A REMARK ON $A+B$ AND $A-A$ FOR COMPACT SETS IN $\mathbb{R}^n$

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Abstract
We prove in particular that if $A \subset \mathbb{R}^n$ is a compact convex set, and $B \subset \mathbb{R}^n$ is an arbitrary compact set, then $\mu(A-A) \ll \frac{\mu(A+B)^2}{\sqrt{n}\mu(A)}$, provided that $\mu(B) \geq \mu(A)$.

1. Introduction

A well–known Ruzsa triangle inequality states that for any finite subsets of an abelian group we have

$$|A-B| \leq \frac{|A+C||C+B|}{|C|}\,;$$

in particular, if $B = A$ and $C = B$, then

$$|A-A| \leq \frac{|A+B|^2}{|B|}\,.$$

The aim of this note is to prove a sharp, up to a dimension–independent constant, form of the above inequality for a compact convex set $A \subset \mathbb{R}^n$, and an arbitrary compact set $B \subset \mathbb{R}^n$, provided that $\mu(A) \geq \mu(B)$.

2. Result

For a set $A \subset \mathbb{R}^n$ and $x \in A-A$ let $A_x = A \cap (A-x)$. Our main tool is the following lemma proved in [4, Lemma 5]. We recall its proof as it is very simple.

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Lemma 1. *Let $A, B \subset \mathbb{R}^n$ be compact sets. Then*

$$\int_{A-A} \mu(A_x + B)\, dx \le \mu(A+B)^2 \,. \tag{1}$$

Proof. We apply the Koester-Katz transform: if $x \in A - A$ then

$$A_x + B \subseteq (A+B)_x \,.$$

Therefore, we have

$$\int_{A-A} \mu(A_x + B) dx \le \int_{A+B-A-B} \mu((A+B)_x) dx = \mu(A+B)^2 \,,$$

and the assertion follows. □

We also need a lower bound for the size of A_x for a convex set A, see [5, Section 3]. We also give the proof for the sake of completeness.

Lemma 2. *Let $A \subset \mathbb{R}^n$ be a compact convex set and $r \in [0,1]$ be any real number. Then for all $x \in r(A-A)$ the following holds:*

$$\mu(A_x) \ge (1-r)^n \mu(A) \,. \tag{2}$$

Proof. Write $x = ra_1 - ra_2$, where $a_1, a_2 \in A$ and let $a \in A$ be any element. By convexity, $(1-r)a + ra_1 \in A$ and $(1-r)a + ra_1 = (1-r)a + ra_2 + x \in A + x$. Thus $(1-r)A + ra_1 \subseteq A \cap (A+x)$ and the result follows. □

Finally, we recall the Brunn-Minkowski inequality, see [5, Section 3].

Theorem 1. *Let $A, B \subset \mathbb{R}^n$ be non–empty compact sets. Then*

$$\mu(A+B)^{1/n} \ge \mu(A)^{1/n} + \mu(B)^{1/n} \,.$$

Now we can formulate our main result.

Theorem 2. *Let $A \subset \mathbb{R}^n$ be a compact convex set, and let $B \subset \mathbb{R}^n$ be an arbitrary compact set. Then*

$$(1 + \omega + \cdots + \omega^{[\sqrt{n}]}) \mu(B)^{1-1/n} \mu(A)^{1/n} \mu(A-A) \ll \mu(A+B)^2 \,, \tag{3}$$

where $\omega = (\mu(A)/\mu(B))^{1/n}$. In particular, if $\mu(A) \ge \mu(B)$ then

$$\mu(A-A) \ll \frac{\mu(A+B)^2}{\sqrt{n}\mu(A)^{1/n}\mu(B)^{1-1/n}} \,, \tag{4}$$

and if $\mu(B) \geq \mu(A)$ *then*

$$\mu(A-A) \ll \frac{\mu(A+B)^2}{\sqrt{n}\mu(A)}\,. \tag{5}$$

Proof. Let $\alpha = \mu(B)/\mu(A)$. Applying (1) and the Brunn-Minkowski inequality, we get

$$\begin{aligned}\mu^2(A+B) &\geq \int_{A-A} \mu(B+A_x)\,dx \geq \int_{A-A} \left(\mu(B)^{1/n} + \mu(A_x)^{1/n}\right)^n dx \\ &= \alpha \sum_{k=0}^{n} \binom{n}{k} \int_{A-A} \alpha^{-k/n} \mu(A)^{(n-k)/n} \mu(A_x)^{k/n}\,dx\,.\end{aligned}$$

