



CONSECUTIVE INTEGERS AND THE COLLATZ CONJECTURE

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Abstract

Pairs of consecutive integers have the same height in the Collatz problem with surprising frequency. Garner gave a conjectural family of conditions for exactly when this occurs. Our main result is an infinite family of counterexamples to Garner's conjecture.

1. Introduction

The Collatz function C is a recursively defined function on the positive integers given by the following definition.

$$C^k(n) = \begin{cases} n, & \text{if } k = 0 \\ C^{k-1}(n)/2, & \text{if } C^{k-1}(n) \text{ is even} \\ 3 * C^{k-1}(n) + 1, & \text{if } C^{k-1}(n) \text{ is odd.} \end{cases}$$

The famed Collatz conjecture states that, under the Collatz map, every positive integer converges to one [2]. The *trajectory* of a number is the path it takes to reach one. For example, the trajectory of three is

$$3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1.$$

The *parity vector* of a number is its trajectory considered modulo two. So the parity vector of three is $\langle 1, 0, 1, 0, 0, 0, 1 \rangle$.

Because applying the map $n \mapsto 3n+1$ to an odd number will always yield an even number, it is sometimes more convenient to use the following alternate definition of the Collatz map, often called T in the literature.

$$T^k(n) = \begin{cases} n, & \text{if } k = 0 \\ T^{k-1}(n)/2, & \text{if } T^{k-1}(n) \text{ is even} \\ (3 * T^{k-1}(n) + 1)/2, & \text{if } T^{k-1}(n) \text{ is odd.} \end{cases}$$

With this new definition, the trajectory of three becomes $3 \rightarrow 5 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$ and its T parity vector is $\langle 1, 1, 0, 0, 0, 1 \rangle$.

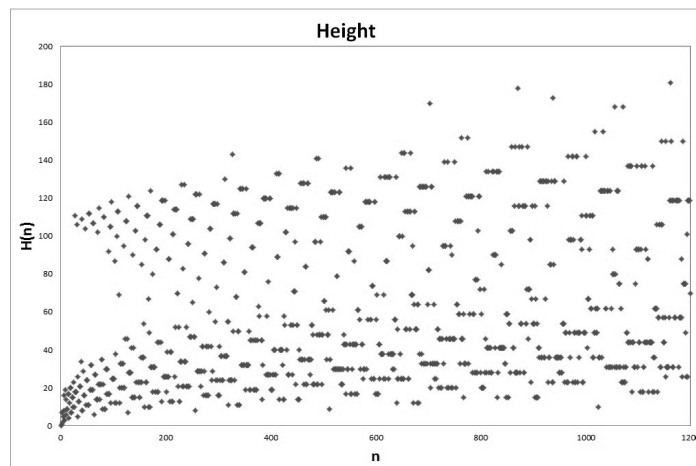
Since the Collatz conjecture states that, for every positive integer n , there exists a non-negative integer k such that $C^k(n) = 1$, it is natural to ask for the smallest such value of k . This k is called the *height* of n and denoted $H(n)$. So, for example, the height of three is seven because it requires seven iterations of the map C for three to reach seven. In this paper, height is used only in association with the map C , never the map T .

It turns out that consecutive integers frequently have the same height. Garner made a conjecture that attempts to predict, in terms of the map T and its parity vectors, exactly which pairs have the same height [1]. He proved that his condition is sufficient to guarantee two consecutive numbers will have the same height, but only surmised that it is a necessary condition.

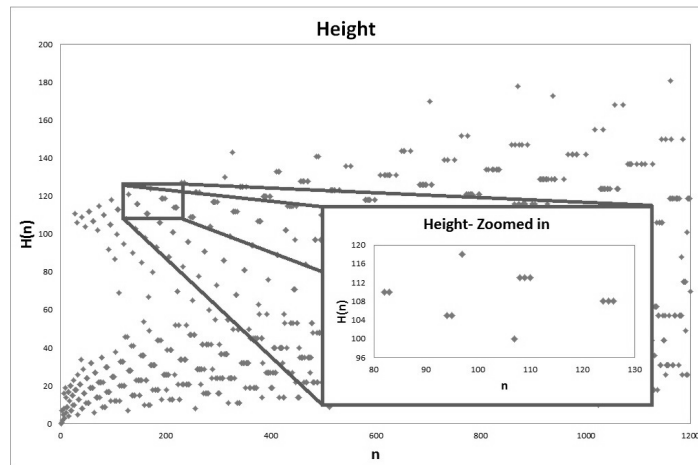
The main idea in this paper is that phrasing Garner’s conjecture in terms of the map C reveals an easier-to-verify implication of Garner’s conjecture, namely, that if two consecutive integers have the same height, then they must reach 4 and 5 (mod 8) at the same step of their trajectory (see Proposition 1). Because this condition is much easier to check than the conclusion of Garner’s conjecture, we were able to find an infinite family of pairs of consecutive integers that do not satisfy this condition, and, hence, constitute counterexamples to Garner’s conjecture (see Theorem 4.1).

