#A19 INTEGERS 15 (2015)

GENERALIZED FOLDING LEMMAS IN THE FIELD OF FORMAL SERIES AND THEIR APPLICATIONS

Jittinart Rattanamoong
Department of Mathematics, Faculty of Science, Srinakharinwirot University, Bangkok, Thailand
jittinart@g.swu.ac.th

Vichian Laohakosol
Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, Thailand
fscivil@ku.ac.th

Tuangrat Chaichana
Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok, Thailand
tuangrat.c@chula.ac.th

Received: 8/18/14, Accepted: 4/27/15, Published: 5/8/15

Abstract

The classical folding lemma is extended, in the field of formal series, to two-tier and three-tier folding lemmas covering all possible shapes of the words enclosing one and two middle terms. The two-fold and three-fold continued fraction identities so obtained are applied to derive a number of explicit continued fractions of certain series expansions, including those related to exponential elements.

1. Introduction

Following [11], a continued fraction is an object of the form

$$[a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

Its n^{th} convergent is defined as

$$\frac{C_n}{D_n} := [a_0; a_1, \ldots, a_n] \qquad (n = 0, 1, 2, \ldots).$$

The classical Folding Lemma (first appeared in [4]; also see [11, Proposition 2]) asserts that

$$\frac{C_n}{D_n} + \frac{(-1)^n}{yD_n^2} = \left[a_0; \overrightarrow{w}_n, y - D_{n-1}/D_n\right] = \left[a_0; \overrightarrow{w}_n, y, -\overleftarrow{w}_n\right], \tag{1}$$

where $\overrightarrow{w}_n$ is an abbreviation for the word $a_1, a_2, \ldots, a_n$, and accordingly, $-\overleftarrow{w}_n$ denotes the word $-a_n, -a_{n-1}, \ldots, -a_1$. The Folding Lemma, as mentioned for example in [12], is useful in the determining of explicit shapes of expressions because the Folding Lemma makes its easy, given the expansion of a partial sum of a series, to adjust a continued fraction for an appended term if the intervening gap is wide enough, especially in the function field case. It is also well-known [2, 12] that the Folding Lemma provides an unusal explanation for the symmetry in certain continued fractions. In particular, Cohn [2] terms the right-hand continued fraction in (1) as having 2-fold symmetry and says similarly that the continued fraction $\left[a_0; \overrightarrow{w}_n, y_1, -\overleftarrow{w}_n, y_2, \overrightarrow{w}_n\right]$ has 3-fold symmetry, etc. The Folding Lemma can also be found (but in a disguised way) in the papers [3, 7]. The two main objectives of our work here are first to extend the classical Folding Lemma one step further by proving the two-tier and three-tier Folding Lemmas, which correspond to all possible shapes of the 2-fold and 3-fold symmetric continued fractions in the field of formal Laurent series, and second to illustrate their versatility by applying them to establish old and new results about explicit continued fractions.

Throughout, we let $F((x^{-1}))$ denote the field of formal Laurent series over a field F equipped with a degree valuation $|\cdot|$ defined by $|x^{-1}| = e^{-1}$. It is well-known [6] that each element $\xi \in F((x^{-1}))$ can be uniquely written as a (Ruban) continued fraction of the form

$$\xi = [a_0; a_1, a_2, \ldots],$$

where $a_0 \in F[x]$ and $a_n \in F[x] \setminus F$ $(n \geq 1)$. Such a continued fraction is finite if and only if $\xi \in F(x)$. Define two sequences (C_n), (D_n) by

$$\begin{aligned} &C_{-1} = 1, \quad C_0 = a_0, \quad C_{n+1} = a_{n+1}C_n + C_{n-1} \quad (n \geq 0) \\ &D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = a_{n+1}D_n + D_{n-1} \quad (n \geq 0). \end{aligned}$$

The following proposition, whose induction proof is omitted, collects basic properties needed throughout.

Proposition 1. *Let* $n \in \mathbb{N} \cup \{0\}$, $\beta \in F((x^{-1})) \setminus \{0\}$.

(i) We have $\frac{\beta C_n + C_{n-1}}{\beta D_n + D_{n-1}} = [a_0; a_1, a_2, \ldots, a_n, \beta]$.

(ii) We have $C_n/D_n = [a_0; a_1, a_2, \ldots, a_n]$, *called the* n^{th} *convergent.*

(iii) We have $D_nC_{n-1} - C_nD_{n-1} = (-1)^n$, *so that* C_n *and* D_n *are relatively prime.*

(iv) *If* $C_n/D_n = [a_0; a_1, a_2, \ldots, a_n]$*, then* $D_n/D_{n-1} = [a_n; a_{n-1}, \ldots, a_2, a_1]$*.*

(v) *If* $C_n/D_n = [0; a_1, a_2, \ldots, a_n]$*, then* $C_n/C_{n-1} = [a_n; a_{n-1}, \ldots, a_3, a_2]$ $(n \geq 2)$*.*

The next result is classically known as the Folding Lemma.

Lemma 1. *Let* $y \in F[x] \setminus \{0\}$*, and let*

$$\frac{C_n}{D_n} := [0; a_1, a_2, \ldots, a_n] = \left[0; \overrightarrow{X}_n\right] \quad (n \in \mathbb{N}).$$

Then

$$\left[0; \overrightarrow{X}_n, y, -\overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y}.$$

Proof. By Proposition 1, we get

$$\begin{aligned}\left[0; \overrightarrow{X}_n, y, -\overleftarrow{X}_n\right] &= [0; a_1, a_2, \ldots, a_n, y, -a_n, -a_{n-1}, \ldots, -a_1] \\ &= [0; a_1, a_2, \ldots, a_n, y - D_{n-1}/D_n] \\ &= \frac{(D_n y - D_{n-1})\, C_n + D_n C_{n-1}}{(D_n y - D_{n-1})\, D_n + D_n D_{n-1}} \\ &= \frac{C_n\,(D_n y) + (-1)^n}{D_n\,(D_n y)}.\end{aligned}$$

□

2. Two-tier and Three-tier Folding Lemmas

In this section, we first extend Lemma 1 by exhibiting the four identities corresponding to all possible patterns of the two words enclosing one middle term.

Lemma 2 (Two-tier Folding Lemma). *Let* $y \in F[x] \setminus \{0\}$*, and let*

$$\frac{C_n}{D_n} := [0; a_1, a_2, \ldots, a_n] = \left[0; \overrightarrow{X}_n\right] \quad (n \in \mathbb{N}).$$

Then

(1) $$\left[0; \overrightarrow{X}_n, y, \overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\,(y + (C_n + D_{n-1})/D_n)}, \tag{2}$$

(2) $$\left[0; \overrightarrow{X}_n, y, -\overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\,(y + (D_{n-1} - C_n)/D_n)}, \tag{3}$$

(3) $$\left[0; \overrightarrow{X}_n, y, \overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\,(y + 2D_{n-1}/D_n)}, \tag{4}$$

(4) $$\left[0; \overrightarrow{X}_n, y, -\overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y}. \tag{5}$$

Proof. Observe first that the identity (5) is simply Lemma 1. Since the proofs of (2)–(4) are quite similar, we give only that of (3). By Proposition 1, we have

$$\begin{aligned}\left[0; \overrightarrow{X}_n, y, -\overrightarrow{X}_n\right] &= [0; a_1, a_2, \dots, a_n, y, -a_1, -a_2, \dots, -a_n] \\ &= [0; a_1, a_2, \dots, a_n, y - C_n/D_n] = \frac{(D_n y - C_n)\, C_n + D_n C_{n-1}}{(D_n y - C_n)\, D_n + D_n D_{n-1}} \\ &= \frac{C_n\,(D_n y - C_n + D_{n-1}) + (-1)^n}{D_n\,(D_n y - C_n + D_{n-1})} = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\,(y + (D_{n-1} - C_n)/D_n)}.\end{aligned}$$

□

The four identities in Lemma 2 will be referred to as *two-fold continued fractions of types 1 to 4*, respectively. Next, we derive analogous results for *three-fold continued fractions*.

Lemma 3 (Three-tier Folding Lemma). *Let $y_1, y_2 \in F[x] \setminus \{0\}$ and C_n/D_n be as in Lemma 2. Then*

$$(1)\quad \left[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, \overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{C_n + D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{C_n + D_{n-1}}{D_n}}},$$

$$(2)\quad \left[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, -\overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{C_n + D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{D_{n-1} - C_n}{D_n}}},$$

$$(3)\quad \left[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, \overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{C_n + D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{2D_{n-1}}{D_n}}},$$

$$(4)\quad \left[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, -\overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{C_n + D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2}},$$

$$(5)\quad \left[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{D_{n-1} - C_n}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{C_n - D_{n-1}}{D_n}}},$$

$$(6)\quad \left[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, -\overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{D_{n-1} - C_n}{D_n}\right) + \frac{(-1)^n}{y_2 - \frac{C_n + D_{n-1}}{D_n}}},$$

$$(7)\quad \left[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, \overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{D_{n-1} - C_n}{D_n}\right) + \frac{(-1)^n}{y_2}},$$

$$(8)\quad \left[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, -\overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{D_{n-1} - C_n}{D_n}\right) + \frac{(-1)^n}{y_2 - \frac{2D_{n-1}}{D_n}}},$$

$$(9)\quad \left[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, \overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{2D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{2C_n}{D_n}}},$$

$$(10)\quad \left[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, -\overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{2D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2}},$$

$$(11)\quad \left[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, \overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{2D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{C_n + D_{n-1}}{D_n}}},$$

$$(12)\quad \left[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, -\overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{2D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{C_n - D_{n-1}}{D_n}}},$$

$$(13)\quad \left[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, \overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y_1 + \frac{(-1)^n}{y_2}},$$

$$(14)\quad \left[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, -\overrightarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y_1 + \frac{(-1)^n}{y_2 - \frac{2C_n}{D_n}}},$$

$$(15)\quad \left[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, \overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y_1 + \frac{(-1)^n}{y_2 + \frac{D_{n-1} - C_n}{D_n}}},$$

$$(16)\quad \left[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, -\overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y_1 + \frac{(-1)^n}{y_2 - \frac{C_n + D_{n-1}}{D_n}}}.$$

Proof. (1) From the two-fold continued fraction of type 1 (i.e., (2)), using Proposition 1, we get

$$\left[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, \overrightarrow{X}_n\right] = \left[0; \overrightarrow{X}_n, y_1 + \frac{C_n}{D_n} + \frac{(-1)^n}{D_n\left(D_n y_2 + C_n + D_{n-1}\right)}\right]$$
$$= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{C_n + D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{C_n + D_{n-1}}{D_n}}}.$$

To prove (2)–(4), we start from the two-fold continued fractions of types 2-4 (i.e., (3)–(5)) and proceed analogously as above.

Using the fact that

$$\left[0; -\overrightarrow{W}_k\right] = -\left[0; \overrightarrow{W}_k\right], \tag{6}$$

the identities (5), (6), (7) and (8) follow from the proofs of (2), (1), (4) and (3), respectively. We give only a detailed proof of (5). From (6), we have

$$\left[0; -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n\right] = -\left[0; \overrightarrow{X}_n, -y_2, -\overrightarrow{X}_n\right].$$

Applying the two-fold continued fraction of type 2, we get

$$\left[0; -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n\right] = -\frac{C_n}{D_n} + \frac{(-1)^n}{D_n\left(D_n y_2 - D_{n-1} + C_n\right)}.$$

The same proof as for (2) leads to

$$\left[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n\right] = \left[0; \overrightarrow{X}_n, y_1 - \frac{C_n}{D_n} + \frac{(-1)^n}{D_n\left(D_n y_2 - D_{n-1} + C_n\right)}\right]$$
$$= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{D_{n-1} - C_n}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{C_n - D_{n-1}}{D_n}}},$$

which is (5).

