# EVERY NATURAL NUMBER IS THE SUM
# OF FORTY-NINE PALINDROMES

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**Abstract**

It is shown that the set of decimal palindromes is an additive basis for the natural numbers. Specifically, we prove that every natural number can be expressed as the sum of forty-nine (possibly zero) decimal palindromes.

## 1. Statement of Result

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of natural numbers (including zero). Every number $n \in \mathbb{N}$ has a unique decimal representation of the form

$$n = \sum_{j=0}^{L-1} 10^j \delta_j, \tag{1}$$

where each $\delta_j$ belongs to the digit set

$$\mathcal{D} := \{0, 1, 2, \ldots, 9\},$$

and the leading digit $\delta_{L-1}$ is nonzero whenever $L \geq 2$. In what follows, we use diagrams to illustrate the ideas; for example,

$$n = \boxed{\delta_{L-1}} \ \boxed{\cdots} \ \boxed{\delta_1} \ \boxed{\delta_0}$$

represents the relation (1). The integer $n$ is said to be a *decimal palindrome* if its decimal digits satisfy the symmetry condition

$$\delta_j = \delta_{L-1-j} \qquad (0 \leq j < L).$$

Denoting by $\mathcal{P}$ the collection of all decimal palindromes in $\mathbb{N}$, the aim of this note is to show that $\mathcal{P}$ is an additive basis for $\mathbb{N}$.

**Theorem 1.** *The set $\mathcal{P}$ of decimal palindromes is an additive basis for the natural numbers $\mathbb{N}$. Every natural number is the sum of forty-nine (possibly zero) decimal palindromes.*

The proof is given in the next section. It is unlikely that the second statement is optimal; a refinement of our method may yield an improvement. No attempt has been made to generalize this theorem to bases other than ten; for large bases, this should be straightforward, but small bases may present new obstacles (for example, obtaining the correct analogue of Lemma 4 may be challenging in the binary case, where the only nonzero digit is the digit one). We remark that arithmetic properties of palindromes (in various bases) have been previously investigated by many authors; see[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the references therein.

## 2. The Proof

### 2.1. Notation

For every $n \in \mathbb{N}$, let $L(n)$ (the "length" of $n$) denote the number of decimal digits $L$ in the expansion (1); in particular, $L(0) := 1$.

For any $\ell \in \mathbb{N}$ and $d \in \mathcal{D}$, we denote

$$p_\ell(d) := \begin{cases} 0 & \text{if } \ell = 0; \\ d & \text{if } \ell = 1; \\ 10^{\ell-1}d + d & \text{if } \ell \geq 2. \end{cases} \qquad (2)$$

Note that $p_\ell(d)$ is a palindrome, and $L(p_\ell(d)) = \ell$ if $d \neq 0$. If $\ell \geq 2$, then the decimal expansion of $p_\ell(d)$ has the form

$$p_\ell(d) = \boxed{d \mid 0\cdots0 \mid d}$$

with $\ell - 2$ zeros nested between two copies of the digit $d$.

More generally, for any integers $\ell \geq k \geq 0$ and $d \in \mathcal{D}$, let

$$p_{\ell,k}(d) := 10^k p_{\ell-k}(d) = \begin{cases} 0 & \text{if } \ell = k; \\ 10^k d & \text{if } \ell = k+1; \\ 10^{\ell-1}d + 10^k d & \text{if } \ell \geq k+2. \end{cases}$$

If $\ell \geq k+2$, then the decimal expansion of $p_{\ell,k}(d)$ has the form

$$p_{\ell,k}(d) = \boxed{d \mid 0\cdots0 \mid d \mid 0\cdots0}$$

with $\ell - k - 2$ zeros nested between two copies of the digit $d$, followed by $k$ copies of the digit zero.

