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ON NONMEASURABLE SUBGROUPS OF THE REAL LINE

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Abstract. We prove that, for every nonzero σ–finite measure μ defined on the real line R and invariant (or quasiinvariant) under all translations of R, there exists a subgroup of R nonmeasurable with respect to μ. Some generalizations of this result are discussed, too, and several problems related to them are posed.

Let R be the real line and let λ be the classical Lebesgue measure on R. In this paper we shall consider some λ–nonmeasurable subsets of R having additional algebraic properties. Also, we shall consider some subsets of R having an additional algebraic structure and nonmeasurable with respect to various invariant measures defined on R.

It is well known that there exist many interesting examples of Lebesgue nonmeasurable subsets of R. In particular, it can be shown (with the aid of an uncountable form of the Axiom of Choice) that there are Lebesgue nonmeasurable subgroups of the additive group of R. For instance, using the method of transfinite recursion and applying Hamel bases of R, it is not difficult to construct two subsets V_1 and V_2 of R such that

1) $V_1 \cap V_2 = \{0\}$, $V_1 + V_2 = R$;

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2) V_1 and V_2 are vector spaces over the field Q of all rational numbers and, in particular, they are subgroups of the additive group of R;

3) V_1 and V_2 are Bernstein subsets of R.

Obviously, relation 3) implies that the subgroups V_1 and V_2 of the additive group of the real line are nonmeasurable with respect to the Lebesgue measure λ.

The construction of the above–mentioned groups V_1 and V_2 is similar to the classical Bernstein construction and is essentially based on some specific topological properties of the real line. In connection with this fact it is reasonable to find a construction of a nonmeasurable subgroup of R in purely algebraic terms and in terms of an invariant measure structure. More precisely, let μ be an arbitrary nonzero σ–finite R–invariant (or, more generally, R–quasiinvariant) measure defined on a σ–algebra of subsets of R. The following question arises in a natural way: does there exist a subgroup of R nonmeasurable with respect to the measure μ? It turns out that the answer to this question is positive. Moreover, it can be proved that there exists a subset V of the real line, such that V is a vector space over the field Q and V is nonmeasurable with respect to μ. Here we shall give a short proof of this fact. The method that we shall use in our further consideration is taken from paper [4] where several applications of Hamel bases to the theory of invariant measures are presented. Notice that this method can be applied in many other situations (see, in particular, Remark 3 below). Actually, the countable chain condition and the quasiinvariance of μ (i.e. invariance of the class of all μ–measurable sets and invariance of the class of all μ–measure zero sets) yield the desired result.

In our considerations we use the standard terminology and notation of point set theory (see, e.g., [9], [8] and [13]). Also, we use some well known facts from group theory, concerning the algebraic structure of infinite commutative groups (see, for instance, [7]).

First of all let us notice that, applying a Hamel basis of the real line R, it is not difficult to represent the additive group of R as a direct sum

$$R = G_1 + G_2 \ (G_1 \cap G_2 = \{0\}),$$

where G_1 and G_2 are some vector spaces over Q and, in addition, $card(G_1) = \omega_1$.

Now, let us consider the following family of sets:

$$\{Y + G_2 : Y \ is\ a\ countable\ subset\ of\ G_1\}.$$

Denote by the symbol I the ideal of sets generated by this family. It is clear that

1) I is a σ–ideal of subsets of R;

2) I is invariant under the group of all translations of R (moreover, I is invariant under the group of all isometric transformations of R);

3) for each set $Z \in I$, there exists an uncountable family $\{g_\alpha : \alpha < \omega_1\}$ of elements from G_1 such that $\{g_\alpha + Z : \alpha < \omega_1\}$ is a family of pairwise disjoint sets.

Starting with these properties of the ideal I and applying the standard methods of measure theory (see, e.g., [5] or [10]), we immediately obtain the following statement.