If the n^{th} convergent of $\left[0; \overrightarrow{X}_n\right]$ is C_n/D_n, then from Proposition 1 the two consecutive $(n-1)^{\text{th}}$ and n^{th} convergents of $\left[0; \overleftarrow{X}_n\right]$ are C_{n-1}/C_n and D_{n-1}/D_n, respectively. Substituting these into Lemma 2, we obtain reverse forms of the two-fold continued fractions of types 1 to 4 as

$(1)'$ $\left[0; \overleftarrow{X}_n, y_2, \overleftarrow{X}_n\right] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2\left(y_2 + \frac{D_{n-1}+C_n}{D_n}\right)}$,

$(2)'$ $\left[0; \overleftarrow{X}_n, y_2, -\overleftarrow{X}_n\right] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2\left(y_2 + \frac{C_n-D_{n-1}}{D_n}\right)}$,

$(3)'$ $\left[0; \overleftarrow{X}_n, y_2, \overrightarrow{X}_n\right] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2(y_2+2C_n/D_n)}$,

$(4)'$ $\left[0; \overleftarrow{X}_n, y_2, -\overrightarrow{X}_n\right] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2 y_2}$.

The identities (9), (10), (11) and (12) are thus obtained in a manner similar to the proofs of (3), (4), (1) and (2), respectively. We give only a detailed proof of (9). Using $(3)'$, we have

$$\begin{aligned}\left[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, \overrightarrow{X}_n\right] &= \left[0; \overrightarrow{X}_n, y_1 + \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n\left(D_n y_2 + 2C_n\right)}\right] \\ &= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \frac{2D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2+2C_n/D_n}}.\end{aligned}$$

Similarly, using (6) and $(4)'$, we get (13) by the same proof as (4); using (6) and $(3)'$, we get (14) by the same proof as (3); using (6) and $(2)'$, we get (15) by the same proof as (2); using (6) and $(1)'$, we get (16) by the same proof as (1). □

3. Folding Lemmas and Series Expansions

If a continued fraction of a finite sum of n terms is known, the two-tier Folding Lemma enables us to determine a continued fraction of the sum with $n+1$ terms explicitly as seen in the next theorem.

Theorem 1. *Let $Y \in F[x] \setminus \{0\}$. If $C_{k_\ell}/D_{k_\ell} = \left[0; \overrightarrow{X}_{k_\ell}\right]$ $(\ell \in \mathbb{N})$ is the ${k_\ell}^{th}$ convergent of the continued fraction representing $\sum_{i=1}^{\ell} 1/\alpha_i$ $(\alpha_i \in F[x] \setminus F)$, then*

$$(1)\quad \left[0; \overrightarrow{X}_{k_\ell}, Y, \overrightarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i,\ \ \alpha_{\ell+1} = (-1)^{k_\ell}\left(\left(C_{k_\ell}D_{k_\ell} + D_{k_\ell-1}D_{k_\ell}\right) + D_{k_\ell}^2 Y\right) \tag{7}$$

$$(2)\quad \left[0; \overrightarrow{X}_{k_\ell}, Y, -\overrightarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i,\ \ \alpha_{\ell+1} = (-1)^{k_\ell}\left(\left(D_{k_\ell-1}D_{k_\ell} - C_{k_\ell}D_{k_\ell}\right) + D_{k_\ell}^2 Y\right)$$

$$(3)\quad \left[0; \overrightarrow{X}_{k_\ell}, Y, \overleftarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i,\ \ \alpha_{\ell+1} = (-1)^{k_\ell}\left(2D_{k_\ell-1}D_{k_\ell} + D_{k_\ell}^2 Y\right)$$

$$(4)\quad \left[0; \overrightarrow{X}_{k_\ell}, Y, -\overleftarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i,\ \ \alpha_{\ell+1} = (-1)^{k_\ell} D_{k_\ell}^2 Y.$$

Proof. Since the proofs of these identities are similar, here we prove only the identity (1). By (2) , we get

$$\left[0; \overrightarrow{X}_{k_\ell}, Y, \overrightarrow{X}_{k_\ell}\right] = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2\left(Y + \frac{C_{k_\ell}+D_{k_\ell-1}}{D_{k_\ell}}\right)} = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{\alpha_{\ell+1}} = \sum_{i=1}^{\ell+1}\frac{1}{\alpha_i}.$$

□

Now, we state a three-fold analogue of Theorem 1.

Theorem 2. *Let* $Y_1, Y_2 \in F[x]\setminus\{0\}$. *If* $C_{k_\ell}/D_{k_\ell} = \left[0; \overrightarrow{X}_{k_\ell}\right]$ $(\ell \in \mathbb{N})$ *is the* ${k_\ell}^{th}$ *convergent of the continued fraction representing* $\sum_{i=1}^{\ell} 1/\alpha_i$, $\alpha_i \in F[x]\setminus F$, *then*

(1) $\left[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, \overrightarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i$,

$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2\left(Y_1 + \frac{C_{k_\ell}+D_{k_\ell-1}}{D_{k_\ell}}\right) + \frac{1}{Y_2+\left(C_{k_\ell}+D_{k_\ell-1}\right)/D_{k_\ell}}$

(2) $\left[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, -\overrightarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i$,

$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2\left(Y_1 + \frac{C_{k_\ell}+D_{k_\ell-1}}{D_{k_\ell}}\right) + \frac{1}{Y_2+(D_{k_\ell-1}-C_{k_\ell})/D_{k_\ell}}$

(3) $\left[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, \overleftarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i$,

$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2\left(Y_1 + \frac{C_{k_\ell}+D_{k_\ell-1}}{D_{k_\ell}}\right) + \frac{1}{Y_2+2D_{k_\ell-1}/D_{k_\ell}}$

(4) $\left[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, -\overleftarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i$,

$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2\left(Y_1 + \frac{C_{k_\ell}+D_{k_\ell-1}}{D_{k_\ell}}\right) + \frac{1}{Y_2}$

(5) $\left[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overrightarrow{X}_{k_\ell}, Y_2, \overrightarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i$,

$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2\left(Y_1 + \frac{D_{k_\ell-1}-C_{k_\ell}}{D_{k_\ell}}\right) + \frac{1}{Y_2+(C_{k_\ell}-D_{k_\ell-1})/D_{k_\ell}}$

(6) $\left[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overrightarrow{X}_{k_\ell}, Y_2, -\overrightarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i$,

$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2\left(Y_1 + \frac{D_{k_\ell-1}-C_{k_\ell}}{D_{k_\ell}}\right) + \frac{1}{Y_2-(C_{k_\ell}+D_{k_\ell-1})/D_{k_\ell}}$

(7) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,-\overrightarrow{X}_{k_\ell},Y_2,\overleftarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2\left(Y_1+\frac{D_{k_\ell-1}-C_{k_\ell}}{D_{k_\ell}}\right)+\frac{1}{Y_2}$

(8) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,-\overrightarrow{X}_{k_\ell},Y_2,-\overleftarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2\left(Y_1+\frac{D_{k_\ell-1}-C_{k_\ell}}{D_{k_\ell}}\right)+\frac{1}{Y_2-2D_{k_\ell-1}/D_{k_\ell}}$

(9) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,\overleftarrow{X}_{k_\ell},Y_2,\overrightarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2\left(Y_1+\frac{2D_{k_\ell-1}}{D_{k_\ell}}\right)+\frac{1}{Y_2+2C_{k_\ell}/D_{k_\ell}}$

(10) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,\overleftarrow{X}_{k_\ell},Y_2,-\overrightarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2\left(Y_1+\frac{2D_{k_\ell-1}}{D_{k_\ell}}\right)+\frac{1}{Y_2}$

(11) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,\overleftarrow{X}_{k_\ell},Y_2,\overleftarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2\left(Y_1+\frac{2D_{k_\ell-1}}{D_{k_\ell}}\right)+\frac{1}{Y_2+(C_{k_\ell}+D_{k_\ell-1})/D_{k_\ell}}$

(12) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,\overleftarrow{X}_{k_\ell},Y_2,-\overleftarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2\left(Y_1+\frac{2D_{k_\ell-1}}{D_{k_\ell}}\right)+\frac{1}{Y_2+(C_{k_\ell}-D_{k_\ell-1})/D_{k_\ell}}$

(13) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,-\overleftarrow{X}_{k_\ell},Y_2,\overrightarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2Y_1+\frac{1}{Y_2}$

(14) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,-\overleftarrow{X}_{k_\ell},Y_2,-\overrightarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2Y_1+\frac{1}{Y_2-2C_{k_\ell}/D_{k_\ell}}$

(15) $\left[0;\overrightarrow{X}_{k_\ell},Y_1,-\overleftarrow{X}_{k_\ell},Y_2,\overleftarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}1/\alpha_i,$

$\alpha_{\ell+1}=(-1)^{k_\ell}D_n^2Y_1+\frac{1}{Y_2+(D_{k_\ell-1}-C_{k_\ell})/D_{k_\ell}}$

(16) $\left[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overleftarrow{X}_{k_\ell}, Y_2, -\overleftarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} 1/\alpha_i,$

$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 Y_1 + \frac{1}{Y_2 - (C_{k_\ell} + D_{k_\ell - 1})/D_{k_\ell}}.$

Proof. The identities are proved by the same method as that of Theorem 1, but appealing instead to the identities in Lemma 3. □

4. Applications

In this section, we derive some known and some new explicit continued fractions of series expansions as applications.

4.1. Two-fold Continued Fraction of Type 1

In this subsection, we work in the field $\mathbb{F}_q((x^{-1}))$ of formal Laurent series over the finite field $\mathbb{F}_q$, where q is a prime power. The notation and basic results follow closely those in Carlitz [1]. For $i \in \mathbb{N}$, let

$$[i] := x^{q^i} - x, \; d_0 := 1, \; d_i := [i] d_{i-1}^q. \tag{8}$$

It is known that $[i]$ is the product of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree dividing i, and d_i is the product of all monic polynomials in $\mathbb{F}_q[x]$ of degree i.

Remark 1. From (8), for all $i \geq 1$, we have:

(1) $d_i = [1][2] \cdots [i] d_1^{q-1} d_2^{q-1} \cdots d_{i-1}^{q-1}$.

(2) $d_i = [i][i-1]^q [i-2]^{q^2} \cdots [2]^{q^{i-2}} [1]^{q^{i-1}}$.

The exponential element for $\mathbb{F}_q[x]$ is defined by

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}, \quad e := e(1).$$

Taking $\alpha_\ell = x^m d_{\ell-1}$ in (4) of Theorem 1, we obtain the following proposition which is [9, Theorem 1].

Proposition 2. *Let (x_n) be a sequence defined recursively by $x_1 = [0; z^{-q}[1]]$, and when $x_n = [a_0; a_1, \ldots, a_{2^n-1}]$, set*

$$x_{n+1} = \left[a_0; a_1, \ldots, a_{2^n-1}, -z^{-q^n(q-2)} d_{n+1}/d_n^2, -a_{2^n-1}, \ldots, -a_1\right].$$

We have

$$x_n = \sum_{i=1}^{n} \frac{z^{q^i}}{d_i}.$$

In particular, $e(z) = z + \lim_{n\to\infty} x_n$ *and for* $q = 2$*,*

$$e = \left[1; \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \dots \right].$$

(More explicitly, for $n > 0$ *the* n^{th} *partial quotient is* $x^{2^{u_n}} - x$ *with* u_n *being the exponent of the highest power of* 2 *dividing* $2n$*. The sequence of partial quotients is the well-known "ruler sequence".)*

Specializing $F = \mathbb{F}_2$ in the last proposition yields the next proposition which is [10, Theorem 4].