2. Heights of Consecutive Integers

Recall that the smallest non-negative k such that $C^k(n) = 1$ is called the *height* of n and denoted $H(n)$. The following is a graph of the height H as a function of n .



The striking regularity in the above graph is the starting point for our studies, but remains largely elusive. If one naïvely searches for curves of best fit to the visible curves therein, one quickly runs into a problem. What appear to be distinct points in the above graph are actually clusters of points, as can be seen below. Thus, it is not entirely clear which points one ought to work with when trying to find a curve of best fit.



This leads to the surprising observation that many consecutive integers have the same height. This is counterintuitive because if two integers are consecutive then they are of opposite parity, so the Collatz map initially causes one to increase ($n \mapsto 3n + 1$) and the other to decrease ($n \rightarrow \frac{n}{2}$). How, then, do they reach one in the same number of iterations? We give a sufficient congruence condition to guarantee two consecutive numbers will have the same height, and show that an all-encompassing theorem like Garner conjectured in [1] is not possible. In fact, we show the situation is much more complicated than Garner originally thought.

The first pair of consecutive integers with the same height is twelve and thirteen. We see that for both numbers, $C^3(n) = 10$. Clearly, once their trajectories coincide, they will stay together and have the same height. This happens because twelve follows the path

$$12 \rightarrow 6 \rightarrow 3 \rightarrow 10,$$

and thirteen follows the path

$$13 \rightarrow 40 \rightarrow 20 \rightarrow 10.$$

Now we seek to generalize this. It turns out that twelve and thirteen merely form the first example of a general phenomenon, namely, numbers that are 4 and 5 (mod 8)

always coincide after the third iteration. The following result agrees with what Garner found using parity vectors [1].

Theorem 2.1. *If $n > 4$ is congruent to 4 (mod 8), then n and $n + 1$ coincide at the third iteration and, hence, have the same height.*

Proof. Suppose $n > 4$ and $n \equiv 4 \pmod{8}$. Then $n = 8k + 4$, for some $k \in \mathbb{N}$. Then, because $8k + 4$ and $4k + 2$ are even, while $2k + 1$ is odd, the trajectory of n under the map C is

$$8k + 4 \rightarrow 4k + 2 \rightarrow 2k + 1 \rightarrow 6k + 4.$$

Because $n + 1 = 8k + 5$ is odd, and $24k + 16$ and $12k + 8$ are even, the trajectory of $n + 1$ under the map C is

$$8k + 5 \rightarrow 24k + 16 \rightarrow 12k + 8 \rightarrow 6k + 4.$$

Therefore, n and $n + 1$ coincide at the third iteration. □

3. Garner’s Conjecture

Garner wanted to generalize this to predict all possible pairs of consecutive integers that coincide. Since he used the map T (defined in Section 1) instead of the map C , we will do the same in this section except during the proof of Proposition 1. He observed that whenever two consecutive integers have the same height, their parity vectors appear to end in certain pairs of corresponding *stems* immediately before coinciding. He defined a *stem* as a parity vector of the form

$$s_i = \langle 0, \underbrace{1, 1, \dots, 1}_{i \text{ 1's}}, 0, 1 \rangle,$$

and the *corresponding stem* as

$$s'_i = \langle 1, \underbrace{1, 1, \dots, 1}_{i \text{ 1's}}, 0, 0 \rangle.$$

LaTourette used the following definitions of a stem and a block in her senior thesis [3], which we adhere to here as well. In what follows, we write $T_w(n)$ to mean apply the sequence of steps indicated by the parity vector w to the input n using the map T .

Definition 1. (LaTourette) A pair of parity sequences s and s' of length k are *corresponding stems* if, for any integer x , $T_s(x) = T_{s'}(x + 1)$ and, for any initial subsequences v and v' of s and s' of equal length, $|T_v(x) - T_{v'}(x + 1)| \neq 1$ and $T_v(x) \neq T_{v'}(x + 1)$.

Definition 2. (LaTourette) A *block* prefix is a pair of parity sequences b and b' , each of length k , such that for all positive integers x , $T_b(x) + 1 = T_{b'}(x + 1)$.

In his conclusion, Garner conjectured that all corresponding stems will be of the form s_i and s'_i listed above. LaTourette conjectured the same.

Conjecture 1. (Garner) Any pair of consecutive integers of the same height will have parity vectors for the non-overlapping parts of their trajectories ending in s_i and s'_i [1].