To estimate the size of A_x we use Lemma 2. After integration by parts, we obtain

$$\begin{aligned}\mu^2(A+B) &\geq \mu(B) \sum_{k=0}^{n} \binom{n}{k} k\alpha^{-k/n} \int_0^1 (1-r)^{k-1} \mu(r(A-A))\,dr \\ &= \mu(B)\mu(A-A) \sum_{k=1}^{n} \binom{n}{k} k\alpha^{-k/n} \int_0^1 (1-r)^{k-1} r^n\,dr \\ &= \mu(B)\mu(A-A) \sum_{k=1}^{n} \binom{n}{k} k\alpha^{-k/n} \mathcal{B}(k, n+1)\,,\end{aligned}$$

where $\mathcal{B}(\cdot,\cdot)$ is the beta function. Thus

$$\mu^2(A+B) \geq \mu(B)\mu(A-A) \sum_{k=1}^{n} \alpha^{-k/n} \frac{(n!)^2}{(n-k)!(n+k)!} := \mu(B)\mu(A-A) \times \sigma\,.$$

One can calculate the last sum σ using the gamma function or hypergeometric series, but we use a rather crude estimate. Put $\Delta = [\sqrt{n}] + 1$; then

$$\sigma = \sum_{k=1}^{n} \alpha^{-\frac{k}{n}} \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right) \prod_{j=1}^{k} \left(1 + \frac{j}{n}\right)^{-1} = \sum_{k=1}^{n} \alpha^{-\frac{k}{n}} \left(1 + \frac{k}{n}\right)^{-1} \prod_{j=1}^{k-1} \left(1 - \frac{2j}{n+j}\right).$$

Using inequalities $\ln(1-x) \geq -2x$ for $0 \leq x \leq 0.5$ and $k \leq n$, we obtain

$$\sigma \geq \frac{1}{2} \sum_{k=1}^{\Delta} \alpha^{-k/n} \exp\left(-\sum_{j=1}^{k-1} \frac{4j}{n+j}\right) \geq \frac{1}{2} \sum_{k=1}^{\Delta} \alpha^{-k/n} \exp\left(-\frac{2k^2}{n}\right) \gg \sum_{k=1}^{\Delta} \omega^k\,.$$

This gives us (3). To see (4), it is enough to observe that if $\mu(A) \geq \mu(B)$ then $\sum_{k=1}^{\Delta} \omega^k \geq \sqrt{n}$. To get (5), take any subset B' of B such that $\mu(B') = \mu(A)$ and apply (4); then

$$\mu(A-A) \ll \frac{\mu(A+B')^2}{\sqrt{n}\mu(A)} \leq \frac{\mu(A+B)^2}{\sqrt{n}\mu(A)}\,.$$

This completes the proof. □

Remark 1. Estimate (4) is tight; see [2] or [3] (discussion after Corollary 8.3). Indeed, consider the n–dimensional simplex

$$A = A_L = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_j \geq 0, \sum_{j=1}^{n} x_j \leq L\},$$

where L is a parameter. Then $\mu(A+A) = 2^n \mu(A)$ and $\mu(A-A) = \binom{2n}{n}\mu(A)$ (to obtain the last formula, one can count the number of integer points in A, say, and approximate $\mu(A-A)$ by

$$\sum_{a+b+c=n} \binom{n}{a,b,c}\binom{L}{a}\binom{L}{b} \sim \frac{L^n}{n!}\sum_{m=0}^{n}\binom{n}{m}^2 = \frac{L^n}{n!}\binom{2n}{n} = \mu(A)\binom{2n}{n};$$

see [1]. Here a, b and c, are the number of possibilities for the positive, negative and zero coordinates in $A-A$, respectively). Hence

$$\mu(A-A) \gg \frac{\mu(A+A)^2}{\sqrt{n}\mu(A)}.$$

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