Proposition 3. *Over* $\mathbb{F}_2((x^{-1}))$*, for* $m > 2$*, with* $\overrightarrow{X}_{(m)}$ *defined by* $\sum_{i=0}^{m-2} 1/d_i x^m = \left[0, \overrightarrow{X}_{(m)}\right]$*, we have*

$$\frac{e}{x^m} = \left[0; \overrightarrow{X}_{(m)}, x^{2^{m-1}-m}, \overrightarrow{X}_{(m)}, x^{2^m-m}, \overrightarrow{X}_{(m)}, x^{2^{m-1}-m}, \overrightarrow{X}_{(m)}, x^{2^{m+1}-m}, \dots\right].$$

With $\overrightarrow{X}$ *denoting the word* $x^2+1, x, x+1$*, we have*

$$\frac{e}{x^2} = \left[0; \overrightarrow{X}, x^2, \overrightarrow{X}, x^6, \overrightarrow{X}, x^2, \overrightarrow{X}, x^{14}, \dots\right].$$

The identities so obtained above allow us to deduce a good deal of new explicit continued fractions. Here we give those of $e/(x+1)^m$ and $e/\left(x(x+1)\right)^m$ for $q = 2$ and $m \geq 2$, using the two-fold continued fraction of type 1. For the proofs, we need:

Lemma 4. *Let* $m, t \in \mathbb{N}$*;* q *a prime power. If a monic* $f(x) \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ *is such that* $f(x) \mid [1]$*, then*

$$\gcd\left(d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1, f(x)^m d_t\right) = 1.$$

Proof. Suppose that the assertion is false. Then there exists a prime $p \in \mathbb{F}_q[x]$ such that

$$p \mid \left(d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1\right) \quad \text{and} \quad p \mid f(x)^m d_t.$$

Using the expression for d_t in Remark 1 (1), since $p \mid f(x)^m d_t$, we get

$$p \mid f(x), \text{ or } p \mid [r] \text{ for some } 1 \leq r \leq t, \text{ or } p \mid d_s \text{ for some } 1 \leq s \leq t-1.$$

Again, Remark 1 (1) leads to

$$\begin{aligned}
&d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1 \\
&= \left([1][2]\cdots[t]d_1^{q-1}d_2^{q-1}d_3^{q-1}\cdots d_{t-1}^{q-1}\right) + \left([1][2]\cdots[t]d_1^{q-2}d_2^{q-1}d_3^{q-1}\cdots d_{t-1}^{q-1}\right) \\
&+ \left([1][2]\cdots[t]d_1^{q-1}d_2^{q-2}d_3^{q-1}\cdots d_{t-1}^{q-1}\right) + \cdots + \left([1][2]\cdots[t]d_1^{q-1}d_2^{q-1}\cdots d_{t-2}^{q-1}d_{t-1}^{q-2}\right) + 1.
\end{aligned}$$

If $p \mid f(x)$ or $p \mid [r]$ $(1 \le r \le t)$, then $p \mid 1$ which is a contradiction, and so $p \mid d_s$ $(1 \le s \le t-1)$. We treat two separate cases. If $q \ge 3$, then $p \mid 1$, a contradiction. If $q = 2$, from the above expressions, for $1 \le s \le t-2$, since $d_i \mid d_{i+1}$ $(i \ge 0)$, then $p \mid 1$, a contradiction, and so $s = t-1$. We apply Remark 1 (1) again to deduce that $p \mid [r]$ for some $1 \le r \le t-1$, or $p \mid d_s$ for some $1 \le s \le t-2$, both of which have already been ruled out. □

For $N \ge 2$, let

$$\begin{aligned} \mathbf{L}_1 &= 2 & \mathbf{R}_1 &= 2 \\ \mathbf{L}_2 &= 2+1 & \mathbf{R}_2 &= 2^2 \\ &\vdots & &\vdots \\ \mathbf{L}_N &= 2^{N-1}+1 & \mathbf{R}_N &= 2^N. \end{aligned}$$

Then, provided $M \ne N$, we have

$$\{n \in \mathbb{N}; n \ge 2\} = \left(\bigcup_{N \ge 1} [\mathbf{L}_N, \mathbf{R}_N] \right) \cap \mathbb{Z}, \;\; [\mathbf{L}_N, \mathbf{R}_N] \cap [\mathbf{L}_M, \mathbf{R}_M] = \emptyset.$$

For a fixed integer $m > 1$, there clearly exists a unique $N \in \mathbb{N}$ such that $m \in [\mathbf{L}_N, \mathbf{R}_N]$.

Theorem 3. *Over* $\mathbb{F}_2((x^{-1}))$, *if*

$$\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x+1)^m d_i} =: \left[0; \overrightarrow{X}_{k_\ell}\right] \quad (\ell \ge 1),$$

then

$$\sum_{i=0}^{(N-1)+\ell+1} \frac{1}{(x+1)^m d_i} = \left[0; \overrightarrow{X}_{k_\ell}, \frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \overrightarrow{X}_{k_\ell}\right].$$

In particular,

$$\frac{e}{(x+1)^m} = \left[0; \underbrace{\underbrace{\overrightarrow{X}_{k_1}}, \frac{[N+1]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \underbrace{\overrightarrow{X}_{k_1}}}, \frac{[N+2]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \dots\right].$$

Proof. For $\ell \ge 1$, let $C_{k_\ell}/D_{k_\ell} := \left[0; \overrightarrow{X}_{k_\ell}\right]$ be the k_ℓ^{th} convergent of the continued fraction of $\sum_{i=0}^{(N-1)+\ell} 1/(x+1)^m d_i$. From

$$\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x+1)^m d_i} = \frac{d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1}{(x+1)^m d_{(N-1)+\ell}},$$

since

$$d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1$$

and $(x+1)^m d_{(N-1)+\ell}$ are monic polynomials over $\mathbb{F}_2$, using Lemma 4 and $C_{k_\ell},\ D_{k_\ell}$ being relatively prime, we get

$$\begin{aligned} C_{k_\ell} &= d_{(N-1)+\ell}\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \\ D_{k_\ell} &= (x+1)^m d_{(N-1)+\ell}. \end{aligned}$$

Claim. We have

$$D_{k_\ell - 1} = (x+1)d_{(N-1)+\ell} + C_{k_\ell} \quad (\ell \geq 1).$$

Proof of Claim. Let $Q = (x+1)d_{(N-1)+\ell} + C_{k_\ell}$ and $P = \left(1 + C_{k_\ell} Q\right)/D_{k_\ell}$. Then

$$PD_{k_\ell} - QC_{k_\ell} = \left(\frac{1 + C_{k_\ell}Q}{D_{k_\ell}}\right) D_{k_\ell} - QC_{k_\ell} = 1.$$

We first show that $P \in \mathbb{F}_2[x]$. From Remark 1 (2), we get

$$(x+1) \mid [i],\ (x+1)^{2^i - 1} \mid d_i \qquad (i \in \mathbb{N}), \tag{9}$$

and

$$\begin{aligned} P &= \frac{1 + C_{k_\ell}Q}{D_{k_\ell}} = \frac{1 + C_{k_\ell}\left((x+1)d_{(N-1)+\ell} + C_{k_\ell}\right)}{D_{k_\ell}} \\ &= \left\{(x+1)d_{(N-1)+\ell}\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right)\right. \\ &\quad \left. + \left(d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1^2} + \frac{d_{(N-1)+\ell}}{d_2^2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}^2}\right)\right\} / (x+1)^m. \end{aligned} \tag{10}$$

From (9), it follows that $(x+1)d_{(N-1)+\ell} \equiv 0 \ \left(\bmod\ (x+1)^{2^{(N-1)+\ell}}\right)$ and

$$\begin{aligned} &\frac{(x+1)d_{(N-1)+\ell}}{d_j} + \frac{d_{(N-1)+\ell}}{d_{j-1}^2} = \frac{d_{(N-1)+\ell}}{d_j}\left((x+1) + (x^{2^j} + x)\right) \\ &= \frac{d_{(N-1)+\ell}}{d_j}(x+1)^{2^j} \equiv 0 \ \left(\bmod\ (x+1)^{2^{(N-1)+\ell}}\right) \quad \left(j \in \{1, 2, \ldots, (N-1)+\ell\}\right). \end{aligned}$$

Since $m \leq 2^N \leq 2^{(N-1)+\ell}$ $(\ell \in \mathbb{N})$, we get $P \in \mathbb{F}_2[x]$.

From $PD_{k_\ell} - QC_{k_\ell} = 1$ and $C_{k_\ell - 1}D_{k_\ell} - D_{k_\ell - 1}C_{k_\ell} = 1$, we get

$$PD_{k_\ell} - QC_{k_\ell} = C_{k_\ell - 1}D_{k_\ell} - D_{k_\ell - 1}C_{k_\ell},\ C_{k_\ell}\left(D_{k_\ell - 1} - Q\right) = D_{k_\ell}\left(C_{k_\ell - 1} - P\right).$$

Since C_{k_ℓ} and D_{k_ℓ} are relatively prime, by (10) we see that

$$\deg P = \deg d_{(N-1)+\ell} + 1 - m \le \deg d_{(N-1)+\ell} - 1 < \deg d_{(N-1)+\ell} = \deg C_{k_\ell}.$$

By definition, $\deg(C_{k_\ell - 1} - P) < \deg C_{k_\ell}$. Thus $C_{k_\ell - 1} = P$, and so

$$D_{k_\ell - 1} = Q = (x+1) d_{(N-1)+\ell} + C_{k_\ell},$$

and the claim is proved.

Next, we show that

$$\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} \in \mathbb{F}_2[x] \setminus \{0\} \quad (\ell \ge 1).$$

This is immediate from $\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} = \frac{(x+1)^{2^{N+\ell}}}{(x+1)^m}$ and $2^{N+\ell} > m$.

Applying Lemma 2 (1), we get

$$\begin{aligned} &\left[0; \overrightarrow{X}_{k_\ell}, \frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \overleftarrow{X}_{k_\ell}\right] \\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{D_{k_\ell}^2 \left(\left(\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} \right) + \left(\frac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}} \right) \right)} \\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x+1)^m d_{(N-1)+(\ell+1)}} = \sum_{i=0}^{(N-1)+\ell+1} \frac{1}{(x+1)^m d_i}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=0}^{(N-1)+1} \frac{1}{(x+1)^m d_i} &= \left[0; \overrightarrow{X}_{k_\ell}\right] \\ \sum_{i=0}^{(N-1)+2} \frac{1}{(x+1)^m d_i} &= \left[0; \overrightarrow{X}_{k_1}, \frac{[N+1]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \overleftarrow{X}_{k_1}\right]. \end{aligned}$$

Continuing, we finally get

$$\frac{e}{(x+1)^m} = \left[0; \overrightarrow{X}_{k_1}, \frac{[N+1]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \overleftarrow{X}_{k_1}, \frac{[N+2]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \dots\right].$$

□

Theorem 4. *Over* $\mathbb{F}_2((x^{-1}))$, *if*

$$\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x(x+1))^m d_i} =: \left[0; \overrightarrow{X}_{k_\ell}\right] \quad (\ell \ge 1),$$

then

$$\sum_{i=0}^{(N-1)+\ell+1} \frac{1}{(x(x+1))^m d_i} = \left[0; \overrightarrow{X}_{k_\ell}, \frac{[N+\ell]+[N]}{(x(x+1))^m}, \overrightarrow{X}_{k_\ell}\right].$$

In particular,

$$\frac{e}{(x(x+1))^m} = \left[0; \underbrace{\underbrace{\overrightarrow{X}_{k_1}}, \frac{[N+1]+[N]}{(x(x+1))^m}, \underbrace{\overrightarrow{X}_{k_1}}}, \frac{[N+2]+[N]}{(x(x+1)))^m}, \dots\right].$$

Proof. Let $C_{k_\ell}/D_{k_\ell} := \left[0; \overrightarrow{X}_{k_\ell}\right]$ be the k_ℓ^{th} convergent of the continued fraction of $\sum_{i=0}^{(N-1)+\ell} 1/(x(x+1))^m d_i$. Consider

$$\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x(x+1))^m d_i} = \frac{d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1}{(x(x+1))^m d_{(N-1)+\ell}}.$$

Using Lemma 4, since C_{k_ℓ} and D_{k_ℓ} are relatively prime, we see that

$$d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1$$

and $(x(x+1))^m d_{(N-1)+\ell}$ are monic polynomials over $\mathbb{F}_2$. Thus,

$$\begin{aligned} C_{k_\ell} &= d_{(N-1)+\ell}\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \\ D_{k_\ell} &= (x(x+1))^m d_{(N-1)+\ell}. \end{aligned}$$

Claim. *We have* $D_{k_\ell-1} = [N]d_{(N-1)+\ell} + C_{k_\ell} \quad (\ell \geq 1)$.

