AN ATLAS OF N- AND P-POSITIONS IN 'NIM WITH A PASS'

Richard M. Low
Department of Mathematics, San Jose State University, San Jose, California
richard.low@sjsu.edu

W.H. Chan[1]
Department of Mathematics and Information Technology, The Hong Kong Institute of Education, Tai Po, Hong Kong
waihchan@ied.edu.hk

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Abstract

Perhaps the most famous combinatorial game is Nim, which was completely analyzed by C.L. Bouton in 1902. Since then, the game of Nim has been the subject of many research papers. In Guy and Nowakowski's *Unsolved Problems in Combinatorial Games*, the following entry is found: "David Gale would like to see an analysis of Nim played with the option of a single pass by either of the players, which may be made at any time up to the penultimate move. It may not be made at the end of the game. Once a player has passed, the game is as in ordinary Nim. The game ends when all heaps have vanished." In this paper, we analyze this particular variant of Nim.

1. Introduction and Some Preliminaries

Having its humble beginnings in the context of recreational mathematics, combinatorial game theory has matured into an active area of research. Along with its natural appeal, the subject has applications to complexity theory, logic, graph theory and biology. For these reasons, combinatorial games have caught the attention of many people and the large body of research literature on the subject continues to increase. The interested reader is directed to [1, 2, 3, 5, 7, 8, 10], and to A. Fraenkel's excellent bibliography [6].

A *combinatorial* game is one of complete information and no element of chance is involved in gameplay. Each player is aware of the game position at any point in the game. Under *normal play*, two players alternate taking turns and a player loses

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when he cannot make a move. An *impartial* combinatorial game is one where both players have the same options from any position. In a *short* game, a position is never repeated and there are only a finite number of other positions which can be reached.

Perhaps the most famous short impartial combinatorial game is Nim, which is played in the following manner:

- There are n heaps, each containing a finite number of stones. Two players alternate turns, each time choosing a heap and removing any number (≥ 1) of stones in that heap. The player who cannot make a move loses the game.

Notation. We introduce the following notation for a game position (initial or otherwise) for Nim. Let $k \geq 1$ and $t_i \geq 1$, for all $1 \leq i \leq k$, and t_i not necessarily distinct. Then, $(t_1, t_2, \ldots, t_k)$ denotes the game position corresponding to k heaps of sizes $t_1, t_2, \ldots, t_k$. When convenient, we will use additional subscripts on the t_i to indicate multiple heaps of size t_i. Note that t_{i0} denotes zero heaps of size t_i. For example, $(1, 2_3, 4_2)$ denotes the game position (in Nim) corresponding to heaps of sizes 1, 2, 2, 2, 4, and 4.

In 1902, Bouton [4] gave a beautiful mathematical analysis and complete solution for Nim. Since then, the game of Nim has been the subject of many mathematics research papers. Within the literature, studies on Nim variants with modified rule sets, Nim played on different configurations (circular, triangular and rectangular), and Nim played on graphs can be found. As of this writing, a keyword search for "Nim" yields 98 entries in the MathSciNet database.

In this paper, we will use some important ideas and standard notation from combinatorial game theory in the analysis of 'Nim with a Pass'. For a more complete overview, the interested reader is directed to [2, 3, 5].

First, we recall a definition and some concepts from combinatorial game theory (CGT).

Definition. A *P-position* is a position which is winning for the previous player (who has just moved). An *N-position* is a position which is winning for the next player (who is about to make a move).

In finite impartial combinatorial games (under normal play), P-positions and N-positions have the following properties:

- All terminal positions are P-positions.
- From every N-position, there is a move leading to a P-position.
- From every P-position, every move leads to an N-position.

- A game Γ equals $0 = \{|\} \iff \Gamma$ is a P-position.

For regular Nim (under normal play), Bouton [4] showed that the game played on heaps of size $x_1, x_2, \ldots, x_k$ is a P-position if and only if $\sum x_i = 0_2$ (BitXor). As an example, suppose an initial game position of Nim is comprised of two heaps of sizes 1 and 2. Since $1 = 1_2$ and $2 = 10_2$, $1 + 2 = 11_2 \neq 0_2$. Hence, this is an N-position. Of course, we can see this directly: On his first move, P1 removes one stone from the heap of size 2. This leaves P2 to play on two heaps of size 1, which is clearly a losing position for P2.

