.$$

For $1 \le i \le 7$ define the quantities $c_i = c_i(g, k, m)$ as

$$\begin{aligned}
c_1 &= U_{3gk}U_{5gk}U_{6gk}U_{7gk}U_{9gk}U_{12gk}, \\
c_2 &= (-1)^m V_{3gk}V_{6gk}V_{9gk}, \\
c_3 &= (-1)^{m+1} V_{6gk} \big(2(-1)^{gk}V_{18gk} + 2(-1)^{gk}V_{14gk} + 3V_{12gk} + 4(-1)^{gk}V_{10gk} \\
&\quad + 2V_{8gk} + 7(-1)^{gk}V_{6gk} + 4V_{4gk} + 6(-1)^{gk}V_{2gk} + 6\big), \\
c_4 &= (-1)^m V_{gk}V_{3gk}V_{6gk} \big((-1)^{gk}V_{18gk} + V_{16gk} + 2(-1)^{gk}V_{14gk} + 4V_{12gk} \\
&\quad + 5(-1)^{gk}V_{10gk} + 6V_{8gk} + 9(-1)^{gk}V_{6gk} + 9V_{4gk} + 10(-1)^{gk}V_{2gk} + 11\big).
\end{aligned}$$

We then have

Theorem 11. *Take* $(m_1, m_2, m_3, m_4, m_5, m_6) = (3g, 5g, 6g, 7g, 9g, 12g)$. *Then*

$$\begin{aligned} e_W c_1 \left(S_7(n) - S_7(2)\right) &= U_{k(n-2)} \left(c_2\Omega(0,3g) + c_3\Omega(3g,5g) + c_4\Omega(5g,6g) \right. \\ &\quad \left. - c_4\Omega(6g,7g) - c_3\Omega(7g,9g) - c_2\Omega(9g,12g)\right). \end{aligned}$$

8. Where the Summand has Eight Factors in the Denominator

Let $0 < m_1 < m_2 < m_3 < m_4 < m_5 < m_6 < m_7$ be integers, and define the sum

$$S_8(k,m,n,m_1,\ldots,m_7) = \sum_{i=1}^{n-1} \frac{\overline{W}_{k(i+m_1)+m}\overline{W}_{k(i+m_2)+m}\overline{W}_{k(i+m_5)+m}\overline{W}_{k(i+m_6)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}\cdots W_{k(i+m_7)+m}}.$$

For $1 \le i \le 8$ define the quantities $c_i = c_i(g,k,m)$ by

$$\begin{aligned} c_1 &= U_{gk}U_{2gk}U_{3gk}U_{5gk}U_{6gk}U_{7gk}U_{8gk}, \\ c_2 &= (-1)^m V_{gk}V_{2gk}V_{6gk}V_{7gk}, \\ c_3 &= (-1)^{m+1}V_{gk}^2 V_{6gk}\left(2(-1)^{gk}V_{10gk} - V_{8gk} + 3(-1)^{gk}V_{6gk} - 4V_{4gk}\right. \\ &\quad \left. + 6(-1)^{gk}V_{2gk} - 2\right), \\ c_4 &= (-1)^m V_{gk}^2\left(2V_{18gk} - 2(-1)^{gk}V_{16gk} + 3V_{14gk} - 3(-1)^{gk}V_{12gk}\right. \\ &\quad \left. + 10V_{10gk} - 4(-1)^{gk}V_{8gk} + 8V_{6gk} - 8(-1)^{gk}V_{4gk} + 13V_{2gk} - 2(-1)^{gk}\right), \\ c_5 &= (-1)^{m+1}V_{gk}V_{5gk}\left((-1)^{gk}V_{16gk} - 2V_{14gk} + 2(-1)^{gk}V_{12gk} + V_{10gk}\right. \\ &\quad \left. + 4(-1)^{gk}V_{8gk} - 4V_{6gk} + 3(-1)^{gk}V_{4gk} + V_{2gk} + 8(-1)^{gk}\right). \end{aligned}$$