Proof of Claim. Let $Q = [N]d_{(N-1)+\ell} + C_{k_\ell}$ and $P = (1 + C_{k_\ell}Q)/D_{k_\ell}$. Thus,

$$PD_{k_\ell} - QC_{k_\ell} = \left(\frac{1 + C_{k_\ell}Q}{D_{k_\ell}}\right) D_{k_\ell} - QC_{k_\ell} = 1.$$

We first show that $P \in \mathbb{F}_2[x]$. In $\mathbb{F}_2[x]$, we have

$$x(x+1) \mid [i] \quad (i \in \mathbb{N}). \tag{11}$$

From Remark 1 (2), since $x(x+1) \mid [i] \ (i \in \mathbb{N})$, we have

$$(x(x+1))^{2^i-1} \mid d_i \quad (i \in \mathbb{N}). \tag{12}$$

Now we consider

$$
\begin{aligned}
P &= \frac{1+C_{k_\ell}Q}{D_{k_\ell}} = \frac{1+C_{k_\ell}\left([N]d_{(N-1)+\ell}+C_{k_\ell}\right)}{D_{k_\ell}}\\
&= \left\{[N]d_{(N-1)+\ell}\left(1+\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_{(N-1)+\ell-1}}+\frac{1}{d_{(N-1)+\ell}}\right)\right.\\
&\qquad \left.+\left(d_{(N-1)+\ell}+\frac{d_{(N-1)+\ell}}{d_1^2}+\frac{d_{(N-1)+\ell}}{d_2^2}+\cdots+\frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}^2}\right)\right\} / \left(x(x+1)\right)^m .
\end{aligned} \tag{13}
$$

For fixed $\ell \geq 1$ and $j \in \{1, 2, \ldots, (N-1)+\ell\}$, we get

$$
x^{2^N} + x^{2^j} \equiv 0 \pmod{\left(x(x+1)\right)^{2^{\min\{N,j\}}}}
$$

and

$$
\begin{aligned}
2^{(N-1)+\ell} - 2^j + 2^{\min\{N,j\}} &= \begin{cases} 2^{(N-1)+\ell} & \text{if } \min\{N,j\}=j \\ 2^{(N-1)+\ell} - 2^j + 2^N & \text{if } \min\{N,j\}=N \end{cases}\\
&\geq 2^N.
\end{aligned} \tag{14}
$$

By (11), (12) and (14), it follows that $[N]d_{(N-1)+\ell} \equiv 0 \pmod{\left(x(x+1)\right)^{2^{(N-1)+\ell}}}$, and

$$
\begin{aligned}
&\frac{[N]d_{(N-1)+\ell}}{d_j} + \frac{d_{(N-1)+\ell}}{d_{j-1}^2} = \frac{d_{(N-1)+\ell}}{d_j}\left([N]+[j]\right)\\
&= \frac{d_{(N-1)+\ell}}{d_j}\left(x^{2^N}+x^{2^j}\right) \equiv 0 \pmod{\left(x(x+1)\right)^{2^N}} \quad (j \in \{1,2,\ldots,(N-1)+\ell\},
\end{aligned}
$$

i.e., $P \in \mathbb{F}_2[x]$.

From $PD_{k_\ell} - QC_{k_\ell} = 1,\ C_{k_\ell-1}D_{k_\ell} - D_{k_\ell-1}C_{k_\ell} = 1$, we get

$$
PD_{k_\ell} - QC_{k_\ell} = C_{k_\ell-1}D_{k_\ell} - D_{k_\ell-1}C_{k_\ell},\ C_{k_\ell}\left(D_{k_\ell-1}-Q\right) = D_{k_\ell}\left(C_{k_\ell-1}-P\right).
$$

Since C_{k_ℓ} and D_{k_ℓ} are relatively prime, by (13) we have

$$
\deg P = \deg d_{(N-1)+\ell} + 2^N - 2m \leq \deg d_{(N-1)+\ell} - 2 < \deg d_{(N-1)+\ell} = \deg C_{k_\ell}.
$$

By definition, $\deg(C_{k_\ell-1} - P) < \deg C_{k_\ell}$. Thus, $C_{k_\ell-1} = P$ and so $D_{k_\ell-1} = Q = [N]d_{(N-1)+\ell} + C_{k_\ell}$, which proves the claim.

Next, we show that

$$
\frac{[N+\ell]+[N]}{\left(x(x+1)\right)^m} \in \mathbb{F}_2[x] \setminus \{0\} \quad (\ell \geq 1).
$$

This follows from

$$\frac{[N+\ell]+[N]}{(x(x+1))^m} = \frac{\left(x^{2^{N+\ell}}+x\right)+\left(x^{2^N}+x\right)}{(x(x+1))^m} = \frac{x^{2^N}(x+1)^{2^N(2^\ell-1)}}{(x(x+1))^m},$$

$2^\ell - 1 \geq 1$, and $2^N \geq m$. Applying Lemma 2 (1), we get

$$\left[0; \overrightarrow{X}_{k_\ell}, \frac{[N+\ell]+[N]}{(x(x+1))^m}, \overrightarrow{X}_{k_\ell}\right] = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{D_{k_\ell}^2\left(\left(\frac{[N+\ell]+[N]}{(x(x+1))^m}\right)+\left(\frac{C_{k_\ell}+D_{k_\ell-1}}{D_{k_\ell}}\right)\right)}$$

$$= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x(x+1))^m d_{(N-1)+(\ell+1)}} = \sum_{i=0}^{(N-1)+\ell+1} \frac{1}{(x(x+1))^m d_i}.$$

Thus

$$\sum_{i=0}^{(N-1)+1} \frac{1}{(x(x+1))^m d_i} = \left[0; \overrightarrow{X}_{k_\ell}\right]$$

$$\sum_{i=0}^{(N-1)+2} \frac{1}{(x(x+1))^m d_i} = \left[0; \overrightarrow{X}_{k_1}, \frac{[N+1]+[N]}{(x(x+1))^m}, \overrightarrow{X}_{k_1}\right].$$

Continuing analogously, we arrive at

$$\frac{e}{(x(x+1))^m} = \left[0; \overrightarrow{X}_{k_1}, \frac{[N+1]+[N]}{(x(x+1))^m}, \overrightarrow{X}_{k_1}, \frac{[N+2]+[N]}{(x(x+1))^m}, \ldots\right].$$

□

4.2. Two-fold Continued Fraction of Type 3

In this subsection, we work in the field of formal series over a field F of characteristic 0. For $n \in \mathbb{N}$, a continued fraction $[a_0, ; a_1, a_2, \ldots, a_n]$ is said to be *palindromic* if the word $a_1, a_2, \ldots, a_n$ is equal to its reversal. It is not hard to see that if a continued fraction $[a_0, ; a_1, a_2, \ldots, a_n] = C_n/D_n$ is palindromic, then $C_n = D_{n-1}$. Let

$$f(T) = T(T+2)(T-2)g(T) - T^2 + 2 \in (F[x])[T],$$

be monic in T, with monic $g(T) \in (F[x])[T]$. Let

$$f_0(T) = T,\ \ f_n(T) = f\left(f_{n-1}(T)\right)\ \ (n \geq 1),$$

i.e., $f_n = f_1 \circ f_1 \circ \cdots \circ f_1$ (n composites). Let $A_0(T) = 1,\ B_0(T) = f_0(T) = T$ and for $n \geq 1$, let

$$A_n(T) = (-1)^n + \sum_{m=1}^{n} (-1)^{m+1} f_m(T) f_{m+1}(T) \cdots f_n(T) = (-1)^n + f_n(T) A_{n-1}(T) \tag{15}$$

$$B_n(T) = f_0(T) f_1(T) \cdots f_n(T). \tag{16}$$

It follows that

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_0(T)f_1(T)\cdots f_n(T)} = \frac{A_\ell(T)}{B_\ell(T)}$$
$$\sum_{n=0}^{\ell+1} \frac{(-1)^n}{f_0(T)f_1(T)\cdots f_n(T)} = \frac{A_\ell(T)}{B_\ell(T)} + \frac{(-1)^{\ell+1}}{f_0(T)f_1(T)\cdots f_{\ell+1}(T)}.$$

Lemma 5. *For $\ell, i \in \mathbb{N} \cup \{0\}$, if $f(T) \in (F[x])\,[T] \setminus \{0\}$, then*

$$A_\ell(f_i(T)) \equiv \pm\, A_{\ell+i}(T) \pmod{f_i(T)}\,.$$

Proof. The case $i = 0$ is trivial. If $i > 0$ and $\ell = 0$, then the desired result follows from the definition of A_0. For $\ell, i \geq 1$, from

$$\begin{aligned} A_\ell(f_i(T)) &= (-1)^\ell + \sum_{m=1}^{\ell} (-1)^{m+1} f_m(f_i(T)) \cdots f_\ell(f_i(T)) \\ &= (-1)^\ell + \sum_{m=i+1}^{\ell+i} (-1)^{m+1-i} f_m(T) f_{m+1}(T) \cdots f_{\ell+i}(T), \end{aligned}$$

we get

$$\begin{aligned} A_{\ell+i}(T) &= (-1)^{\ell+i} + \sum_{m=1}^{i} (-1)^{m+1} f_m(T) \cdots f_{\ell+i}(T) \\ &\qquad + \sum_{m=i+1}^{\ell+i} (-1)^{m+1} f_m(T) \cdots f_{\ell+i}(T) \\ &= (-1)^i A_\ell(f_i(T)) + \sum_{m=1}^{i} (-1)^{m+1} f_m(T) \cdots f_{\ell+i}(T) \\ &\equiv \pm\, A_\ell(f_i(T)) \pmod{f_i(T)}\,. \end{aligned}$$

□

Replacing T with a nonzero polynomial $Z(x) := Z$ in $F[x]$, we get:

Lemma 6. *If $Z \in F[x] \setminus \{0\}$, then $Z \mid \left(A_\ell^2(Z) - 1\right)$ for all $\ell \in \mathbb{N} \cup \{0\}$.*