Next, for any integers $\ell, k \in \mathbb{N}$, $\ell \geq k + 4$, and digits $a, b \in \mathcal{D}$, we denote

$$q_{\ell,k}(a,b) := p_{\ell,k}(a) + p_{\ell-1,k}(b) = 10^{\ell-1}a + 10^{\ell-2}b + 10^k(a+b). \qquad (3)$$

Using $\lfloor \cdot \rfloor$ to denote the floor function, we have $a + b = 10 \cdot \lfloor (a+b)/10 \rfloor + \delta_0(a+b)$ for all $a, b \in \mathcal{D}$, where $\lfloor (a+b)/10 \rfloor$ takes the value 0 or 1, and $\delta_0(a+b)$ is the one's digit of $a + b$. Hence, one sees that the decimal expansion of $q_{\ell,k}(a,b)$ has the form

$$q_{\ell,k}(a,b) = \boxed{\begin{array}{c|c|c|c|c|c} a & b & 0\cdots0 & \lfloor(a+b)/10\rfloor & \delta_0(a+b) & 0\cdots0 \end{array}}$$

with $\ell - k - 4$ zeros nested between the digits $a, b$ and the digits of $a + b$, followed by $k$ copies of the digit zero. For example, $q_{10,2}(7,8) = 7800001500$.

## 2.2. Handling Small Integers

Let $f : \mathcal{D} \to \mathcal{D}$ be the function whose values are provided by the following table:

| $d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|---|---|---|---|---|---|---|---|---|
| $f(d)$ | 0 | 1 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |

We begin our proof of Theorem 1 with the following observation.

**Lemma 1.** *Every number $f(d)$ is a palindrome, and $10^j d - f(d)$ is a palindrome for every integer $j \geq 1$.*

*Proof.* This is easily seen if $d = 0$ or $j = 1$. For $j \geq 2$ and $d = 1$, the number $10^j d - f(d) = 10^j - 1$ is a repunit of the form $9 \cdots 9$, hence a palindrome. Finally, for $j \geq 2$ and $2 \leq d \leq 9$, the number $10^j d - f(d)$ is a palindrome that has a decimal expansion of the form

$$10^j d - f(d) = \boxed{\begin{array}{c|c|c} d-1 & 9\cdots9 & d-1 \end{array}}$$

with $j - 2$ nines nested between two copies of the digit $d - 1$. $\qquad\square$

**Lemma 2.** *If $n$ is a natural number with at most $K$ nonzero decimal digits, then $n$ is the sum of $2K + 1$ palindromes.*

*Proof.* Starting with the expansion (1) we write

$$n = \delta_0 + \sum_{j \in \mathcal{J}} 10^j \delta_j,$$

where

$$\mathcal{J} := \{1 \leq j < L : \delta_j \neq 0\}.$$

Since

$$n = \delta_0 + \sum_{j \in \mathcal{J}} f(\delta_j) + \sum_{j \in \mathcal{J}} (10^j \delta_j - f(\delta_j)), \qquad (4)$$

Lemma 1 implies that $n$ is the sum of $2|\mathcal{J}| + 1$ palindromes. Since zero is a palindrome, we obtain the stated result by adding $2K - 2|\mathcal{J}|$ additional zeros on the right side of (4). $\qquad\square$

Lemma 2 implies, in particular, that $n \in \mathbb{N}$ is a sum of 49 palindromes whenever $L(n) \leq 24$. Therefore, we can assume that $L(n) \geq 25$ in the sequel.

### 2.3. Reduction to $\mathbb{N}_{\ell,0}(5+; c)$

For each $n \in \mathbb{N}$, let $L$ and $\{\delta_j\}_{j=0}^{L-1}$ be defined as in (1). Let $\mathbb{N}_{\ell,k}(5+; c)$ denote the set of integers $n$ for which $L = \ell$, $\delta_{\ell-1} \geq 5$, $\delta_k = c$, and $10^k \mid n$. In other words, $\mathbb{N}_{\ell,k}(5+; c)$ is the set of natural numbers $n$ that have a decimal expansion of the form

$$n = \boxed{\delta_{\ell-1}}\ \boxed{*\ \cdots\ *}\ \boxed{c}\ \boxed{0\cdots0}$$

with $\ell - k - 2$ arbitrary digits nested between the digit $\delta_{\ell-1} \geq 5$ and the digit $c$, followed by $k$ copies of the digit zero.