In this paper, *Nim** will be used to denote David Gale's 'Nim with a Pass' game. Here is how it is played.

- Nim* is played like ordinary Nim, with the option of a single pass which can be used by exactly one player. Once the pass option is used, it cannot be used again and the game continues in ordinary Nim fashion. The pass option can be used at any time, up to the penultimate move. It cannot be used at the end of the game. The player who cannot make a move loses the game.

Notation. We introduce the following notation for a game position (initial or otherwise) for Nim*, where the pass option has not yet been used. Let $k \geq 1$ and $t_i \geq 1$, for all $1 \leq i \leq k$, and t_i not necessarily distinct. Then, $[t_1, t_2, \ldots, t_k]$ denotes the game position corresponding to k heaps of sizes $t_1, t_2, \ldots, t_k$. When convenient, we will use additional subscripts on the t_i to indicate multiple heaps of size t_i. Note that t_{i0} denotes zero heaps of size t_i. For example, $[1, 2_3, 4_2]$ denotes the game position (in Nim*) corresponding to heaps of sizes 1, 2, 2, 2, 4, and 4, with the pass option available.

2. The Relationship Between Nim and Nim*

Combinatorial game theory can be used effectively in the analysis of a game which is a disjoint sum of smaller games. For example, the Nim game $(1, 3_2)$ can be viewed as a disjoint sum of smaller Nim games (1) and (3_2). Unfortunately, Nim* cannot be viewed (in the natural way, or possibly in any way) as a disjoint sum of smaller Nim* games. When we have a collection of heaps in Nim*, the pass option can only be used once during the entire game and not once on each heap. On a different note, it is tempting to think that the Nim* game $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ is merely equal to the Nim game $(1_{m_1+1}, 2_{m_2}, \ldots, k_{m_k})$. However, this is not the case since the pass option cannot be used as a last move in Nim*. Because of these difficulties, the powerful tools in combinatorial game theory cannot be directly applied to the analysis of Nim*.

In [9], Morrison, Friedman and Landsberg used a dynamical systems approach to analyze Nim* on three heaps. Their findings indicate a deep and rich complexity in 3-heap Nim* and suggest that obtaining a complete analytical solution (in the spirit of Bouton) may be intractable.

Nevertheless, in this paper, we give a partial analysis of Nim*. We begin with a simple Lemma.

Lemma 1. *If* $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ *is a P-position in Nim*, then* $(1_{m_1}, 2_{m_2}, \ldots, k_{m_k})$ *is an N-position in Nim.*

Proof. Let $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ be a P-position in Nim*. Then, any move made from this position leads to an N-position in Nim*. In particular, a pass leads to $(1_{m_1}, 2_{m_2}, \ldots, k_{m_k})$, an N-position in Nim. □

Note that the converse of Lemma 1 is not necessarily true. For example, $(1, 2)$ is an N-position in Nim and $[1, 2]$ is a P-position (see Theorem 1) in Nim*, whereas $(1, 3)$ is an N-position in Nim and $[1, 3]$ is an N-position (see Theorem 1) in Nim*.

3. Nim* Played on Two Heaps

Lemma 2. *Let* $m \geq 1$ *and odd. Then, Nim* played on* $[m, m+1]$ *is a P-position.*

Proof. (Induction on $m \geq 1$ and odd.)

- Let $m = 1$. Then, we have two heaps of sizes 1 and 2. If P1 passes, then P2 removes one stone from the heap of size 2, which leaves a losing position for P1. If P1 initially removes an entire heap (either of size 1 or 2), then P2 removes the other heap and wins. However, if P1 removes one stone from a heap of size 2, then P2 passes and leaves P1 to move from a losing position. Therefore, the claim is true for $m = 1$.

Now, assume claim is true for $m = 1, 3, \ldots, 2k - 1$.