Garner gave no bound on the length of stem involved, though, so searching for counterexamples by computer was a lengthy task. The big innovation in this paper is that using the map C instead of the map T yields a much simpler implication of Garner’s conjecture, which makes it possible to search for counterexamples.

Proposition 1. *If n and $n + 1$ have parity vectors for the non-overlapping parts of their trajectories ending in s_i and s'_i , and k is the smallest positive integer such that $C^k(n) = C^k(n + 1)$, then $C^{k-3}(n) \equiv 4 \pmod{8}$ and $C^{k-3}(n + 1) = C^{k-3}(n) + 1$ or $C^{k-3}(n + 1) \equiv 4 \pmod{8}$ and $C^{k-3}(n) = C^{k-3}(n + 1) + 1$.*

Proof. To see this, we must change the Garner stems to be consistent with the map C . Converting the parity vectors simply involves inserting an extra ‘0’ after each ‘1’. So Garner’s stems in terms of the map C now look like

$$s_i = \langle 0, \underbrace{1, 0, 1, 0, \dots, 1, 0, 0, 1, 0}_{i \text{ } 1,0's} \rangle,$$

and

$$s'_i = \langle 1, \underbrace{0, 1, 0, 1, 0, \dots, 1, 0, 0, 0}_{i \text{ } 1,0's} \rangle.$$

Now we will rearrange this more strategically. We have

$$s_i = \langle \underbrace{0, 1, 0, 1, \dots, 0, 1, 0, 0, 1, 0}_{i \text{ } 0,1's} \rangle,$$

and

$$s'_i = \langle \underbrace{1, 0, 1, 0, \dots, 1, 0, 1, 0, 0, 0}_{i \text{ } 1,0's} \rangle.$$

The point of these stems is that the trajectories coincide right after this vector. Since both end with a ‘0’, they have coincided one step before the end, so we can simply omit the last ‘0’. Now the corresponding stems are only $\langle 0, 0, 1 \rangle$ and $\langle 1, 0, 0 \rangle$, with repeated blocks in front of them. Terras[4] proved that there is a bijection between the set of integers modulo 2^k and the set of parity vectors of length k . The

Proof. Consider the parity vectors of 3067 and 3068 up to the point where they coincide. We know that there will be a pair with the same parity vectors for every integer of the form $2^{19}m + 3067$ by Terras's bijection[4]. Each of these pairs will coincide in the same way that 3067 and 3068 do and, thus, have the same height. Therefore, there are infinitely many counterexamples to Garner's conjecture. \square

5. Conclusion

At this point, we look at those numbers that do not have the stems Garner predicted to see why they coincide. To salvage Garner's conjecture, we seek to expand the list of possible stems. To see what is going on, we have no choice but to examine the trajectories of 3067 and 3068, side by side (See Appendix A).

We can see that there are no other places within the trajectories where their values have a difference of one. Therefore, by the current definition of a stem, the entire parity vector of length 27 (up until they coincide at 1384) is a new stem. However, by this logic, the next counterexample, 4088 and 4089, has a new stem of length 30. The next pair, 6135 and 6136, has a stem of length 28. It would be ridiculous to have only one stem (of length 3) before 3067 and to suddenly add dozens more of varying lengths. Instead, we look for some new type of stems within these counterexample, stems that do not start with consecutive integers. The trajectories of all three pairs listed above coincide at 1384. In fact, they have the same 22 elements leading up to that. Thus, it is tempting to label that beginning as the stem. But if we look further, the consecutive integers 32743 and 32744 join that group just 5 steps before coinciding at 1384. Therefore, the situation is much more complicated than Garner's stems. It would be interesting to know if there is some pattern similar to what Garner conjectured, perhaps with a much-expanded list of stems, that explains every pair of consecutive numbers that converges together. However, we have found no such simple salvage of Garner's conjecture.

We have shown that pairs of integers of the form $8m + 4$ and $8m + 5$ have coinciding trajectories after 3 steps (and therefore have the same height). We have also shown that all pairs that obey Garner's conjecture ultimately reduce down to the 4 and 5 (mod 8) case before coinciding. This allowed us to find that 3067 and 3068 form the smallest of an infinite family of counterexamples to Garner's longstanding conjecture [1].

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References

- [1] L. Garner. On heights in the collatz $3n+1$ problem. *Discrete Mathematics*, 55:57–64, 1985.
- [2] J. Lagarias. The $3x+1$ problem: An annotated bibliography (1963 - 1999). Available online at [arXiv:math/0608208](https://arxiv.org/abs/math/0608208), 74 pages.
- [3] K. LaTourette. Explorations of the collatz conjecture. Moravian College Senior Honors Thesis, 2007. Undergraduate Thesis.
- [4] R. Terras. A stopping time problem on the positive integers. *Acta Arithm.*, 30(3):241–252, 1976.