Lemma 1. *Let μ be an arbitrary σ–finite R–invariant (respectively, R–quasiinvariant) measure defined on the real line R. Then there exists a measure ν, also defined on R, such that*

1) ν is an extension of μ;

2) ν is an R–invariant (respectively, R–quasiinvariant) measure;

3) the ideal I is contained in $dom(\nu)$;

4) for each set $Z \in I$, we have the equality $\nu(Z) = 0$.

We need also the following auxiliary proposition.

Lemma 2. *Let μ be a σ–finite measure defined on some σ–algebra of subsets of a basic set E and let $\{Z_\alpha : \alpha < \omega_1\}$ be an uncountable family of μ–measurable subsets of E. Furthermore, suppose that $n > 0$ is a fixed natural number and suppose that, for every n–element subset D of ω_1, the equality*

$$\mu(\cap\{Z_\alpha : \alpha \in D\}) = 0$$

holds. Then there exists an uncountable subset A of ω_1 such that $\mu(Z_\alpha) = 0$, for each ordinal α from A.

The proof of this lemma is not difficult. It can be carried out by induction on n. Actually, a result much stronger than Lemma 2 can be established in terms of σ–algebras with σ–ideals satisfying the countable chain condition (for details, see [6]).

Now, we are going to establish the following statement.

Theorem 1. *Let μ be an arbitrary nonzero σ–finite R–invariant (or, more generally, R–quasiinvariant) measure defined on the real line R. Then there exists a subset V of R such that V is a vector space over the field Q and V is nonmeasurable with respect to the given measure μ.*

Proof. Without loss of generality we may assume that the ideal I of subsets of R is contained in $dom(\mu)$ (see Lemma 1). Therefore, for each set $Z \in I$, we have the equality $\mu(Z) = 0$.

Let X be a Hamel basis for the vector space G_1. Obviously, the equality $card(X) = \omega_1$ holds. Let us consider a family

$$\{X_{n,\xi} : n < \omega,\ \xi < \omega_1\}$$

of those subsets of X which form the classical Ulam matrix in X. (Information about this matrix is given, e.g., in the well known books [9], Chapter 5, and [8], Chapter 2.)

Let us recall that the Ulam matrix has the following two properties:

a) for each ordinal number $\xi < \omega_1$, the set

$$X \setminus \cup\{X_{n,\xi} : n < \omega\}$$

is at most countable;

b) for each natural number n, the family

$$\{X_{n,\xi} : \xi < \omega_1\}$$

consists of pairwise disjoint sets.

Let us put

$$Y_{n,\xi} = \cup\{X_{k,\xi} : k \leq n\}.$$

It is clear that, for each ordinal $\xi < \omega_1$, the family $\{Y_{n,\xi} : n < \omega\}$ of subsets of X is increasing with respect to inclusion and the complement of the union of this family is a countable subset of X. From this fact we can easily deduce that, if an ordinal $\xi < \omega_1$ is fixed, then there exists a natural number n (certainly depending on ξ) such that the following inequality is fulfilled:

$$\mu^*(lin(Y_{n,\xi}) + G_2) > 0,$$

where μ^* denotes the outer measure associated with μ and $lin(Y_{n,\xi})$ denotes the vector space (over the field Q) generated by the set $Y_{n,\xi}$.

Since ω is a countable set and ω_1 is an uncountable set, we can conclude that there exist a natural number m and an uncountable subset B of ω_1 such that, for all ordinals $\xi \in B$, we have

$$\mu^*(lin(Y_{m,\xi}) + G_2) > 0.$$

Now, let us put

$$V_\xi = lin(Y_{m,\xi}) + G_2 \ (\xi \in B).$$

Taking into account the fact that the sets in any given row of the Ulam matrix are pairwise disjoint, we deduce that, for every $(m + 2)$–element subset D of B, the equality

$$\cap\{Y_{m,\xi} : \xi \in D\} = \emptyset$$

holds. Furthermore, taking into account the fact that the set X is a Hamel basis of the space G_1, it is easy to verify that, for every $(m+2)$–element subset D of B, the equality