- Let us consider Nim* played on heaps of sizes $2k+1$ and $2k+2$. If P1 passes, then P2 removes one stone from the heap of size $2k+2$, which leaves a losing position for P1. If P1 initially removes an entire heap (either of size $2k+1$ or $2k+2$), then P2 removes the other heap and wins. On the other hand, if P1 removes one stone from the heap of size $2k+2$, then P2 passes, leaving P1 to play on the losing position $(2k+1, 2k+1)$. However, if P1 removes stones (not all of them) from a single heap and leaves j stones remaining in that heap, then P2 removes the appropriate number of stones in the other heap, leaving a losing position for P1 to play on. This can be accomplished (by Induction Hypothesis). Thus, the claim is true for $m = 2k + 1$.

□

Lemma 3. *Let $m \geq 1$ and odd. Then, Nim* played on $[m,n]$, where $1 \leq m \leq n$ and $n \neq m+1$, is an N-position.*

Proof. Suppose that we have two heaps of sizes m and n. If $n = m$, then this is clearly an N-position (since P1 merely passes on his first turn). Now, let $n \geq m+2$. Then, $[m,n]$ is still an N-position (since P1 can remove stones from the heap of size n, thus leaving $[m,m+1]$ for P2 to move from). □

Lemma 4. *Let $m \geq 2$ and even. Then, Nim* played on $[m,n]$, where $2 \leq m \leq n$, is an N-position.*

Proof. Suppose that we have two heaps of sizes m and n. If $n = m$, then this is clearly an N-position (since P1 merely passes on his first turn). Now, let $n \geq m+1$. Then, $[m,n]$ is still an N-position (since P1 can remove stones from the heap of size n, thus leaving $[m,m-1]$ for P2 to move from). □

Theorem 1. *Suppose that Nim* is played on $[m,n]$, where $1 \leq m \leq n$. If m is odd and $n = m+1$, then this is a P-position. Otherwise, it is an N-position.*

Proof. This follows immediately from Lemmas 2, 3 and 4. □

4. Nim* Played on Three Heaps

Lemma 5. *Suppose that Nim* is played on $[1,2,n]$, where $n \geq 2$. Then, this is an N-position.*

Proof. Suppose that we have three heaps of sizes 1, 2 and n. Here, P1 removes the entire heap of size n, which leaves $[1,2]$ (a losing position for P2; by Theorem 1). □

Lemma 6. *Suppose that Nim* is played on $[1,m,n]$. If $m = n = 1,3,4,5,\ldots$, then this is a P-position.*

Proof. Let $m = n = 1$. Then, we have three heaps of size 1. If P1 passes, then P2 removes an entire heap, which leaves a losing position for P1. If P1 initially removes an entire heap, then P2 passes and wins. Therefore, the claim is true for $m = n = 1$.

- Let $m = n = 3$. Then, we have three heaps of sizes 1, 3 and 3. If P1 passes, then P2 removes the heap of size 1, which leaves a losing position for P1. If P1 removes the heap of size 1, then P2 passes (and eventually wins). If P1 removes a stone from a heap of size 3, this leaves $[1,2,3]$ (an N-position for P2, by Lemma 5). However, if P1 removes two stones from a heap, this leaves $[1_2, 3]$, (a winning position for P2, since he can then leave $[1_3]$ for P1 to move from). Finally, if P1 removes an entire heap of size 3, this leaves $[1, 3]$ (a winning position for P2, since he can then leave $[1, 2]$ and for P1 to move from). Therefore, the claim is true for $m = n = 3$.

Assume that the claim is true for $m = 3, 4, 5, \ldots, k$. Now, consider $m = n = k + 1$. Then, we have three heaps of sizes 1, $k+1$ and $k+1$. If P1 passes, then P2 removes the heap of size 1 and wins. If P1 removes the heap of size 1, then P2 passes and wins. If P1 removes stones from a heap of size $k + 1$, there are three possible types of positions for P2 to move from:

- $[1, k + 1]$. Here, P2 should remove $k - 1$ stones from the heap of size $k + 1$. This leaves $[1, 2]$, a losing position for P1.
- $[1, j, k + 1]$, where $1 \le j \le k$ and $j \ne 2$. Here, P2 should remove stones from the heap of size $k + 1$ so as to leave $[1, j, j]$. By the induction hypothesis, this is a losing position for P1.
- $[1, 2, k + 1]$. Here, P2 should remove the entire heap of size $k + 1$. This leaves $[1, 2]$, a losing position for P1.