Proof. Since $f_1(0) = 2$, by induction we have $f_\ell(0) = -2$ $(\ell \geq 2)$. To prove the lemma, it suffices to show that $A_\ell(0) = \pm 1$ for all $\ell \in \mathbb{N} \cup \{0\}$. Clearly, $A_0(Z) = 1$. Thus,

$$A_1(0) = (-1)^1 + f_1(0) \cdot A_0(0) = -1 + 2 \cdot 1 = 1.$$

By induction, we get $A_\ell(0) = (-1)^{\ell+1}$ $(\ell \geq 1)$, and the desired result follows. □

Lemma 7. *If $Z \in F[x]\backslash F$, then $B_\ell(Z) \neq 0,\ B_\ell(Z) \mid (A_\ell^2(Z){-}1)$ for all $\ell \in \mathbb{N}\cup\{0\}$.*

Proof. We have

$$2 \leq |f_0(Z)| < |f_1(Z)| < |f_2(Z)| < \cdots ,$$

and (16) implies that $B_\ell(Z) \neq 0$ ($\ell \in \mathbb{N} \cup \{0\}$). Now from Lemma 6, we get

$$f_\ell(Z) \mid (A_\ell^2(f_\ell(Z)) - 1) \quad (\ell \geq 0). \tag{17}$$

We also have from Lemma 5 that for $\ell, i \in \mathbb{N} \cup \{0\}$, either

$$A_\ell^2(f_i(Z)) = A_{\ell+i}^2(Z) + 2Df_i(Z)A_{\ell+i}(Z) + D^2 f_i^2(Z),$$

or

$$A_\ell^2(f_i(Z)) = A_{\ell+i}^2(Z) - 2Df_i(Z)A_{\ell+i}(Z) + D^2 f_i^2(Z),$$

for some $D \in F[x]$. By (17), we have $f_i(Z) \mid (A_{\ell+i}^2(Z) - 1)$. Specifically,

$$f_i(Z) \mid (A_{(\ell-i)+i}^2(Z) - 1) = A_\ell^2(Z) - 1 \quad (i = 0, 1, \ldots \ell). \tag{18}$$

It remains to prove that

$$B_\ell(Z) = f_0(Z)f_1(Z)\ldots f_\ell(Z) \mid (A_\ell^2(Z) - 1). \tag{19}$$

For $0 \leq j < k$, since $f_k(Z) = f_{k-j}(f_j(Z)) \equiv f_{k-j}(0) \pmod{f_j(Z)}$, and

$$f_{k-j}(0) = \begin{cases} 2 & \text{for } k = j+1 \\ -2 & \text{for } k > j+1, \end{cases}$$

we deduce that, for all $j \neq k$,

$$\gcd(f_j(Z), f_k(Z)) = \gcd(f_j(Z), 2) \in F, \tag{20}$$

i.e., $f_j(Z), f_k(Z)$ are relatively prime. Hence, (19) follows from (18) and (20). □

An analogoue of Tamura's result [8] in the field of formal Laurent series reads:

Theorem 5. *If $Z \in F[x] \setminus F$ is monic, then $1/f_0(Z) = [0; Z]$, and if*

$$\sum_{n=0}^{\ell-1} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = \left[0; \overrightarrow{X}_{k_\ell}\right] \quad (\ell \geq 1)$$

is a palindromic continued fraction, then

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = \left[0; \overrightarrow{X}_{k_\ell}, u_\ell(Z), \overleftarrow{X}_{k_\ell}\right], \tag{21}$$

where

$$u_\ell(Z) = (-1)^{\ell-1}\frac{f_\ell(Z)}{B_{\ell-1}(Z)} - 2\frac{A_{\ell-1}(Z)}{B_{\ell-1}(Z)}, \quad A_{\ell-1}(Z) = C_{k_\ell}, \quad B_{\ell-1}(Z) = D_{k_\ell}$$

with C_{k_ℓ}/D_{k_ℓ} *being the* ${k_\ell}^{th}$ *convergent of* $\left[0; \overrightarrow{X}_{k_\ell}\right]$.

In particular, the continued fraction representing the corresponding infinite sum takes the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = [0; Z, u_1(Z), Z, u_2(Z), Z, u_1(Z), Z, u_3(Z), \ldots].$$

Proof. For $\ell \in \mathbb{N}$, let $\alpha_\ell = (-1)^{\ell-1}f_0(Z)f_1(Z)\cdots f_{\ell-1}(Z)$ and let $C_{k_\ell}/D_{k_\ell} =: \left[0; \overrightarrow{X}_{k_\ell}\right]$ be the ${k_\ell}^{\text{th}}$ convergent of the continued fraction of

$$\begin{aligned}\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_\ell} &= \frac{1}{f_0(Z)} + \frac{-1}{f_0(Z)f_1(Z)} + \cdots + \frac{(-1)^{\ell-1}}{f_0(Z)f_1(Z)\cdots f_{\ell-1}(Z)} \\ &= \sum_{n=0}^{\ell-1} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)}.\end{aligned}$$

Clearly, $1/f_0(Z) = [0; Z] =: C_{k_1}/D_{k_1}$, so k_1 is an odd positive integer. From Lemma 7, we know that $A_{\ell-1}(Z)$ and $B_{\ell-1}(Z)$ are relatively prime. Since $A_{\ell-1}(Z), B_{\ell-1}(Z)$ are monic (in Z) and $B_{\ell-1}(Z)$, D_{k_ℓ} are monic (in Z), we infer that

$$B_{\ell-1}(Z) = D_{k_\ell} \tag{22}$$

and so $A_{\ell-1}(Z) = C_{k_\ell}$. Since $\left[0; \overrightarrow{X}_{k_\ell}\right]$ is palindromic, we have

$$C_{k_\ell} = D_{k_\ell - 1}. \tag{23}$$

Next, we show that if $Z \in F[x] \setminus F$, then $u_\ell(Z) \in F[x] \setminus F$. By (15), we get

$$A_\ell(Z)^2 = (-1)^{2\ell} + 2(-1)^\ell f_\ell(Z)A_{\ell-1}(Z) + f_\ell(Z)^2 A_{\ell-1}(Z)^2. \tag{24}$$

By Lemma 7 and (24), we get

$$\begin{aligned}u_\ell(Z) &= (-1)^{\ell-1}(Z)\frac{f_\ell(Z)}{B_{\ell-1}(Z)} - \frac{2A_{\ell-1}(Z)}{B_{\ell-1}(Z)} \\ &= (-1)^\ell f_\ell(Z)\frac{A_{\ell-1}(Z)^2 - 1}{B_{\ell-1}(Z)} + (-1)^{\ell-1}\frac{A_\ell(Z)^2 - 1}{B_\ell(Z)} \in F[x] \setminus F.\end{aligned}$$

From (22) and (23), we get

$$\begin{aligned}&-\left(2D_{k_\ell-1}D_{k_\ell} + q_{k_\ell}^2 u_\ell\right) \\ &\quad = -\left(2A_{\ell-1}(Z)B_{\ell-1}(Z) + B_{\ell-1}(Z)^2\left((-1)^{\ell-1}\frac{f_\ell(Z)}{B_{\ell-1}(Z)} - 2\frac{A_{\ell-1}(Z)}{B_{\ell-1}(Z)}\right)\right) \\ &\quad = (-1)^\ell B_{\ell-1}(Z)f_\ell(Z) = (-1)^\ell f_0(Z)f_1(Z)\cdots f_\ell(Z) = \alpha_{\ell+1}.\end{aligned}$$

We observe that $\{k_\ell\}_{\ell\geq 1}$ obtained by this process is a sequence of odd positive integers, and so

$$(-1)^{k_\ell}\left(2D_{k_\ell-1}D_{k_\ell}+q_{k_\ell}^2u_\ell\right)=\alpha_{\ell+1}.$$

Using Theorem 1 (3), we get

$$\left[0;\overrightarrow{X}_{k_\ell},u_\ell(Z),\overleftarrow{X}_{k_\ell}\right]=\sum_{i=1}^{\ell+1}\frac{1}{\alpha_i}=\sum_{n=0}^{\ell}\frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)}.$$

□

The above proof with some minor changes is also applicable to some other forms of $f(T)$ such as $T(T+2)(T-2)g(T)+T^2-2$.

4.3. Two-fold Continued Fraction of Type 4

In this subsection we consider $F=\mathbb{F}_q$, the finite field of q elements. Making use of the two-fold continued fraction of type 4, we now extend and complement the works of [10]. We begin with:

Theorem 6. *Let $\{Q_i\}_{i\geq 1}$ be a sequence of nonconstant monic polynomials over the finite field $\mathbb{F}_q$, where q is a prime power. Assume that there exists $N\in\mathbb{N}\cup\{0\}$ such that*

$$Q_1Q_2\cdots Q_{j+1}\mid Q_{j+2}\quad (j\geq N) \tag{25}$$

and that if $N\geq 1$, then

$$\gcd\left((Q_2\cdots Q_{N+1})+(Q_3\cdots Q_{N+1})+\cdots+Q_{N+1}+1,Q_1Q_2\cdots Q_{N+1}\right)=1. \tag{26}$$

If $\sum_{i=1}^{N+\ell}1/Q_1Q_2\cdots Q_i=[0;a_1,a_2,\ldots,a_{k_\ell}]\quad(\ell\geq 1)$, then

$$\sum_{i=1}^{N+\ell+1}\frac{1}{Q_1Q_2\cdots Q_i}=\left[0;a_1,a_2,\ldots,a_{k_\ell},\frac{(-1)^{k_\ell}Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}},-a_{k_\ell},\ldots,-a_2,-a_1\right].$$

Proof. Let $C_{k_\ell}/D_{k_\ell}:=\left[0;\overrightarrow{X}_{k_\ell}\right]$ be the k_ℓ^{th} convergent of the continued fraction of $\sum_{i=1}^{N+\ell}1/Q_1Q_2\cdots Q_i$. We observe that both C_{k_ℓ} and D_{k_ℓ} are monic. From

$$\sum_{i=1}^{N+\ell}\frac{1}{Q_1Q_2\cdots Q_i}=\frac{(Q_2Q_3\cdots Q_{N+\ell})+(Q_3Q_4\cdots Q_{N+\ell})+\cdots+Q_{N+\ell}+1}{Q_1Q_2\cdots Q_{N+\ell}},$$

we assert that

$$\gcd\left((Q_2Q_3\cdots Q_{N+\ell})+(Q_3Q_4\cdots Q_{N+\ell})+\cdots+Q_{N+\ell}+1,Q_1Q_2\cdots Q_{N+\ell}\right)=1.$$

For $N\geq 1$ and $\ell=1$, this is obvious from (26). Next, we treat the remaining cases.

Suppose there exists a prime $p \in \mathbb{F}_q[x]$ such that

$$p \mid \left((Q_2Q_3 \cdots Q_{N+\ell}) + (Q_3Q_4 \cdots Q_{N+\ell}) + \cdots + Q_{N+\ell} + 1 \right), \quad p \mid Q_1Q_2 \cdots Q_{N+\ell}.$$

If $N = 0$, by (25), we have $Q_1Q_2 \cdots Q_i \mid Q_{i+1}$. Since $p \mid (Q_1Q_2 \cdots Q_\ell)$, then $p \mid Q_k$ for some $1 \le k \le \ell$, and so $p \mid Q_jQ_{j+1} \cdots Q_\ell$ for all $2 \le j \le k$. Since $Q_1Q_2 \cdots Q_k \mid Q_{k+t}$ $(1 \le t \le \ell - k)$, we have $Q_k \mid Q_{k+l} \cdots Q_\ell$, and so $p \mid Q_{k+t} \cdots Q_\ell$ $(1 \le t \le \ell - k)$. Since $p \mid \left((Q_2Q_3 \cdots Q_\ell) + (Q_3Q_4 \cdots Q_\ell) + \cdots + Q_\ell + 1 \right)$, then $p \mid 1$, a contradiction. Thus

$$\gcd \left((Q_2Q_3 \cdots Q_\ell) + (Q_3Q_4 \cdots Q_\ell) + \cdots + Q_\ell + 1, Q_1Q_2 \cdots Q_\ell \right) = 1.$$

If $N \ge 1$ and $\ell \ge 2$, since $p \mid (Q_1Q_2 \cdots Q_{N+\ell})$, then $p \mid Q_k$ for some $1 \le k \le N + \ell$. If $p \mid Q_{N+\ell}$, since

$$p \mid \left((Q_2Q_3 \cdots Q_{N+\ell}) + (Q_3Q_4 \cdots Q_{N+\ell}) + \cdots + Q_{N+\ell} + 1 \right),$$

then $p \mid 1$, a contradiction.