$$\cap\{V_\xi : \xi \in D\} = G_2$$

holds, too. Obviously, we have

$$\mu(G_2) = 0.$$

Suppose now that all vector spaces V_ξ $(\xi \in B)$ are measurable with respect to μ. Then, applying Lemma 2, we conclude that there exists an ordinal $\xi \in B$ such that $\mu(V_\xi) = 0$. But the latter relation gives us a contradiction with the relation $\mu^*(V_\xi) > 0$. Consequently, there exists an ordinal number $\beta \in B$ such that the corresponding vector space V_β is nonmeasurable with respect to the measure μ. Hence, if we put $V = V_\beta$, then the set V is the required one. Moreover, the same argument shows us that there exists an uncountable subset C of B such that all vector spaces V_ξ $(\xi \in C)$ are nonmeasurable with respect to μ. Thus, the proof of Theorem 1 is complete. □

Remark 1. Let us consider again the real line R as a vector space over the field Q of all rational numbers. Let H be a fixed uncountable vector subspace of R. Let μ be an arbitrary nonzero σ–finite H–invariant (or, more generally, H–quasiinvariant) measure defined on R. It can be shown, using an argument similar to the proof of Theorem 1, that there exists a vector subspace of R nonmeasurable with respect to the measure μ.

Indeed, without loss of generality we may assume that $card(H) = \omega_1$. Furthermore, since H is a vector space over Q, we can write

$$R = G_1 + G_2 \ (G_1 \cap G_2 = \{0\}),$$

where $G_1 = H$ and G_2 also is a vector space over Q. Now, it is clear that we can apply here the same argument as in the proof of Theorem 1 (actually, we used in the proof of this theorem only the G_1–quasiinvariance of the given measure μ). In this way we obtain a vector subspace of R (over Q) nonmeasurable with respect to μ.

Remark 2. Let μ be an arbitrary nonzero σ–finite R–invariant (or R–quasiinvariant) measure defined on R. The following question arises naturally: does there exist a Hamel basis in R nonmeasurable with respect to μ? It turns out that this question is undecidable in set theory ZFC. Indeed, it can be proved (see, e.g., [1] and [13]) that the next two assertions are equivalent:

1) the Continuum Hypothesis;

2) there exists a countable family of Hamel bases in R such that the union of this family coincides with the set $R \setminus \{0\}$.

From this fact it can be deduced that the next two assertions also are equivalent:

1) the Continuum Hypothesis;

3) for every nonzero σ–finite R–invariant (or R–quasiinvariant) measure μ defined on R, there exists a Hamel basis in R nonmeasurable with respect to μ.

Indeed, first let us observe that if μ is an arbitrary σ–finite R–invariant (R–quasiinvariant) measure on R and H is a μ–measurable Hamel basis in R, then we have the equality $\mu(H) = 0$. Taking into account this observation and the equivalence of assertions 1) and 2), we see that 1) implies 3). Conversely, suppose that 3) is fulfilled and consider the R–invariant σ–ideal J of subsets of R, generated by the family of all Hamel bases in R. Then one can easily verify that J is not a proper σ–ideal (otherwise there exists a probability R–invariant measure ν on R such that $J \subset dom(\nu)$). Thus, J coincides with the family of all subsets of R, i.e. $R \in J$. From this fact it is not difficult to deduce assertion 2) and, consequently, the Continuum Hypothesis.

The equivalence of 1) and 3) immediately implies that the question posed above is undecidable in set theory ZFC.

Notice that Theorem 1 can be generalized to a wide class of uncountable groups. In particular, it can be generalized to the class of all uncountable commutative groups. Of course, in order to obtain the corresponding result for uncountable commutative groups, we need some deep theorems concerning the algebraic structure of those groups.