**Lemma 3.** *Let $n \in \mathbb{N}$, and put $L := L(n)$ as in (1). If $L \geq 5$, then $n$ is the sum of two palindromes and an element of $\mathbb{N}_{\ell,0}(5+; c)$ with some $\ell \in \{L-1, L\}$ and $c \in \mathcal{D}$.*

*Proof.* Let $\{\delta_j\}_{j=0}^{L-1}$ be defined as in (1). If the leading digit $\delta_{L-1}$ exceeds four, then $n \in \mathbb{N}_{L,0}(5+; \delta_0)$, and there is nothing to prove (since zero is a palindrome).

Now suppose that $\delta_{L-1} \leq 4$. Put $m := 10\delta_{L-1} + \delta_{L-2} - 6$, and observe that $4 \leq m \leq 43$. If $4 \leq m \leq 9$, then using (2) we see that

$$\begin{aligned}
n - p_{L-1}(m) &= n - (10^{L-2}m + m) \\
&= \sum_{j=0}^{L-1} 10^j \delta_j - 10^{L-2}(10\delta_{L-1} + \delta_{L-2} - 6) - m \\
&= 6 \cdot 10^{L-2} + \sum_{j=0}^{L-3} 10^j \delta_j - m.
\end{aligned}$$

As the latter number clearly lies in the interval $(5 \cdot 10^{L-2}, 7 \cdot 10^{L-2})$, and thus its leading digit is five or six, we deduce that $n - p_{L-1}(m)$ lies in $\mathbb{N}_{L-1,0}(5+; c)$, where $c \equiv (\delta_0 - m) \bmod 10$. Since $p_{L-1}(m)$ is a palindrome, this yields the desired result for $4 \leq m \leq 9$.

In the case that $10 \leq m \leq 43$, we write $m = 10a + b$ with digits $a, b \in \mathcal{D}$, $a \neq 0$.

Using (3) we have

$$
\begin{aligned}
n - q_{L,0}(a,b) &= n - (10^{L-1}a + 10^{L-2}b + a + b) \\
&= n - (10^{L-2}m + a + b) \\
&= \sum_{j=0}^{L-1} 10^j \delta_j - 10^{L-2}(10\delta_{L-1} + \delta_{L-2} - 6) - a - b \\
&= 6 \cdot 10^{L-2} + \sum_{j=0}^{L-3} 10^j \delta_j - a - b.
\end{aligned}
$$

As before, the latter number has five or six as its leading digit, so we conclude that $n - q_{L,0}(a,b)$ lies in $\mathbb{N}_{L-1,0}(5+;c)$, where $c \equiv (\delta_0 - a - b) \bmod 10$. Since $q_{L,0}(a,b)$ is the sum of two palindromes, we are done in this case as well. $\qquad\square$

### 2.4. Inductive Passage from $\mathbb{N}_{\ell,k}(5+;c_1)$ to $\mathbb{N}_{\ell-1,k+1}(5+;c_2)$

**Lemma 4.** *Let $\ell, k \in \mathbb{N}$, $\ell \geq k + 6$, and $c_1 \in \mathcal{D}$ be given. Given $n \in \mathbb{N}_{\ell,k}(5+;c_1)$, one can find digits $a_1, \ldots, a_{18}, b_1, \ldots, b_{18} \in \mathcal{D} \setminus \{0\}$ and $c_2 \in \mathcal{D}$ such that the number*

$$
n - \sum_{j=1}^{18} q_{\ell-1,k}(a_j, b_j)
$$

*lies in the set $\mathbb{N}_{\ell-1,k+1}(5+;c_2)$.*

*Proof.* Fix $n \in \mathbb{N}_{\ell,k}(5+;c_1)$, and let $\{\delta_j\}_{j=0}^{\ell-1}$ be defined as in (1) (with $L := \ell$). Let $m$ be the three-digit integer formed from the first three digits of $n$; that is,

$$
m := 100\delta_{\ell-1} + 10\delta_{\ell-2} + \delta_{\ell-3}.
$$

Clearly, $m$ is an integer in the range $500 \leq m \leq 999$, and we have

$$
n = \sum_{j=k}^{\ell-1} 10^j \delta_j = 10^{\ell-3}m + \sum_{j=k}^{\ell-4} 10^j \delta_j. \tag{5}
$$