Thus by induction, the claim is established. □

Corollary 1. *Suppose that Nim* is played on* $[1, m, n]$*, where* $1 \le m < n$*. Then, this is an N-position.*

Proof. If $m = 2$, then this is an N-position (by Lemma 5). Now, let $m \ne 2$. On his first move, P1 removes $n - m$ stones from the heap of size n. This leaves $[1, m, m]$, a losing position for P2 to play on. □

Theorem 2. *Suppose that Nim* is played on* $[1, m, n]$*, where* $1 \le m \le n$*. If* $m = n = 1, 3, 4, 5, \ldots$*, then this is a P-position. Otherwise, it is an N-position.*

Proof. This follows immediately from Lemmas 5, 6 and Corollary 1. □

The dynamics underlying Nim* appear to be very complex. A computer program was prepared and used to compute the P-positions in Nim* on three heaps. For three heaps of sizes s, m and n, where $2 \le s \le m \le n$, the P-positions show some order. However for $s = 10$, there is no obvious pattern for the P-positions. For the convenience of the reader, we list the P-positions for 3-heap Nim* ($[s, m, n]$, where $2 \le s \le m \le n$ and $s \in \{2, 3, 4, \ldots, 9\}$).

- $s = 2$. $[2, 2, 2]$, $[2, 3, 5]$, $[2, 4, 7]$, $[2, 6, 8]$, $[2, 4k+1, 4k+4]$ and $[2, 4k+2, 4k+3]$, where $k = 2, 3, 4, \ldots$

- $s = 3$. $[3, 6, 7]$, $[3, 8, 9]$, $[3, 4k + 2, 4k + 4]$ and $[3, 4k + 3, 4k + 5]$, where $k = 2, 3, 4, \ldots$
- $s = 4$. $[4, 5, 8]$, $[4, 6, 9]$, $[4, 8k + 2, 8k + 5]$, $[4, 8k + 3, 8k + 6]$, $[4, 8k + 4, 8k + 7]$ and $[4, 8k + 8, 8k + 9]$, where $k = 1, 2, 3, 4, \ldots$
- $s = 5$. $[5, 7, 9]$, $[5, 10, 14]$, $[5, 11, 15]$, $[5, 12, 13]$, $[5, 16, 18]$, $[5, 17, 19]$, $[5, 8k + 4, 8k + 5]$, $[5, 8k + 6, 8k + 9]$, $[5, 8k + 7, 8k + 10]$ and $[5, 8k + 8, 8k + 11]$, where $k = 2, 3, 4, \ldots$
- $s = 6$. [6, 10, 15], [6, 11, 16], [6, 12, 14], $[6, 8k + 5, 8k + 9]$, $[6, 8k + 10, 8k + 14]$, $[6, 8k + 11, 8k + 15]$ and $[6, 8k + 12, 8k + 16]$, where $k = 1, 2, 3, 4, \ldots$
- $s = 7$. [7, 10, 16], [7, 11, 17], [7, 12, 18], [7, 13, 15], [7, 14, 19], $[7, 4k, 4k + 2]$ and $[7, 4k + 1, 4k + 3]$, where $k = 5, 6, 7, \ldots$
- $s = 8$. [8, 10, 17], [8, 11, 18], [8, 12, 16], [8, 13, 19], [8, 14, 20], [8, 15, 21], $[8, 8k+6, 8k+10]$, $[8, 8k+7, 8k+11]$, $[8, 8k+8, 8k+12]$ and $[8, 8k+9, 8k+13]$, where $k = 2, 3, 4, \ldots$
- $s = 9$. [9, 11, 19], [9, 13, 18], [9, 14, 17], $[9, 10k+5, 10k+10]$, $[9, 10k+6, 10k+11]$, $[9, 10k + 12, 10k + 17]$, $[9, 10k + 13, 10k + 18]$ and $[9, 10k + 14, 10k + 19]$, where $k = 1, 2, 3, \ldots$

Note that this list can be verified by straight-forward (but tedious) induction and case analysis.