Let $(G, +)$ be an arbitrary uncountable commutative group and let μ be an arbitrary nonzero σ–finite G–invariant (or G–quasiinvariant) measure defined on G. Obviously, without loss of generality, we may assume that μ is a probability G-quasiinvariant measure on G. According to the well known result from the theory of commutative groups (see, e.g., [7], p. 148), we have the equality

$$G = \cup\{\Gamma_k : k \in \omega\},$$

where $\{\Gamma_k : k \in \omega\}$ is an increasing (with respect to inclusion) countable family of subgroups of G such that every group Γ_k can be represented as a direct sum of cyclic groups. Now, only two cases are possible.

1. For all natural numbers k the inequality $card(G/\Gamma_k) \geq \omega_1$ holds. It is easy to see that there exists a natural number p such that $\mu^*(\Gamma_p) > 0$. On the other hand, the group G contains an uncountable family of pairwise

disjoint translates of the group Γ_p. Hence, the group Γ_p is not measurable with respect to the given measure μ.

2. There exists a natural number k such that $card(G/\Gamma_k) \leq \omega$. Let us fix a number k with this property. If the group Γ_k is nonmeasurable with respect to μ, then there is nothing to prove. Suppose now that Γ_k is a μ–measurable subset of G. Then we have $\mu(\Gamma_k) > 0$. Let ν denote the restriction of the measure μ to the group Γ_k. Obviously, ν is a nonzero finite Γ_k–quasiinvariant measure defined on the group Γ_k. Since Γ_k is uncountable and is a direct sum of cyclic groups, we can write

$$\Gamma_k = G_1 + G_2 \ (G_1 \cap G_2 = \{0\}),$$

where G_1 and G_2 are some subgroups of Γ_k satisfying the following relations:
1) $card(G_1) = \omega_1$;
2) G_1 is a direct sum of cyclic groups;
3) G_2 also is a direct sum of cyclic groups.

Now, we can apply to the group $\Gamma_k = G_1 + G_2$ and to the measure ν the same construction as in the proof of Theorem 1. In such a way we get a subgroup Γ of the group Γ_k, nonmeasurable with respect to the measure ν. But it is clear that Γ is also a subgroup of the original group G and that Γ is nonmeasurable with respect to the original measure μ.

Thus, we obtained the following result.

Theorem 2. *Let $(G, +)$ be an uncountable commutative group and let μ be a nonzero σ–finite G–invariant (or G–quasiinvariant) measure defined on G. Then there exists a subgroup of G nonmeasurable with respect to μ.*

The next example gives us a nontrivial application of Theorem 2 to the Haar measure defined on a commutative locally compact topological group.

Example. Let $(G, +)$ be an uncountable commutative σ–compact locally compact topological group and let μ be the completion of the Haar measure on G. Suppose also that Γ is an arbitrary thick subgroup of G with respect to the measure μ, i.e.

$$\mu_*(G \setminus \Gamma) = 0.$$

Let us denote by ν the trace of the measure μ on the group Γ. Then ν is a nonzero σ–finite Γ–invariant measure defined on Γ. Applying the result of Theorem 2, we conclude that there exists a subgroup of the group Γ nonmeasurable with respect to the measure ν. In particular, if we take $\Gamma = G$, then we get a subgroup of G nonmeasurable with respect to the original measure μ.

It is also clear that these results are true for any commutative nondiscrete locally compact topological group $(G, +)$ equipped with the completion of the Haar measure on G.

Notice that there is a commutative nondiscrete locally compact topological group $(G, +)$ without dense proper subgroups (see [11]). Evidently, such a group G does not contain dense subgroups nonmeasurable with respect to the completion μ of the Haar measure on G. This fact shows directly that we cannot apply the classical Bernstein construction to G, in order to obtain a μ–thick proper subgroup of G. Thus, we see that an approach based on the corresponding combinatorial properties of the Ulam transfinite matrix is more preferable because it yields the existence of nonmeasurable subgroups.