We then have

Theorem 12. *Let* $(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = (g, 2g, 3g, 5g, 6g, 7g, 8g)$. *Then*

$$\begin{aligned} e_W c_1 \left(S_8(n) - S_8(2)\right) &= U_{k(n-2)} \left(c_2\Omega(0,g) + c_3\Omega(g,2g) + c_4\Omega(2g,3g) \right. \\ &\quad + c_5\Omega(3g,5g) + c_4\Omega(5g,6g) + c_3\Omega(6g,7g) \\ &\quad \left. + c_2\Omega(7g,8g)\right). \end{aligned}$$

9. A Sample Proof

After some preparation, in the form of certain identities, we will give a proof of Theorem 6. In the identities that follow the parameters are taken to be integers.

$$W_m = W_0 U_{m+1} + W_{-1} U_m. \tag{1}$$

Identity (1) can be proved easily with the use of the closed forms, which are given at the end of this section. By translating the starting values for $\{W_n\}$ we have

$$W_{n+m} = W_n U_{m+1} + W_{n-1} U_m. \tag{2}$$

It follows from (2) that

$$U_{n+m} = U_n U_{m+1} + U_{n-1} U_m, \tag{3}$$

and

$$V_{n+m} = V_n U_{m+1} + V_{n-1} U_m. \tag{4}$$

Simson's identity states that

$$U_n^2 - U_{n-1} U_{n+1} = (-1)^{n+1}. \tag{5}$$

Using Simson's identity to replace $(-1)^{n+1}$ by $U_n^2 - U_{n-1}U_{n+1}$ it is easy to prove that

$$U_{n+1}^2 - pU_{n+1}U_n - U_n^2 + (-1)^{n+1} = 0. \tag{6}$$

We remind the reader (see the introduction) that $\overline{W}_n = W_{n-1} + W_{n+1}$. We also note that $U_{-n} = (-1)^{n+1} U_n$, and $V_{-n} = (-1)^n V_n$, where both identities can be established with the use of the closed forms.

We also require the following result which can be established with the use of the closed forms.

$$U_{k(n-1)} W_{k(i+n)+m} - U_{k(n-2)} W_{k(i+n+1)+m} = (-1)^{kn} U_k W_{k(i+2)+m}. \tag{7}$$

The following finite sum occurs in [2] and is a special case of Theorem 1 in [1]. For $n \geq 2$

$$\sum_{i=1}^{n-1} \frac{(-1)^{ki}}{W_{ki+m} W_{k(i+1)+m}} = \frac{(-1)^k U_{k(n-1)}}{U_k W_{k+m} W_{kn+m}}. \tag{8}$$

Finally we require an identity involving Ω that is required for the proofs of all the theorems in this paper. We state this identity as a lemma and give its proof. Following the convention that we adopted earlier (see the paragraph before the statement of Theorem 2) we take $\Omega(n)$ to mean $\Omega(k, m, n, l_1, l_2)$.

Lemma 1. *With Ω as defined in Section 1, we have*

$$D\,(k, n) = U_{k(n-1)} \Omega(n+1) - U_{k(n-2)} \Omega(n) = \frac{(-1)^{k(n+l_1)} U_{k(l_2-l_1)}}{W_{k(n+l_1)+m} W_{k(n+l_2)+m}}.$$

Proof. The difference $D(k,n) = U_{k(n-1)}\Omega(n+1) - U_{k(n-2)}\Omega(n)$ can be calculated with the use of (7). This difference is

$$\begin{aligned} D(k,n) &= \sum_{i=l_1}^{l_2-1} \frac{(-1)^{ki}}{W_{k(i+2)+m}} \left(\frac{U_{k(n-1)}}{W_{k(i+n+1)+m}} - \frac{U_{k(n-2)}}{W_{k(i+n)+m}} \right) \\ &= (-1)^{kn} U_k \sum_{i=l_1}^{l_2-1} \frac{(-1)^{ki}}{W_{k(i+n)+m} W_{k(i+n+1)+m}} \\ &= (-1)^{kn} U_k \sum_{i=1}^{l_2-l_1} \frac{(-1)^{k(i+l_1-1)}}{W_{k(i+l_1-1+n)+m} W_{k(i+l_1+n)+m}}. \end{aligned}$$