Assume that $p \mid Q_k$ for some $1 \le k \le N+\ell-1$. Using (25) when $j = N+\ell-2 \ge N$, we get $Q_1Q_2 \cdots Q_{N+\ell-1} \mid Q_{N+\ell}$, which implies that $p \mid Q_{N+\ell}$, again we have a contradiction. Thus,

$$\gcd \left((Q_2Q_3 \cdots Q_{N+\ell}) + (Q_3Q_4 \cdots Q_{N+\ell}) + \cdots + Q_{N+\ell} + 1, Q_1Q_2 \cdots Q_{N+\ell} \right) = 1.$$

Since C_{k_ℓ} and D_{k_ℓ} are relatively prime, and all Q_i are monic, we have $D_{k_\ell} = Q_1Q_2 \cdots Q_{N+\ell}$. For $\ell \ge 1$, using (25) with $j = N + \ell - 1 \ge N$, we get

$$(-1)^{k_\ell} Q_{N+\ell+1} / Q_1Q_2 \cdots Q_{N+\ell} \in \mathbb{F}_q[x] \setminus \{0\}.$$

Applying Lemma 2 (4), we get

$$\left[0; a_1, a_2, \ldots, a_{k_\ell}, \frac{(-1)^{k_\ell} Q_{N+\ell+1}}{Q_1Q_2 \cdots Q_{N+\ell}}, -a_{k_\ell}, \ldots, -a_2, -a_1 \right] = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2 \frac{(-1)^{k_\ell} Q_{N+\ell+1}}{Q_1Q_2 \cdots Q_{N+\ell}}}$$

$$= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{(Q_1Q_2 \cdots Q_{N+\ell})^2 \frac{(-1)^{k_\ell} Q_{N+\ell+1}}{Q_1Q_2 \cdots Q_{N+\ell}}} = \sum_{i=1}^{N+\ell+1} \frac{1}{Q_1Q_2 \cdots Q_i}.$$

□

Note that Theorem 6 is contained in the following proposition of [5] which can also be proved by taking $\alpha_{\ell-1} = P_{\ell+1}$ $(\ell \ge I + 1)$ in Theorem 1 (4).

Proposition 4. *Let I be a fixed positive integer, $\{k_i\}_{i \ge 1}$ a sequence of positive integers, $\{c_i\}_{i \ge I}$ a sequence of nonzero polynomials over $\mathbb{F}_q$, subject to the condition* *that if $I = 1$, then c_1 and those c_i $(i \geq 2)$ for which $k_i = 2$ are nonconstant polynomials over $\mathbb{F}_q$. Let the sequence $\{P_i\}_{i\geq 1}$ be defined by*

$$P_1 = 1,\ P_2, P_3, \ldots, P_I \in \mathbb{F}_q[x] \setminus \mathbb{F}_q; \quad P_u = c_{u-1} P_{u-1}^{k_{u-1}} P_{u-2}^{k_{u-2}} \cdots P_{u-I}^{k_{u-I}} \quad (u \geq I+1),$$

and let

$$E(u) = \sum_{i=1}^{u} \frac{1}{P_i} \quad (u \in \mathbb{N}).$$

Assume that

(i) if $I \geq 2$, then $P_2 \mid P_3 \mid \cdots \mid P_I$;

(ii) $k_i \geq 2$ for all $i \geq I$.

If $E(u) = [a_0; a_1, a_2, \ldots, a_n]$ $(u \geq I+1)$, then there exists $\beta \in \mathbb{F}_q \setminus \{0\}$ such that

$$E(u+1) = [a_0; a_1, a_2, \ldots, a_n, \beta s_u, -a_n, \ldots, -a_2, -a_1],$$

where $s_u = c_u P_u^{k_u - 1} / c_{u-1} P_{u-I}^{k_{u-I}}$.

Now we apply Theorem 6 to determine explicit continued fractions of $e/f(x)$, where $f(x)$ is a nonconstant monic polynomial satisfying $f(x) \mid [1]$.

Corollary 1. *Let $f(x) \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ be monic, q a prime power. If $f(x) \mid [1]$, then*

$$\frac{e}{f(x)} = \left[0; \underbrace{\underbrace{f(x)}, \frac{-[1]}{f(x)}, \underbrace{-f(x)}}, \frac{-[2]d_1^{q-2}}{f(x)}, \underbrace{f(x), \frac{[1]}{f(x)}, -f(x)}, \frac{-[3]d_2^{q-2}}{f(x)}, \ldots \right].$$

Proof. Let $Q_1 = f(x)$ and $Q_{i+1} = d_i/d_{i-1}$ $(i \in \mathbb{N})$. Observe that

$$Q_{i+1} = \frac{d_i}{d_{i-1}} = \frac{[i]d_{i-1}^q}{d_{i-1}} = [i]d_{i-1}^{q-1} \in \mathbb{F}_q[x] \setminus \mathbb{F}_q.$$

Since $Q_2 = d_1 = [1]$ and $f(x) \mid [1]$, we have $Q_1 \mid Q_2$. For $i \geq 2$, from

$$Q_1 Q_2 Q_3 \cdots Q_i = f(x) \frac{d_1}{d_0} \frac{d_2}{d_2} \cdots \frac{d_{i-1}}{d_{i-2}} = f(x) d_{i-1}$$

and $Q_{i+1} = d_i/d_{i-1}$, we get

$$\frac{Q_{i+1}}{Q_1 Q_2 Q_3 \cdots Q_i} = \frac{d_i/d_{i-1}}{f(x)d_{i-1}} = \frac{[i]d_{i-1}^q}{f(x)d_{i-1}^2} = \frac{[i]d_{i-1}^{q-2}}{f(x)}.$$

We treat two separate cases.

If $q \geq 3$, since $f(x) \mid [1]$ and $[1] \mid d_i$ $(i \in \mathbb{N})$, then $f(x) \mid d_i$ $(i \in \mathbb{N})$, which implies that $Q_1 Q_2 Q_3 \cdots Q_i \mid Q_{i+1}$.

If $q = 2$, since $[1] \mid [i]$ $(i \in \mathbb{N})$, then $f(x) \mid [i]$, and so $Q_1Q_2Q_3 \cdots Q_i \mid Q_{i+1}$. Applying Theorem 6 with $N = 0$, we get

$$\frac{1}{f(x)} = [0; f(x)]$$
$$\frac{1}{f(x)} + \frac{1}{f(x)d_1} = \left[0; f(x), \frac{-[1]}{f(x)}, -f(x)\right]$$
$$\frac{1}{f(x)} + \frac{1}{f(x)d_1} + \frac{1}{f(x)d_2} = \left[0; f(x), \frac{-[1]}{f(x)}, -f(x), \frac{-[2]d_1^{q-2}}{f(x)}, f(x), \frac{[1]}{f(x)}, -f(x)\right].$$

Continuing in the same manner, we finally arrive at

$$\frac{e}{f(x)} = \left[0; f(x), \frac{-[1]}{f(x)}, -f(x), \frac{-[2]d_1^{q-2}}{f(x)}, f(x), \frac{[1]}{f(x)}, -f(x), \frac{-[3]d_2^{q-2}}{f(x)}, \dots\right].$$

□

We now show, using Corollary 1, how to derive explicit continued fractions of

$$e/x,\ e/\left(x^{q-1}-1\right),\ e/(x-1),\ e/\left(x^{q-2}+x^{q-3}+\cdots+1\right),\ e/x(x-1),$$
$$e/x\left(x^{q-2}+x^{q-3}+\cdots+1\right),\ e/[1]$$

for $q \geq 2$ being a prime power, and show, using Theorem 6, how to find explicit continued fractions of

$$e/x^m,\ e/(x-1)^m,\ e/\left(x(x-1)\right)^m,\ e/\left(x^{q-1}-1\right)^m,\ e/\left(x^{q-2}+x^{q-3}+\cdots+1\right)^m,$$
$$e/\left(x\left(x^{q-2}+x^{q-3}+\cdots+1\right)\right)^m,\ e/[1]^m,$$

for a prime power $q \geq 3$ and integer $m \geq 2$. To this end, we need to use two appropriate partitions of positive integers.

4.3.1. Partition 1

For a prime power $q \geq 3$, let

$$\begin{aligned}
L_1 &= 2 & R_1 &= q-1 \\
L_2 &= q & R_2 &= q^2-q-1 \\
L_3 &= q^2-q & R_3 &= q^3-q^2-q-1 \\
&\vdots & &\vdots \\
L_N &= q^{N-1}-q^{N-2}-\cdots-q^2-q & R_N &= q^N-q^{N-1}-\cdots-q-1 \quad (N \geq 3).
\end{aligned}$$

Observe that for $M \neq N$, we have

$$\{n \in \mathbb{N}; n \geq 2\} = \left(\bigcup_{N \geq 1} [L_N, R_N]\right) \cap \mathbb{Z},\ [L_N, R_N] \cap [L_M, R_M] = \emptyset.$$

For a fixed integer $m > 1$, there exists a unique N in $\mathbb{N}$ such that $m \in [L_N, R_N]$.

Corollary 2. *Let $q > 2$ be a prime power. We have*

(1) $$\frac{e}{x^m} = \left[0; \underbrace{\underbrace{\overrightarrow{X}_{k_1}}, u_1, \underbrace{-\overleftarrow{X}_{k_1}}}, u_2, \underbrace{\overrightarrow{X}_{k_1}, -u_1, -\overleftarrow{X}_{k_1}}, u_3, \ldots\right],$$
where $\left[0; \overrightarrow{X}_{k_1}\right] := \sum_{i=0}^{N} \frac{1}{x^m d_i}$, $u_\ell := \frac{(-1)^{k_\ell}[N+\ell] d_{N+\ell-1}^{q-2}}{x^m}$;

(2) $$\frac{e}{(x-1)^m} = \left[0; \underbrace{\underbrace{\overrightarrow{Y}_{k_1}}, v_1, \underbrace{-\overleftarrow{Y}_{k_1}}}, v_2, \underbrace{\overrightarrow{Y}_{k_1}, -v_1, -\overleftarrow{Y}_{k_1}}, v_3, \ldots\right],$$
where $\left[0; \overrightarrow{Y}_{k_1}\right] := \sum_{i=0}^{N} \frac{1}{(x-1)^m d_i}$, $v_\ell := \frac{(-1)^{k_\ell}[N+\ell] d_{N+\ell-1}^{q-2}}{(x-1)^m}$;

(3) $$\frac{e}{(x(x-1))^m} = \left[0; \underbrace{\underbrace{\overrightarrow{Z}_{k_1}}, w_1, \underbrace{-\overleftarrow{Z}_{k_1}}}, w_2, \underbrace{\overrightarrow{Z}_{k_1}, -w_1, -\overleftarrow{Z}_{k_1}}, w_3, \ldots\right],$$
where $\left[0; \overrightarrow{Z}_{k_1}\right] := \sum_{i=0}^{N} \frac{1}{(x(x-1))^m d_i}$, $w_\ell := \frac{(-1)^{k_\ell}[N+\ell] d_{N+\ell-1}^{q-2}}{(x(x-1))^m}$.