Let us denote

$$
\mathcal{S} := \{19, 29, 39, 49, 59\}.
$$

In view of the fact that

$$
9\mathcal{S} := \underbrace{\mathcal{S} + \cdots + \mathcal{S}}_{\text{nine copies}} = \{171, 181, 191, \ldots, 531\},
$$

it is possible to find an element $h \in 9\mathcal{S}$ for which $m - 80 < 2h \leq m - 60$. With $h$ fixed, let $s_1, \ldots, s_9$ be elements of $\mathcal{S}$ such that

$$
s_1 + \cdots + s_9 = h.
$$

Finally, let $\varepsilon_1, \ldots, \varepsilon_9$ be natural numbers, each equal to zero or two: $\varepsilon_j \in \{0, 2\}$ for $j = 1, \ldots, 9$. A specific choice of these numbers is given below.

We now put

$$t_j := s_j + \varepsilon_j \qquad \text{and} \qquad t_{j+9} := s_j - \varepsilon_j \qquad (j = 1, \ldots, 9),$$

and let $a_1, \ldots, a_{18}, b_1, \ldots, b_{18} \in \mathcal{D}$ be determined from the digits of $t_1, \ldots, t_{18}$, respectively, via the relations

$$10a_j + b_j = t_j \qquad (j = 1, \ldots, 18).$$

Since

$$\mathcal{S} + 2 = \{21, 31, 41, 51, 61\} \qquad \text{and} \qquad \mathcal{S} - 2 = \{17, 27, 37, 47, 57\},$$

all of the digits $a_1, \ldots, a_{18}, b_1, \ldots, b_{18}$ are *nonzero*, as required.

Using (3) we compute

$$\sum_{j=1}^{18} q_{\ell-1,k}(a_j, b_j) = \sum_{j=1}^{18} \left( 10^{\ell-2} a_j + 10^{\ell-3} b_j + 10^k(a_j + b_j) \right)$$

$$= 10^{\ell-3} \sum_{j=1}^{18} t_j + 10^k \sum_{j=1}^{18} (a_j + b_j)$$

$$= 2h \cdot 10^{\ell-3} + 10^k \sum_{j=1}^{18} (a_j + b_j)$$

since

$$t_1 + \cdots + t_{18} = 2(s_1 + \cdots + s_9) = 2h$$

regardless of the choice of the $\varepsilon_j$'s. Taking (5) into account, we have

$$n - \sum_{j=1}^{18} q_{\ell-1,k}(a_j, b_j) = 10^{\ell-3}(m - 2h) + \sum_{j=k}^{\ell-4} 10^j \delta_j - 10^k \sum_{j=1}^{18} (a_j + b_j), \qquad (6)$$

and since $60 \leq m - 2h < 80$ it follows that the number defined by either side of (6) lies in the set $\mathbb{N}_{\ell-1,k}(5+; c)$, where $c$ is the unique digit in $\mathcal{D}$ determined by the congruence

$$\delta_k - \sum_{j=1}^{18} (a_j + b_j) \equiv c \bmod 10. \qquad (7)$$

To complete the proof, it suffices to show that for an appropriate choice of the $\varepsilon_j$'s we have $c = 0$, for this implies that $n \in \mathbb{N}_{\ell-1,k+1}(5+; c_2)$ for some $c_2 \in \mathcal{D}$. To do this, let $g(r)$ denote the sum of the decimal digits of any $r \in \mathbb{N}$. Then

$$\sum_{j=1}^{18} (a_j + b_j) = \sum_{j=1}^{18} g(t_j) = \sum_{j=1}^{9} g(s_j + \varepsilon_j) + \sum_{j=1}^{9} g(s_j - \varepsilon_j).$$