In the final sum write the upper limit as $(l_2 - l_1 + 1) - 1$ and the denominator of the summand as $W_{ki+k(n+l_1-1)+m} W_{k(i+1)+k(n+l_1-1)+m}$. Lemma 1 then follows with the use of (8). □

With the preparations above we can now give a proof of Theorem 6.

Proof. We remind the reader that all the finite sums in this paper are defined for $n \geq 2$, and so it is for these values of n that the following argument holds. In the statement of Theorem 6, denote the quantity on the left side by $L(n)$ and the quantity on the right side by $R(n)$. With the previously stated restrictions on the relevant parameters, we have

$$L(n+1) - L(n) = \frac{(-1)^{kn} d_1 \overline{W}_{k(n+m_1)+m} \overline{W}_{k(n+m_2)+m}}{W_{kn+m} W_{k(n+m_1)+m} W_{k(n+m_2)+m} W_{k(n+m_1+m_2)+m}}. \quad (9)$$

For $1 \leq i \leq 3$ define $f_i = f_i(k, m, n, m_1, m_2)$ by

$$\begin{aligned} f_1 &= (-1)^{km_2} d_2 U_{m_1 k} W_{kn+m} W_{k(n+m_1)+m}, \\ f_2 &= (-1)^{km_1} d_3 U_{(m_2-m_1)k} W_{kn+m} W_{k(n+m_1+m_2)+m}, \\ f_3 &= d_2 U_{m_1 k} W_{k(n+m_2)+m} W_{k(n+m_1+m_2)+m}. \end{aligned}$$

Then, with the f_i just defined, and with the use of Lemma 1, we have

$$R(n+1) - R(n) = \frac{(-1)^{kn} (f_1 + f_2 + f_3)}{W_{kn+m} W_{k(n+m_1)+m} W_{k(n+m_2)+m} W_{k(n+m_1+m_2)+m}}. \quad (10)$$

We now proceed to prove that

$$d_1 \overline{W}_{k(n+m_1)+m} \overline{W}_{k(n+m_2)+m} = f_1 + f_2 + f_3, \quad (11)$$

and we do this by proving that

$$d_1 \overline{W}_{k(n+m_1)+m} \overline{W}_{k(n+m_2)+m} - (f_1 + f_2 + f_3) = 0. \quad (12)$$

Now, with the use use of (1)-(4), together with the three identities in the two sentences that follow (5) and (6), we express each quantity on the left side of (12) in terms of the seven *primitive* quantities U_{m_1k}, U_{m_1k+1}, U_{m_2k}, U_{m_2k+1}, W_{kn+m}, W_{kn+m-1}, and p. We then expand and observe that this expansion can be factored as

$$d_1\overline{W}_{k(n+m_1)+m}\overline{W}_{k(n+m_2)+m} - (f_1 + f_2 + f_3) = (-1)^{km_1}h_1h_2h_3h_4, \tag{13}$$

where

$$\begin{aligned}
h_1 &= U_{m_1k+1}U_{m_2k} - U_{m_1k}U_{m_2k+1}, \\
h_2 &= U_{m_1k}^2\left(pU_{m_1k} - 2U_{m_1k+1}\right)U_{m_2k}W_{kn+m}, \\
h_3 &= U_{m_1k}\left(U_{m_2k+1}W_{kn+m-1} + U_{m_2k}\left(3W_{kn+m} - 2pW_{kn+m-1}\right)\right) \\
&\quad + U_{m_1k+1}\left(U_{m_2k+1}W_{kn+m} + U_{m_2k}\left(3W_{kn+m-1} + pW_{kn+m}\right)\right), \\
h_4 &= U_{m_2k+1}^2 - pU_{m_2k+1}U_{m_2k} - U_{m_2k}^2 + (-1)^{m_2k+1}.
\end{aligned}$$