5. Nim* Played on Heaps of Sizes 1 and 2

Theorem 3. *Suppose that Nim* is played on* $[1_{m_1}, 2_{m_2}]$. *If* $m_1 + 2m_2 \geq 7$*, then this is a P-position when* m_1 *is odd and* m_2 *is even, and, otherwise, it is an N-position.*

Proof. Let m_1 and m_2 be the numbers of heaps of size 1 and 2, respectively. It is already known that m_1 and m_2 both even is a P-position of Nim. Hence, otherwise, m_1 and m_2 not both even are N-positions of Nim. We list all of the N- and P-positions for $1 \leq m_1 + 2m_2 \leq 8$ in Table 1. Note that our claim does not necessarily hold for $m_1+2m_2 \leq 6$. Observe that our claim holds for $m_1+2m_2 = 7$, and 8.

We induct on $m_1 + 2m_2$. Let $m_1 + 2m_2 \geq 9$. If m_1 is odd and m_2 is even and

(i) P1 uses the pass option, then $m_1' = m_1$ (odd) and $m_2' = m_2$ (even);

(ii) P1 removes a heap of size 1, then $m_1' = m_1 - 1$ (even) and $m_2' = m_2$ (even);

(iii) P1 removes a heap of size 2, then $m_1' = m_1$ (odd) and $m_2' = m_2 - 1$ (odd);

Table 1: N- and P-positions of Nim and Nim* for $1 \leq m_1 + 2m_2 \leq 8$

Game	$m_1 + 2m_2$	m_1	m_2	Position	m_1	m_2	Position
Nim	even	even	even	P	even	odd	N
	odd	odd	even	N	odd	odd	N
Nim*	1	1	0	N			
	2	2	0	N	0	1	N
	3	3	0	P	1	1	P
	4	4	0	N	2	1	N
		0	2	N			
	5	5	0	P	3	1	N
		1	2	N			
	6	6	0	N	4	1	N
		2	2	N	0	3	P
	7	7	0	P	5	1	N
		3	2	P	1	3	N
	8	8	0	N	6	1	N
		4	2	N	2	3	N
		0	4	N			

(iv) P1 removes a stone from a heap of size 2, then $m_1' = m_1 + 1$ (even) and $m_2' = m_2 - 1$ (odd);

This leaves an N-position of Nim for P2 in (i) while in each of (ii), (iii) and (iv), since $m_1 + 2m_2 - 2 \leq m_1' + 2m_2' < m_1 + 2m_2$, this leaves an N-position of Nim* for P2 to play on. Hence, it is a P-position when m_1 is odd and m_2 is even.

Otherwise, if

(i) m_1 is even and m_2 is even, P1 uses the pass option to leave $m_1' = m_1$ (even) and $m_2' = m_2$ (even);

(ii) m_1 is odd and m_2 is odd, P1 removes a heap of size 2 to leave $m_1' = m_1$ (odd) and $m_2' = m_2 - 1$ (even);

(iii) m_1 is even and m_2 is odd, P1 removes a stone from a heap of size 2 to leave $m_1' = m_1 + 1$ (odd) and $m_2' = m_2 - 1$ (even);

This leaves a P-position of Nim for P2 in (i) while in each of (ii) and (iii), since $m_1 + 2m_2 - 2 \leq m_1' + 2m_2' < m_1 + 2m_2$, this leaves a P-position for P2 to play on. Hence, it is an N-position when m_1 is even or (m_1 and m_2 are both odd). □

6. Nim* Played on Heaps of Sizes $1, 2$ and 3

Lemma 7. *Suppose that $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ is a P-position in Nim*. Then, the positions comprised of adding a heap of size r to $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ or adding additional stones to a single heap in $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ are N-positions.*

Proof. In the first instance, P1 removes the heap of size r, which leaves P2 to move on $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ (a P-position). In the second instance, P1 removes the appropriate number of stones and leaves P2 to move on $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$ (a P-position). □

Lemma 8. *Suppose that Nim* is played on $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$, and $m_1 \geq 2$. If m_j is even, for all $1 \leq j \leq k$, then this is an N-position.*

Proof. P1 uses the pass option, which leaves a P-position of Nim for P2 to play on. □

Lemma 9. *Suppose that Nim* is played on $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$, and $m_1 \geq 3$. If m_1 is odd and m_j is even, for all $2 \leq j \leq k$, then this is a P-position.*

Proof. If P1 first uses the pass option, then P2 removes a heap of size 1. This leaves a P-position of Nim for P1 to play on. On the other hand, if P1 first removes a heap of size 1, then P2 uses the pass option and leaves a P-position of Nim for P1 to play on.