For every number $s \in \mathcal{S}$, one readily verifies that

$$g(s+2) + g(s-2) = 2\,g(s) - 9.$$

Therefore, (7) is equivalent to the congruence condition

$$\delta_k - 2\sum_{j=1}^{9} g(s_j) + 9E \equiv c \bmod 10,$$

where $E$ is the number of integers $j \in \{1, \ldots, 9\}$ such that $\varepsilon_j := 2$. As we can clearly choose the $\varepsilon_j$'s so the latter congruence is satisfied with $c = 0$, the proof of the lemma is complete. $\qquad\square$

### 2.5. Proof of Theorem 1

Let $n$ be an arbitrary natural number. To show that $n$ is the sum of 49 palindromes, we can assume that $L := L(n)$ is at least 25, as mentioned in §2.2. By Lemma 3 we can find two palindromes $\widetilde{p}_1, \widetilde{p}_2$ such that the number

$$n_1 := n - \widetilde{p}_1 - \widetilde{p}_2 \tag{8}$$

belongs to $\mathbb{N}_{\ell,0}(5+; c_1)$ for some $\ell \in \{L-1, L\}$ and $c_1 \in \mathcal{D}$. Since $\ell \geq 24$, by Lemma 4 we can find digits $a_1^{(1)}, \ldots, a_{18}^{(1)}, b_1^{(1)}, \ldots, b_{18}^{(1)} \in \mathcal{D} \setminus \{0\}$ and $c_2 \in \mathcal{D}$ such that the number

$$n_2 := n_1 - \sum_{j=1}^{18} q_{\ell-1,0}\big(a_j^{(1)}, b_j^{(1)}\big)$$

lies in the set $\mathbb{N}_{\ell-1,1}(5+; c_2)$. Similarly, using Lemma 4 again we can find digits $a_1^{(2)}, \ldots, a_{18}^{(2)}, b_1^{(2)}, \ldots, b_{18}^{(2)} \in \mathcal{D} \setminus \{0\}$ and $c_3 \in \mathcal{D}$ such that

$$n_3 := n_2 - \sum_{j=1}^{18} q_{\ell-2,1}\big(a_j^{(2)}, b_j^{(2)}\big)$$

belongs to the set $\mathbb{N}_{\ell-2,2}(5+; c_3)$. Proceeding inductively in this manner, we continue to construct the sequence $n_1, n_2, n_3, \ldots$, where each number

$$n_i := n_{i-1} - \sum_{j=1}^{18} q_{\ell-i+1,i-2}\big(a_j^{(i-1)}, b_j^{(i-1)}\big) \tag{9}$$

lies in the set $\mathbb{N}_{\ell-i+1,i-1}(5+; c_i)$. The method works until we reach a specific value of $i$, say $i := \nu$, where $\ell - \nu + 1 < (\nu - 1) + 6$; at this point, Lemma 4 can no longer be applied.

Notice that, since $\ell - \nu + 1 \leq (\nu - 1) + 5$, every element of $\mathbb{N}_{\ell-\nu+1,\nu-1}(5+; c_\nu)$ has at most five nonzero digits. Therefore, by Lemma 2 we can find eleven palindromes $\widetilde{p}_3, \widetilde{p}_4, \ldots, \widetilde{p}_{13}$ such that

$$n_\nu = \widetilde{p}_3 + \widetilde{p}_4 + \cdots + \widetilde{p}_{13}. \tag{10}$$