In connection with Theorem 2, let us remark that it can be extended to a large class of uncountable noncommutative groups, too. On the other hand, this theorem cannot be generalized to the class of all uncountable groups. Indeed, let us consider an arbitrary group G satisfying the following conditions:

(1) the cofinality of the cardinal number $card(G)$ is strictly greater than ω;

(2) for every subgroup H of G we have either $card(H) < card(G)$ or $H = G$.

In particular, if G is a Jonsson group of cardinality ω_1, then it is clear that G satisfies conditions (1) and (2) because ω_1 is a regular cardinal number and G does not contain a proper subgroup of cardinality ω_1. We wish to recall that such a group G was first constructed by Shelah (see [12]).

Now, denote by J the σ–ideal of all subsets Z of G with $card(Z) < card(G)$. Let S be the σ–algebra of subsets of G, generated by the ideal J. Then it is easy to define a probability G–invariant measure μ on S such that $\mu(Z) = 0$ for each set Z from the ideal J. For this measure μ it is also easy to check that there does not exist a subgroup of G nonmeasurable with respect to μ.

In connection with Theorems 1 and 2, the following two similar problems can be posed.

Problem 1. Give a characterization of all uncountable groups G such that, for every nonzero σ–finite G–invariant (or G–quasiinvariant) measure μ defined on G, there exists a subgroup of G nonmeasurable with respect to μ.

Problem 2. Give a characterization of all uncountable groups G such that, for any uncountable subgroup H of G and for any nonzero σ–finite H–invariant (or H–quasiinvariant) measure μ defined on G, there exists a subgroup of G nonmeasurable with respect to μ.

Finally, let us notice that, for every uncountable group G and for every nonzero σ–finite G–invariant (or G–quasiinvariant) measure μ defined on G, there exists a subset of G nonmeasurable with respect to μ. More generally, if G is a group, H is an uncountable subgroup of G and μ is a nonzero σ–finite H–quasiinvariant measure defined on G, then there exists a subset of G nonmeasurable with respect to μ (one can find these results in [3]; see also [2]). In addition, if G is a group and H is an arbitrary subgroup of G, then there exists a nonzero σ–finite G–invariant measure ν on G such that $H \in dom(\nu)$ (in other words, H is not an absolutely nonmeasurable subset of G). It is also known that there exists an absolutely nonmeasurable set in every uncountable commutative group. But the following question remains open: does there exist an absolutely nonmeasurable set in any uncountable group?

Remark 3. Theorems 1 and 2 have some natural analogs for the Baire property. For instance, let H be an uncountable vector subspace (over Q) of the real line R and let T be a topology on R such that

1) T is a Baire space topology;

2) the Suslin number $c(T)$ is equal to ω, i.e. T satisfies the Suslin condition (the countable chain condition);

3) the σ–algebra of all sets having the Baire property with respect to T and the σ–ideal of all first category sets with respect to T are invariant under the group H.

Then it can be proved, by the method described above (using the corresponding analogs of Lemmas 1 and 2 for the Baire property), that there exists a vector subspace of R (over Q) which does not have the Baire property with respect to the topology T.

Actually, this result is a generalization of Theorem 1. Indeed, we can immediately deduce Theorem 1 from this result if we consider a von Neumann topology $T(\mu)$ on R, associated with the completion of a nonzero σ–finite H–invariant (or H–quasiinvariant) measure μ defined on R. We recall that the von Neumann topology $T(\mu)$ is agreed with the completion of μ in the sense of the Baire property and category (see, e.g., [9], Chapter 22). This application of a von Neumann topology is typical. In many cases, if we have a result for the Baire property, we may automatically deduce from it the corresponding result for a measure. In particular, it is well known that the classical (A)–operation preserves the Baire property (in an arbitrary topological space). From this fact we immediately obtain, using a von Neumann topology, that (A)–operation also preserves the class of all measurable sets with respect to each complete σ–finite measure.

Obviously, we can formulate two problems for the Baire property, analogous to Problems 1 and 2 posed above.

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