Proof. (1) Let $Q_1 = x^m$ and $Q_{i+1} = d_i/d_{i-1}$ ($i \in \mathbb{N}$). Observe that

$$Q_{i+1} = \frac{d_i}{d_{i-1}} = \frac{[i]d_{i-1}^q}{d_{i-1}} = [i]d_{i-1}^{q-1} \in \mathbb{F}_q[x] \setminus \mathbb{F}_q \quad (i \geq 1),$$

For $j \geq N$, we write $j = N + h,\ h \geq 0$, to get

$$\frac{Q_{N+h+2}}{Q_1 Q_2 \cdots Q_{N+h+1}} = \frac{d_{N+h+1}/d_{N+h}}{x^m d_{N+h}} = \frac{[N+h+1]d_{N+h}^{q-2}}{x^m}.$$

First, we show that $x^m \mid [N+h+1]d_{N+h}^{q-2} \quad (h \geq 0)$. By Remark 1 (2), we have

$$[N+h+1]d_{N+h}^{q-2} = [N+h+1]\left([N+h][N+h-1]^q[N+h-2]^{q^2}\cdots[1]^{q^{N+h-1}}\right)^{q-2}.$$

Since $x \mid [i]$ ($i \in \mathbb{N}$), we have $x^{(q-2)\left(q^{N+h-1}+q^{N+h-2}+\cdots+q+1\right)+1} \mid [N+h+1]d_{N+h}^{q-2}$. For $h \geq 0$, since

$$\begin{aligned}
&(q-2)\left(q^{N+h-1}+q^{N+h-2}+\cdots+q+1\right)+1 \\
&\quad \geq (q-2)\left(q^{N-1}+q^{N-2}+\cdots+q+1\right)+1 = q^N - q^{N-1} - \cdots - q - 1 \geq m,
\end{aligned}$$

we have $x^m \mid [N+h+1]d_{N+h}^{q-2}$. Thus, Q_i satisfies (25). Using Lemma 4, we get

$$\begin{aligned}
&\gcd\left((Q_2Q_3\cdots Q_{N+1}) + (Q_3Q_4\cdots Q_{N+1}) + \cdots + Q_{N+1} + 1, Q_1Q_2\cdots Q_{N+1}\right) \\
&= \gcd\left(d_N + \frac{d_N}{d_1} + \frac{d_N}{d_2} + \cdots + \frac{d_N}{d_{N-1}} + 1, x^m d_N\right) = 1.
\end{aligned}$$

For $\ell \geq 1$, since

$$\frac{(-1)^{k_\ell} Q_{N+\ell+1}}{Q_1 Q_2 \cdots Q_{N+\ell}} = \frac{(-1)^{k_\ell}[N+\ell] d_{N+\ell-1}^{q-2}}{x^m} = u_\ell,$$

applying Theorem 6, we get

$$\sum_{i=1}^{N+1} \frac{1}{Q_1 Q_2 \cdots Q_i} = \sum_{i=0}^{N} \frac{1}{x^m d_i} = \left[0; \overrightarrow{X}_{k_1}\right]$$
$$\sum_{i=1}^{N+2} \frac{1}{Q_1 Q_2 \cdots Q_i} = \sum_{i=0}^{N+1} \frac{1}{x^m d_i} = \left[0; \overrightarrow{X}_{k_1}, \frac{(-1)^{k_1}[N+1] d_N^{q-2}}{x^m}, -\overleftarrow{X}_{k_1}\right].$$

Continuing the procedure, we arrive at

$$\frac{e}{x^m} = \left[0; \overrightarrow{X}_{k_1}, u_1, -\overleftarrow{X}_{k_1}, u_2, \overrightarrow{X}_{k_1}, -u_1, -\overleftarrow{X}_{k_1}, u_3, \ldots\right].$$

The proofs of (2) and (3) are similarly done by taking $Q_1 = (x-1)^m$ and $Q_1 = (x(x-1))^m$, respectively. □

4.3.2. Partition 2

For a prime power $q \geq 3$, let

$$\begin{aligned}
\mathcal{L}_1 &= 1 & \mathcal{R}_1 &= q-2 \\
\mathcal{L}_2 &= q-1 & \mathcal{R}_2 &= q^2-2q \\
\mathcal{L}_3 &= q^2-2q+1 & \mathcal{R}_3 &= q^3-2q^2 \\
&\vdots & &\vdots \\
\mathcal{L}_N &= q^{N-1}-2q^{N-2}+1 & \mathcal{R}_N &= q^N-2q^{N-1} \quad (N \geq 3).
\end{aligned}$$

Observe that, for $M \neq N$,

$$\mathbb{N} = \left(\bigcup_{N \geq 1} [\mathcal{L}_N, \mathcal{R}_N]\right) \cap \mathbb{Z}, \ [\mathcal{L}_N, \mathcal{R}_N] \cap [\mathcal{L}_M, \mathcal{R}_M] = \emptyset.$$

For a fixed positive integer $m > 1$, there exists a unique N in $\mathbb{N}$ such that $m \in [\mathcal{L}_N, \mathcal{R}_N]$.

Corollary 3. *Let $q > 2$ be a prime power. We have*

(1) $$\frac{e}{(x^{q-1}-1)^m} = \left[0; \underbrace{\underbrace{\overrightarrow{W}_{k_1}}, u_1, \underbrace{-\overleftarrow{W}_{k_1}}}, u_2, \underbrace{\overrightarrow{W}_{k_1}, -u_1, -\overleftarrow{W}_{k_1}}, u_3, \ldots\right],$$

where $\left[0; \overrightarrow{W}_{k_1}\right] := \sum_{i=0}^{N} \frac{1}{(x^{q-1}-1)^m d_i}, \ u_\ell := \frac{(-1)^{k_\ell}[N+\ell] d_{N+\ell-1}^{q-2}}{(x^{q-1}-1)^m};$

(2) $$\frac{e}{(x^{q-2}+x^{q-3}+\cdots+1)^m} = \left[0; \underbrace{\underbrace{\overrightarrow{X}_{k_1}}, v_1, \underbrace{-\overleftarrow{X}_{k_1}}}, v_2, \underbrace{\overrightarrow{X}_{k_1}, -v_1, -\overleftarrow{X}_{k_1}}, v_3, \ldots\right],$$

where $\left[0; \overrightarrow{X}_{k_1}\right] := \sum_{i=0}^{N} \frac{1}{(x^{q-2}+x^{q-3}+\cdots+1)^m d_i}$, $v_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{(x^{q-2}+x^{q-3}+\cdots+1)^m}$;

(3) $$\frac{e}{(x\,(x^{q-2}+x^{q-3}+\cdots+1))^m}$$
$$= \left[0; \underbrace{\underbrace{\overrightarrow{Y}_{k_1}}, w_1, \underbrace{-\overleftarrow{Y}_{k_1}}}, w_2, \underbrace{\overrightarrow{Y}_{k_1}, -w_1, -\overleftarrow{Y}_{k_1}}, w_3, \ldots\right],$$

where $\left[0; \overrightarrow{Y}_{k_1}\right] := \sum_{i=0}^{N} \frac{1}{(x(x^{q-2}+x^{q-3}+\cdots+1))^m d_i}$, $w_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{(x(x^{q-2}+x^{q-3}+\cdots+1))^m}$;

(4) $$\frac{e}{[1]^m} = \left[0; \underbrace{\underbrace{\overrightarrow{Z}_{k_1}}, y_1, \underbrace{-\overleftarrow{Z}_{k_1}}}, y_2, \underbrace{\overrightarrow{Z}_{k_1}, -y_1, -\overleftarrow{Z}_{k_1}}, y_3, \ldots\right],$$

where $\left[0; \overrightarrow{Z}_{k_1}\right] := \sum_{i=0}^{N} \frac{1}{[1]^m d_i}$, $y_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{[1]^m}$.

Proof. (1) Let $Q_1 = \left(x^{q-1}-1\right)^m$ and $Q_{i+1} = d_i/d_{i-1}$ $(i \in \mathbb{N})$. Observe that

$$Q_{i+1} = \frac{d_i}{d_{i-1}} = \frac{[i]d_{i-1}^q}{d_{i-1}} = [i]d_{i-1}^{q-1} \in \mathbb{F}_q[x] \setminus \mathbb{F}_q.$$

For $j \geq N$, we write $j = N+h,\ h \geq 0$, to get

$$\frac{Q_{N+h+2}}{Q_1 Q_2 \cdots Q_{N+h+1}} = \frac{d_{N+h+1}/d_{N+h}}{\left(x^{q-1}-1\right)^m d_{N+h}} = \frac{[N+h+1]d_{N+h}^{q-2}}{\left(x^{q-1}-1\right)^m}.$$

We claim that

$$\left(x^{q-1}-1\right)^m \mid [N+h+1]d_{N+h}^{q-2} \quad (h \geq 0).$$

By Remark 1 (2), we have

$$[N+h+1]d_{N+h}^{q-2} = [N+h+1]\left([N+h][N+h-1]^q[N+h-2]^{q^2}\cdots[1]^{q^{N+h-1}}\right)^{q-2}.$$

Since $\left(x^{q-1}-1\right) \mid [1]$, we have $\left(x^{q-1}-1\right)^{(q-2)q^{N+h-1}} \mid [N+h+1]d_{N+h}^{q-2}$. For $h \geq 0$, since $(q-2)q^{N+h-1} \geq (q-2)q^{N-1} = q^N - 2q^{N-1} \geq m$, then $\left(x^{q-1}-1\right)^m \mid [N+h+1]d_{N+h}^{q-2}$. Thus, Q_i satisfies (25). Using Lemma 4, we get

$$\begin{aligned}&\gcd\left((Q_2Q_3\cdots Q_{N+1}) + (Q_3Q_4\cdots Q_{N+1}) + \cdots + Q_{N+1} + 1, Q_1Q_2\cdots Q_{N+1}\right)\\ &= \gcd\left(d_N + \frac{d_N}{d_1} + \frac{d_N}{d_2} + \cdots + \frac{d_N}{d_{N-1}} + 1, \left(x^{q-1}-1\right)^m d_N\right) = 1.\end{aligned}$$

For $\ell \geq 1$, since

$$\frac{(-1)^{k_\ell} Q_{N+\ell+1}}{Q_1 Q_2 \cdots Q_{N+\ell}} = \frac{(-1)^{k_\ell} [N+\ell] d_{N+\ell-1}^{q-2}}{\left(x^{q-1}-1\right)^m} = u_\ell,$$

applying Theorem 6, we get

$$\sum_{i=1}^{N+1} \frac{1}{Q_1 Q_2 \cdots Q_i} = \sum_{i=0}^{N} \frac{1}{\left(x^{q-1}-1\right)^m d_i} = \left[0; \overrightarrow{W}_{k_1}\right]$$
$$\sum_{i=1}^{N+2} \frac{1}{Q_1 Q_2 \cdots Q_i} = \sum_{i=0}^{N+1} \frac{1}{\left(x^{q-1}-1\right)^m d_i} = \left[0; \overrightarrow{W}_{k_1}, \frac{(-1)^{k_1} [N+1] d_N^{q-2}}{\left(x^{q-1}-1\right)^m}, -\overleftarrow{W}_{k_1}\right]$$

Continuing the procedure, we finally arrive at

$$\frac{e}{\left(x^{q-1}-1\right)^m} = \left[0; \overrightarrow{W}_{k_1}, u_1, -\overleftarrow{W}_{k_1}, u_2, \overrightarrow{W}_{k_1}, -u_1, -\overleftarrow{W}_{k_1}, u_3, \ldots\right].$$

The proofs of (2), (3) and (4) follow similarly by taking

$$Q_1 = \left(x^{q-2} + x^{q-3} + \cdots + 1\right)^m, \ Q_1 = \left(x\left(x^{q-2} + x^{q-3} + \cdots + 1\right)\right)^m, \ Q_1 = [1]^m,$$

respectively. □

The identities for $e/f(x)^m$, $m \in \mathbb{N}$, monic $f(x) \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ in Subsections 4.1 and 4.3 and those known earlier are summarized in the next table.