Now, combining (8), (9) with $i = 2, 3, \ldots, \nu$, and (10), we see that

$$n = \sum_{i=1}^{13} \widetilde{p}_j + \sum_{j=1}^{18} N_j,$$

where

$$N_j := \sum_{i=2}^{\nu} q_{\ell-i+1,i-2}\big(a_j^{(i-1)}, b_j^{(i-1)}\big) \qquad (j = 1, \ldots, 18).$$

To complete the proof of the theorem, it remains to verify that every integer $N_j$ is the sum of two palindromes. Indeed, by (3) we have

$$N_j = \sum_{i=2}^{\nu} p_{\ell-i+1,i-2}\big(a_j^{(i-1)}\big) + \sum_{i=2}^{\nu} p_{\ell-i,i-2}\big(b_j^{(i-1)}\big).$$

Considering the form of the decimal expansions, for each $j$ we see that

$$\sum_{i=2}^{\nu} p_{\ell-i+1,i-2}\big(a_j^{(i-1)}\big) = \boxed{a_j^{(1)} \mid \cdots \mid a_j^{(\nu-1)} \mid 0\cdots 0 \mid a_j^{(\nu-1)} \mid \cdots \mid a_j^{(1)}}$$

which is a palindrome of length $\ell - 1$ (since $a_j^{(1)} \neq 0$) having precisely $2(\nu - 1)$ nonzero entries, and

$$\sum_{i=2}^{\nu} p_{\ell-i,i-2}\big(b_j^{(i-1)}\big) = \boxed{b_j^{(1)} \mid \cdots \mid b_j^{(\nu-1)} \mid 0\cdots 0 \mid b_j^{(\nu-1)} \mid \cdots \mid b_j^{(1)}}$$

which is a palindrome of length $\ell - 2$ (since $b_j^{(1)} \neq 0$), also having precisely $2(\nu - 1)$ nonzero entries.

### References

[1] J.-P. Allouche and J. Shallit, "Sums of digits, overlaps, and palindromes," *Discrete Math. Theor. Comput. Sci.* 4 (2000), no. 1, 1–10.

[2] W. D. Banks, D. N. Hart and M. Sakata, "Almost all palindromes are composite," *Math. Res. Lett.* 11 (2004), no. 5-6, 853–868.

[3] W. D. Banks and I. E. Shparlinski, "Average value of the Euler function on binary palindromes," *Bull. Pol. Acad. Sci. Math.* 54 (2006), no. 2, 95–101.

[4] W. D. Banks, and I. E. Shparlinski, "Prime divisors of palindromes," *Period. Math. Hungar.* 51 (2005), no. 1, 1–10.

[5] A. Bérczes and V. Ziegler, "On simultaneous palindromes," *J. Comb. Number Theory* 6 (2014), no. 1, 37–49.

[6] J. Cilleruelo, F. Luca, and I. E. Shparlinski, "Power values of palindromes," *J. Comb. Number Theory* 1 (2009), no. 2, 101–107.

[7] J. Cilleruelo, R. Tesoro and F. Luca, "Palindromes in linear recurrence sequences," *Monatsh. Math.* 171 (2013), no. 3-4, 433–442.

[8] S. Col, "Palindromes dans les progressions arithmétiques," *Acta Arith.* 137 (2009), no. 1, 1–41. (French)

[9] D. Coogan and H. Gabai, "On palindromes and palindromic primes," *Math. Mag.* 42 (1969), 252–254.

[10] E. Goins, "Palindromes in different bases: A conjecture of J. Ernest Wilkins," *Integers* 9 (2009), A55, 725–734.

[11] S. Hernández Hernández and F. Luca, "Palindromic powers," *Rev. Colombiana Mat.* 40 (2006), no. 2, 81–86.

[12] I. Korec, "Palindromic squares for various number system bases," *Math. Slovaca* 41 (1991), no. 3, 261–276.

[13] F. Luca, "Palindromes in Lucas sequences," *Monatsh. Math.* 138 (2003), no. 3, 209–223.

[14] G. J. Simmons, "On palindromic squares of non-palindromic numbers," *J. Recreational Math.* 5 (1972), no. 1, 11–19.