Now, suppose that P1 first removes t ($1 \leq t \leq l$) stones from a heap of size l. Then, P2 should respond by removing t stones from another heap of size l. This leaves the position where m_l has been decreased by 2 and m_{l-t} has been increased by 2. Note that m_1 is still odd and all of the other m_i are even. The game continues in the following manner: If at any time P1 uses the pass option, P2 responds by removing a heap of size 1, leaving a P-position of Nim for P1 to play on. If at any time P1 removes a heap of size 1, P2 responds by using the pass option, leaving a P-position of Nim for P1 to play on. Each time P1 removes t' stones from a heap of size l' (not of size 1), P2 responds (via symmetric play) by removing t' stones from another heap of size l'.

Note that $m_1 \geq 3$ and odd ($m_1 \neq 1$) is necessary. This prevents the possibility of P1 removing the heap of size 1 (in a game position [1]), where P2 would have no move since the pass option cannot be used as the last move. □

Corollary 2. *Suppose that Nim* is played on $[1_{m_1}, 2_{m_2}, \ldots, k_{m_k}]$, and $m_1 \geq 2$. This is an N-position, if*

(a) exactly one of m_j, $2 \leq j \leq k$, is odd, or

(b) m_1 and exactly two of m_j, $2 \leq j \leq k$, are odd.

Proof. (a) Suppose m_i is odd, and m_j is even if $i \neq j$, for $2 \leq i, j \leq k$.

- If m_1 is even, P1 removes $i-1$ stones from a heap of size i to make $m'_i = m_i - 1$ (even), $m'_1 = m_1 + 1 \geq 3$ (odd), and all $m'_j = m_j$ (even), $2 \leq i \neq j \leq k$.
- If m_1 is odd, P1 removes a heap of size i to make $m'_i = m_i - 1$ (even), $m'_1 = m_1$ (odd) and all $m'_j = m_j$ (even), $2 \leq i \neq j \leq k$.

(b) Suppose m_1 (≥ 3), m_i and m_j ($i < j$) are odd, and all m_l, $2 \leq i \neq l \neq j \leq k$, are even. P1 removes $j - i$ stones from a heap of size j to make $m'_j = m_j - 1$ (even), $m'_i = m_i + 1$ (even), $m'_1 = m_1 \geq 3$ (odd) and all $m'_l = m_l$ (even), $2 \leq i \neq l \neq j \leq k$.

From Lemma 9, each of the cases above leaves a P-position of Nim* for P2 to play on. □

Lemma 10. *Suppose that Nim* is played on* $[1_{m_1}, 2_{m_2}, 3_{m_3}]$ *and* $m_1 + 2m_2 + 3m_3 \geq 10$*. If* $m_1 \geq 4$ *is even and* m_2, m_3 *are odd, then this is a P-position.*

Proof. Let $m_1 \geq 4$ be even and m_2, m_3 both odd. Here are the possible first moves for P1:

(i) If P1 uses the pass option, then $m'_1 = m_1$ (even), $m'_2 = m_2$ (odd) and $m'_3 = m_3$ (odd). This leaves an N-position of Nim for P2.

(ii) If P1 removes one stone from a heap of size 3, then $m'_1 = m_1$ (even), $m'_2 = m_2 + 1$ (even) and $m'_3 = m_3 - 1$ (even). This leaves an N-position for P2, as he simply uses the pass option.

(iii) If P1 removes two stones from a heap of size 3, then $m'_1 = m_1 + 1$ (odd), $m'_2 = m_2$ (odd) and $m'_3 = m_3 - 1$ (even). By Corollary 2, this leaves an N-position for P2.

(iv) If P1 removes a heap of size 3, then $m'_1 = m_1$ (even), $m'_2 = m_2$ (odd) and $m'_3 = m_3 - 1$ (even). By Corollary 2, this leaves an N-position for P2.