$f(x)$ $(m \geq 2)$	$q = 2$	$q \geq 3$
x	Corollary 1	Corollary 1
x^m	Thakur (1996)	Corollary 2
$x^{q-1} - 1$	Corollary 1	Corollary 1
$\left(x^{q-1} - 1\right)^m$	Theorem 3	Corollary 3
$x - 1$	Corollary 1	Corollary 1
$(x-1)^m$	Theorem 3	Corollary 2
$x^{q-2} + x^{q-3} + \cdots + 1$	Thakur (1992)	Corollary 1
$\left(x^{q-2} + x^{q-3} + \cdots + 1\right)^m$	Thakur (1992)	Corollary 3
$x(x-1)$	Corollary 1	Corollary 1
$(x(x-1))^m$	Theorem 4	Corollary 2
$x\left(x^{q-2} + x^{q-3} + \cdots + 1\right)$	Corollary 1	Corollary 1
$\left(x\left(x^{q-2} + x^{q-3} + \cdots + 1\right)\right)^m$	Thakur (1996)	Corollary 3
$[1]$	Corollary 1	Corollary 1
$[1]^m$	Theorem 4	Corollary 3

4.4. Three-fold Continued Fraction of Type 13

In [2], series expansions of real numbers of the form $\sum_{n=0}^{\infty} 1/f^n(x)$, where $f^0(x) = x$ and $f^n(x) = f\left(f^{n-1}(x)\right)$ $(n \geq 1)$ are shown to have explicit continued fractions if and only if $f(x)$ satisfies one of the fourteen congruence conditions. Here, we extend one of these results using Theorem 2 (13). Let

$$f(T) = T^2(T-1)g(T) + 1 \in (F[x])[T]$$

be monic (in T), with monic $g(T) \in (F[x])[T]$. Then, for all $n \geq 1$,

$$f^n(T) = f^{n-1}(T)^2\left(f^{n-1}(T) - 1\right)g_n(T) + 1, \quad g_n(T) := g\left(f^{n-1}(T)\right) \in (F[x])[T].$$

For brevity, let

$$A_n(T) = Tf(T)\cdots f^n(T) \quad (n \geq 0).$$

Theorem 7. *If $Z \in F[x] \setminus F$ is monic, then*

$$\sum_{n=0}^{1} \frac{1}{f^n(Z)} = [0; Z, -g(Z)(Z-1), -Z+1, -Z-1],$$

and for $\ell \geq 2$, if $\sum_{n=0}^{\ell-1} 1/f^n(Z) = \left[0; \overrightarrow{X}_{k_\ell}\right]$, then

$$\sum_{n=0}^{\ell} \frac{1}{f^n(Z)} = \left[0; \overrightarrow{X}_{k_\ell}, \frac{g_\ell(Z)g_{\ell-1}(Z)\left(f^{\ell-2}(Z)-1\right)}{A_{\ell-3}(Z)^2}, -\overleftarrow{X}_{k_\ell}, 1, \overrightarrow{X}_{k_\ell}\right].$$

Proof. For $\ell \geq 1$, let $\alpha_\ell = f^{\ell-1}(Z)$ and let $C_{k_\ell}/D_{k_\ell} =: \left[0; \overrightarrow{X}_{k_\ell}\right]$ be the k_ℓ^{th} convergent of the continued fraction of

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_\ell} = \frac{1}{Z} + \frac{1}{f(Z)} + \cdots + \frac{1}{f^{\ell-1}(Z)} = \sum_{n=0}^{\ell-1} \frac{1}{f^n(Z)}.$$

We claim that

$$D_{k_\ell} = A_{\ell-1}(Z) \quad (\ell \geq 1).$$

From

$$\begin{aligned} f^\ell(Z) - 1 &= f^{\ell-1}(Z)^2\left(f^{\ell-1}(Z)-1\right)g_\ell(Z) \\ &= f^{\ell-1}(Z)^2 f^{\ell-2}(Z)^2\left(f^{\ell-2}(Z)-1\right)g_{\ell-1}(Z)g_\ell(Z) = \ldots \\ &= f^{\ell-1}(Z)^2 f^{\ell-2}(Z)^2 \cdots f(Z)^2 Z^2 (Z-1)\, g(Z)g_2(Z)\cdots g_\ell(Z), \end{aligned}$$

we see that

$$A_{\ell-1}(Z)^2 \mid \left(f^\ell(Z) - 1\right), \tag{27}$$

i.e.,

$$f^\ell(Z) = A_{\ell-1}(Z)^2 B_\ell + 1 \quad \text{for some} \quad B_\ell \in (F[x])[T]. \tag{28}$$

Now

$$\begin{aligned}\frac{C_{k_\ell}}{D_{k_\ell}} &= \frac{1}{Z} + \frac{1}{f(Z)} + \cdots + \frac{1}{f^{\ell-1}(Z)} \\ &= \frac{\left(f(Z)\cdots f^{\ell-1}(Z)\right) + \left(Zf^2(Z)\cdots f^{\ell-1}(Z)\right) + \cdots + \left(Zf(Z)\cdots f^{\ell-2}(Z)\right)}{Zf(Z)\cdots f^{\ell-1}(Z)}.\end{aligned}$$

The numerator and denominator are relatively prime, for if there exists a prime element $p \in F[x]$ dividing them, then $p \mid f^r(Z)$ for some $0 \le r \le \ell - 1$, which implies that

$$p \mid Zf(Z)\cdots f^{r-1}(Z)f^{r+1}(Z)f^{r+2}(Z)\cdots f^{\ell-1}(Z),$$

and so $p \mid Zf(Z)\cdots f^{r-1}(Z)$ or $p \mid f^{r+1}(Z)f^{r+2}(Z)\cdots f^{\ell-1}(Z)$.

If $p \mid Zf(Z)\cdots f^{r-1}(Z)$, using (27), we see that $p \mid \left(f^r(Z) - 1\right)$, contradicting $p \mid f^r(Z)$. Thus,

$$p \mid f^{r+1}(Z)f^{r+2}(Z)\cdots f^{\ell-1}(Z).$$

By (28), we get

$$\begin{aligned}&f^{r+1}(Z)f^{r+2}(Z)\cdots f^{\ell-1}(Z) \\ &\quad = \left(A_r(Z)^2B_{r+1} + 1\right)\left(A_{r+1}(Z)^2B_{r+2} + 1\right)\cdots\left(A_{\ell-2}(Z)^2B_{\ell-1} + 1\right).\end{aligned}$$

Since $f^r(Z) \mid A_j(Z)$ $(r \le j)$ and $p \mid f^r(Z)$, then $p \mid 1$, a contradiction. Thus,

$$\left(f(Z)\cdots f^{\ell-1}(Z)\right) + \left(Zf^2(Z)\cdots f^{\ell-1}(Z)\right) + \cdots + \left(Zf(Z)\cdots f^{\ell-2}(Z)\right)$$

and $Zf(Z)\cdots f^{\ell-1}(Z)$ are relatively prime. Since Z is monic, we get

$$D_{k_\ell} = Zf(Z)\cdots f^{\ell-1}(Z) = A_{\ell-1}(Z) \quad (\ell \ge 1),$$

and the claim is proved.

Next, we consider

$$\begin{aligned}\alpha_{\ell+1} = f^\ell(Z) &= f^{\ell-1}(Z)^2\left(f^{\ell-1}(Z) - 1\right)g_\ell(Z) + 1 \\ &= f^{\ell-1}(Z)^2f^{\ell-2}(Z)^2\left(f^{\ell-2}(Z) - 1\right)g_{\ell-1}(Z)g_\ell(Z) + 1 \\ &= A_{\ell-1}(Z)^2\frac{\left(f^{\ell-2}(Z) - 1\right)g_{\ell-1}(Z)g_\ell(Z)}{A_{\ell-3}(Z)^2} + 1 = A_{\ell-1}(Z)^2Y_1 + \frac{1}{Y_2},\end{aligned}$$

where $Y_1 := \frac{\left(f^{\ell-2}(Z)-1\right)g_{\ell-1}(Z)g_\ell(Z)}{A_{\ell-3}(Z)^2}$, $Y_2 := 1$. By (27), we get $Y_1 \in F[x] \setminus \{0\}$. Let $\overrightarrow{X}_{k_2}$ be the word $Z, -g(Z)(Z-1), -Z+1, -Z-1$. Then

$$\left[0; \overrightarrow{X}_{k_2}\right] = [0; Z, -g(Z)(Z-1), -Z+1, -Z-1] = \frac{1}{Z} + \frac{1}{f(Z)}.$$

We observe that $\{k_\ell\}_{\ell\geq 2}$ obtained by this process is a sequence of even positive integers. Thus, $\alpha_{\ell+1} = (-1)^{k_\ell} D_{k_\ell}^2 Y_1 + 1/Y_2$. Using Theorem 2 (13), we get

$$\left[0; \overrightarrow{X}_{k_\ell}, \frac{g_\ell(Z) g_{\ell-1}(Z)\left(f^{\ell-2}(Z)-1\right)}{A_{\ell-3}(Z)^2}, -\overleftarrow{X}_{k_\ell}, 1, \overrightarrow{X}_{k_\ell}\right] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i} = \sum_{n=0}^{\ell} \frac{1}{f^n(Z)}.$$

□

Acknowlefgement. The second author is supported by a grant from the Faculty of Science, Kasetsart University.

References

[1] L. Carlitz, On certain funcions connected with polynomials in a Galois field, *Duke Math. J.* **1** (1935), 137–168.

[2] H. Cohn, Symmetry and specializability in continued fractions, *Acta Arith.* **75** (1996), 297–320.

[3] W. Leighton and W. T. Scott, A general continued fraction expansion, *Bull. Amer. Math. Soc.* **45** (1939), 596–605.

[4] M. Mendés France, Sur les fractions continues limitées, *Acta Arith.* **23** (1973), 207–215.

[5] P. Riyapan, V. Laohakosol and T. Chaichana, Two types of explicit continued fractions, *Period. Math. Hungar.* **52** (2) (2006), 51–72.

[6] W. M. Schmidt, On continued fractions and diophantine approximation in power series fields, *Acta Arith.* **95** (2000), 139–166.

[7] W. T. Scott and H. S. Wall, Continued fraction expansions for arbitrary power series, *Ann. of Math.* **41** (1940), 328–349.

[8] J. Tamura, Symmetric continued fractions related to certain series, *J. Number Theory* **38** (1991), 251–264.

[9] D. Thakur, Continued fraction for the exponential for $\mathbb{F}_q[T]$, *J. Number Theory* **41** (1992), 150–155.

[10] D. Thakur, Exponential and continued fractions, *J. Number Theory* **59** (1996), 248–261.

[11] A. J. van der Poorten and J. O. Shallit, Folded continued fractions, *J. Number Theory* **40** (1992), 237–250.

[12] A. J. van der Poorten, Symmetry and folding of continued fractions, *J. Théor. Nombres Bordeaux* **14** (2002), 603–611.