Appendix

This chart shows the partial trajectories of the first eight counterexample pairs to Garner’s conjecture.

n	3067	3068	4088	4089	6135	6136	7151	7152	8179	8180	9979	9980	10904	10905	13304	13305	n
1	9202	1534	2044	12268	18406	3068	21454	3576	24538	4090	29938	4990	5452	32716	6652	39916	1
2	4601	767	1022	6134	9203	1534	10727	1788	12269	2045	14969	2495	2726	16358	3326	19958	2
3	13804	2302	511	3067	27610	767	32182	894	36808	6136	44908	7486	1363	8179	1663	9979	3
4	6902	1151	1534	9202	13805	2302	16091	447	18404	3068	22454	3743	4090	24538	4990	29938	4
5	3451	3454	767	4601	41416	1151	48274	1342	9202	1534	11227	11230	2045	12269	2495	14969	5
6	10354	1727	2302	13804	20708	3454	24137	671	4601	767	33682	5615	6136	36808	7486	44908	6
7	5177	5182	1151	6902	10354	1727	72412	2014	13804	2302	16841	16846	3068	18404	3743	22454	7
8	15532	2591	3454	3451	5177	5182	36206	1007	6902	1151	50524	8423	1534	9202	11230	11227	8
9	7766	7774	1727	10354	15532	2591	18103	3022	3451	3454	25262	25270	767	4601	5615	33682	9
10	3883	3887	5182	5177	7766	7774	54310	1511	10354	1727	12631	12635	2302	13804	16846	16841	10
11	11650	11662	2591	15532	3883	3887	27155	4534	5177	5182	37894	37906	1151	6902	8423	50524	11
12	5825	5831	7774	7766	11650	11662	81466	2267	15532	2591	18947	18953	3454	3451	25270	25262	12
13	17476	17494	3887	3883	5825	5831	40733	6802	7766	7774	56842	56860	1727	10354	12635	12631	13
14	8738	8747	11662	11650	17476	17494	122200	3401	3883	3887	28421	28430	5182	5177	37906	37894	14
15	4369	26242	5831	5825	8738	8747	61100	10204	11650	11662	85264	14215	2591	15532	18953	18947	15
16	13108	13121	17494	17476	4369	26242	30550	5102	5825	5831	42632	42646	7774	7766	56860	56842	16
17	6554	39364	8747	8738	13108	13121	15275	2551	17476	17494	21316	21323	3887	3883	28430	28421	17
18	3277	19682	26242	4369	6554	39364	45826	7654	8738	8747	10658	63970	11662	11650	14215	85264	18
19	9832	9841	13121	13108	3277	19682	22913	3827	4369	26242	5329	31985	5831	5825	42646	42632	19
20	4916	29524	39364	6554	9832	9841	68740	11482	13108	13121	15988	95956	17494	17476	21323	21316	20
21	2458	14762	19682	3277	4916	29524	34370	5741	6554	39364	7994	47978	8747	8738	63970	10658	21
22	1229	7381	9841	9832	2458	14762	17224	3277	19682	3997	23989	26242	4369	31985	5329	25270	22
23	3688	22144	29524	4916	1229	7381	51556	8612	9832	9841	11992	71968	13121	13108	95956	15988	23
24	1844	11072	14762	2458	3688	22144	25778	4306	4916	29524	5996	35984	39364	6554	47978	7994	24
25	922	5536	7381	1229	1844	11072	12889	2153	2458	14762	2998	17992	19682	3277	23989	3997	25
26	461	2768	22144	3688	922	5536	38668	6460	1229	7381	1499	8996	9841	9832	71968	11992	26
27	1384	1384	11072	1844	461	2768	19334	3230	3688	22144	4498	4498	29524	4916	35984	5996	27
28	692	692	5536	922	1384	1384	9667	1615	1844	11072	2249	2249	14762	2458	17992	2998	28
29	346	346	2768	461	692	692	29002	4846	922	5536	6748	6748	7381	1229	8996	1499	29
30	173	173	1384	1384	346	346	14501	2423	461	2768	3374	3374	22144	3688	4498	4498	30
31	520	520	692	692	173	173	43504	7270	1384	1384	1687	1687	11072	1844	2249	2249	31
32	260	260	346	346	520	520	21752	3635	692	692	5062	5062	5536	922	6748	6748	32
33	130	130	173	173	260	260	10876	10906	346	346	2531	2531	2768	461	3374	3374	33
34	65	65	520	520	130	130	5438	5453	173	173	7594	7594	1384	1384	1687	1687	34
35	196	196	260	260	65	65	2719	16360	520	520	3797	3797	692	692	5062	5062	35