(v) If P1 removes one stone from a heap of size 2, then $m'_1 = m_1 + 1$ (odd), $m'_2 = m_2 - 1$ (even) and $m'_3 = m_3$ (odd). By Corollary 2, this leaves an N-position for P2.

(vi) If P1 removes a heap of size 2, then $m'_1 = m_1$ (even), $m'_2 = m_2 - 1$ (even) and $m'_3 = m_3$ (odd). By Corollary 2, this leaves an N-position for P2.

(vii) If P1 removes a heap of size 1, then $m'_1 = m_1 - 1$ (odd), $m'_2 = m_2$ (odd) and $m'_3 = m_3$ (odd). By Corollary 2, this leaves an N-position for P2.

In all instances, P1 loses after making his first move, thus establishing the claim. □

Theorem 4. *Suppose that Nim* is played on* $[1_{m_1}, 2_{m_2}, 3_{m_3}]$*, and* $m_1+2m_2+3m_3 \geq 10$*. If* m_1 *is odd,* m_2 *and* m_3 *are even, or* m_1 *is even,* m_2 *and* m_3 *are odd, then this is a P-position; otherwise, it is an N-position.*

Proof. Suppose that $m_1 \geq 3$. If m_1 is odd and m_2, m_3 are both odd, then this is an N-position by Corollary 2. If m_1 is even and m_2, m_3 are both odd, then this is a P-position by Lemma 10. If m_1 is even (or odd) and exactly one of the m_2, m_3 is odd, then this an N-position by Corollary 2. If m_1 is even and m_2, m_3 are both even, then this is an N-position by Lemma 8. If m_1 is odd and m_2, m_3 are both even, then this is a P-position by Lemma 9. Hence, the claim is established for $m_1 \geq 3$.

Now, suppose that $m_1 = 2$. In this case, if m_2, m_3 are both even, then this is an N-position by Lemma 8. If exactly one of the m_2, m_3 is odd, then this is an N-position by Corollary 2. Hence, the claim is established for $m_1 = 2$ (except for the case where m_2, m_3 are both odd).

If $m_1 = 0$ and m_2, m_3 are both even (not all zero), then this is an N-position (since P1 uses the pass option, leaving a P-position of Nim for P2 to play on). Hence, the claim is established for this particular case.

We now induct on $n = m_1 + 2m_2 + 3m_3$ for the remaining cases, (1) $m_1 = 0$ or 2, and m_2, m_3 are both odd, (2) $m_1 = 1$, m_2, m_3 are both even or both odd, and (3) $m_1 = 0$ or 1, and m_2 and m_3 are of different parities. We show the results in these cases, for $1 \leq n \leq 12$, in Table 2. Note that the claim does not necessarily hold for $n \leq 9$. Observe that the claim holds for $n = 10, 11$, and 12.

Let $n \geq 13$. If $m_1 = 0$ or 2, and m_2 and m_3 are both odd and

(i) P1 uses the pass option, then $m_1' = 0$ or 2, and $m_2' = m_2$ (odd) and $m_3' = m_3$ (odd);

(ii) P1 removes one stone from a heap of size 3, then $m_1' = 0$ or 2, $m_2' = m_2 + 1$ (even) and $m_3' = m_3 - 1$ (even);

(iii) P1 removes two stones from a heap of size 3, then $m_1' = 1$ or 3, $m_2' = m_2$ (odd), $m_3' = m_3 - 1$ (even);

(iv) P1 removes a heap of size 3, then $m_1' = 0$ or 2, $m_2' = m_2$ (odd), $m_3' = m_3 - 1$ (even);

(v) P1 removes one stone from a heap of size 2, then $m_1' = 1$ or 3, $m_2' = m_2 - 1$ (even), $m_3' = m_3$ (odd);

(vi) P1 removes a heap of size 2, then $m_1' = 0$ or 2, $m_2' = m_2 - 1$ (even), $m_3' = m_3$ (odd);

(vii) P1 removes a heap of size 1, then $m_1' = 1$, $m_2' = m_2$ (odd), $m_3' = m_3$ (odd).

This leaves an N-position of Nim for P2 in (i) while in each of (ii) to (vii), since $n-3 \leq n' < n$, this leaves an N-position for P2 to play on. Hence, it is a P-position when $m_1 = 0$ or 2, and m_2 and m_3 are both odd.