To assist in the algebra we made use of *Mathematica 8*. It follows from (6) that the factor h_4 is identically zero. This establishes (12), which shows that for $n \geq 2$

$$L(n+1) - L(n) = R(n+1) - R(n). \tag{14}$$

Furthermore, $L(2) = R(2) = 0$. This, together with (14) proves Theorem 6. □

There is a more direct (though less enlightening) method of proof that is effective in the proofs of all the theorems in this paper. This method makes use of the closed form for each sequence. With $\alpha = \left(p + \sqrt{\Delta}\right)/2$ we have, for all integers n,

$$\begin{aligned}
U_n &= \left(\alpha^n + (-1)^{n+1}\alpha^{-n}\right)/\sqrt{\Delta}, \\
V_n &= \alpha^n + (-1)^n\alpha^{-n}, \\
W_n &= \left(\left(b + a\alpha^{-1}\right)\alpha^n + (-1)^{n+1}\left(b - a\alpha\right)\alpha^{-n}\right)/\sqrt{\Delta}, \\
\overline{W}_n &= \left(b + a\alpha^{-1}\right)\alpha^n + (-1)^n\left(b - a\alpha\right)\alpha^{-n}.
\end{aligned}$$

In the proof above the key identity is (12), and this identity is easily established by substitution of the closed forms. Likewise, the proof of each theorem in this paper hinges around the proof of a key identity that is analogous to (12), and each such identity follows immediately by substitution of the appropriate closed forms. The method is mechanical and is not dependent upon and special identities. However, the use of a computer algebra system (in our case *Mathematica 8*) is essential. In implementing this method to prove Theorem 6 we found that a factor of the left side of (12) is $\left((-1)^{2km_1} - 1\right)$, thus establishing (12).

To similarly prove those theorems in this paper that involve e_W we require the fact that $p = \alpha - \alpha^{-1}$.

10. Concluding Comments

As we stated in the introduction, we have discovered many more results than those presented here. In some cases we have chosen to present those results that possess the most pleasing symmetry. For instance, in S_3^U we could have defined the numerator of the summand to be $(-1)^{ki}U_{ki+m}$ or $(-1)^{ki}U_{k(i+m_2)+m}$. We have found closed forms for the finite sums in these cases, but they do not possess the symmetry of the result given in Theorem 2.

In some cases we have chosen not to present the results that possess the most symmetry. Instead we have chosen to present results that highlight what is possible. For instance, in S_8 we could have replaced m_1, m_2, m_5, m_6 in the numerator of the summand by m_1, m_3, m_6, m_7, respectively. Indeed, with the m_i used to define the denominator of the summand of S_8, we could have defined the numerator of the summand of S_8 in $\binom{8}{4}$ different ways. Furthermore, we could have written down the closed form for each of the corresponding finite sums. Here we are assuming that the terms in the numerator are being drawn from the sequence $\left\{\overline{W}_n\right\}$. However, the terms of the numerator could have been drawn from $\{U_n\}$ or $\{V_n\}$. Furthermore, we could have used different multiples of g to define the m_i.

Staying with S_8, we could have defined the numerator of the summand to be 1. We could also have defined the numerator of the summand to be $(-1)^{ki}$ times the product of *two* terms from the sequence $\left\{\overline{W}_n\right\}$, or $(-1)^{ki}$ times the product of *six* terms from the sequence $\left\{\overline{W}_n\right\}$. For each of the two possibilities described in the previous sentence we could have drawn the terms of the numerator from $\{U_n\}$ or $\{V_n\}$.

Comments similar to those in the preceding three paragraphs apply to the finite sums in Sections 4 to 7. In this paper we have presented only finite sums in which the denominator of the summand is a product of at most eight terms. Our investigations have shown that similar results exist for analogous finite sums in which the denominator of the summand is a product of nine or more terms. The scope of this topic is huge.

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