If $m_1 = 1$ and m_2, m_3 are both even and

(i) P1 uses the pass option, then $m'_1 = 1$, $m'_2 = m_2$ (even) and $m'_3 = m_3$ (even);

(ii) P1 removes one stone from a heap of size 3, then $m'_1 = 1$, $m'_2 = m_2 + 1$ (odd) and $m'_3 = m_3 - 1$ (odd);

(iii) P1 removes two stones from a heap of size 3, then $m'_1 = 2$, $m'_2 = m_2$ (even), $m'_3 = m_3 - 1$ (odd);

(iv) P1 removes a heap of size 3, then $m'_1 = 1$, $m'_2 = m_2$ (even), $m'_3 = m_3 - 1$ (odd);

(v) P1 removes one stone from a heap of size 2, then $m'_1 = 2$, $m'_2 = m_2 - 1$ (odd), $m'_3 = m_3$ (even);

(vi) P1 removes a heap of size 2, then $m'_1 = 1$, $m'_2 = m_2 - 1$ (odd), $m'_3 = m_3$ (even);

(vii) P1 removes a heap of size 1, then $m'_1 = 0$, $m'_2 = m_2$ (even), $m'_3 = m_3$ (even).

This leaves an N-position of Nim for P2 in (i) while in each of (ii) to (vii), since $n-3 \leq n' < n$, this leaves an N-position for P2 to play on. Hence, it is a P-position when $m_1 = 1$, and m_2 and m_3 are both even.

Finally, we show that the remaining cases are N-positions.

(i) If $m_1 = 1$, and m_2 and m_3 are odd, P1 removes one stone from a heap of size 3 to leave $m'_1 = 1$, $m'_2 = m_2 + 1$ (even) and $m'_3 = m_3 - 1$ (even).

(ii) If $m_1 = 1$, m_2 is odd and m_3 is even, P1 removes a heap of size 2 to leave $m'_1 = 1$, $m'_2 = m_2 - 1$ (even) and $m'_3 = m_3$ (even).

(iii) If $m_1 = 1$, m_2 is even and m_3 is odd, P1 removes a heap of size 3 to leave $m'_1 = 1$, $m'_2 = m_2$ (even) and $m'_3 = m_3 - 1$ (even).

(iv) If $m_1 = 0$, m_2 is even and m_3 is odd, P1 removes two stones from a heap of size 3 to leave $m'_1 = 1$, $m'_2 = m_2$ (even) and $m'_3 = m_3 - 1$ (even).

(v) If $m_1 = 0$, m_2 is odd and m_3 is even, P1 removes one stone from a heap of size 2 to leave $m'_1 = 1$, $m'_2 = m_2 - 1$ (even) and $m'_3 = m_3$ (even).

In each of (i) to (v), since $n-3 \le n' < n$, this leaves a P-position for P2 to play on. Hence, these last five cases are N-positions.

Thus, the claim is established.

Table 2: N- and P-positions of Nim and Nim* for $1 \le n = m_1 + 2m_2 + 3m_3 \le 12$ with (1) $m_1 = 0$ or 2, and m_2 and m_3 are odd, (2) $m_1 = 1$, m_2 and m_3 are both even or both odd, and (3) $m_1 = 0$ or 1, and m_2 and m_3 are of different parities.

Game	n	m_1	m_2	m_3	Position	m_1	m_2	m_3	Position
Nim	even	even	even	even	P	Otherwise			N
	odd	odd	odd	odd	P				
Nim*	The results for $m_3 = 0$ are shown in Table 1.								
	3	0	0	1	N				
	4	1	0	1	N				
	5	0	1	1	N				
	6	1	1	1	N				
	7	2	1	1	P	1	0	2	P
		0	2	1	N				
	8	1	2	1	N	0	1	2	N
	9	1	1	2	N	0	3	1	N
		0	0	3	N				
	10	1	3	1	N	1	0	3	N
		0	2	2	N				
	11	2	3	1	P	1	2	2	P
		0	4	1	N	0	1	3	P
	12	1	4	1	N	1	1	3	N
		0	3	2	N				

□

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