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Lattice Polytopes Associated to Certain Demazure Modules of sl_{n+1}

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Abstract. Let w be an element of the Weyl group of sl_{n+1}. We prove that for a certain class of elements w (which includes the longest element w_0 of the Weyl group), there exist a lattice polytope $\Delta_i^w \subset \mathbb{R}^{\ell(w)}$, for each fundamental weight ω_i of sl_{n+1}, such that for any dominant weight $\lambda = \sum_{i=1}^n a_i\omega_i$, the number of lattice points in the Minkowski sum $\Delta_\lambda^w = \sum_{i=1}^n a_i \Delta_i^w$ is equal to the dimension of the Demazure module $E_w(\lambda)$. We also define a linear map $A^w : \mathbb{R}^{\ell(w)} \longrightarrow P \otimes_{\mathbb{Z}} \mathbb{R}$ where P denotes the weight lattice, such that char $E_w(\lambda) = e^\lambda \sum e^{-A^w(x)}$ where the sum runs through the lattice points x of Δ_λ^w.

Keywords: lattice polytope, Demazure module, Minkowski sum, character formula

1. Introduction

In this paper, we present some results concerning the first of a two-part programme to prove the existence of degenerations of Schubert varieties of $SL(n)$ into toric varieties (by degeneration of a Schubert variety into a toric variety, we mean a flat deformation where the generic fibre is a Schubert variety and the special fibre is a toric variety). This involves the construction of the lattice polytope which in turn, in the second part of the programme, will provide the toric variety into which the corresponding Schubert variety degenerates. In this direction, Gonciulea and Lakshmibai [10] recently proved such degenerations for Schubert varieties in an arbitrary miniscule G/P, as well as the class of Kempf varieties in the flag variety $SL(n)/B$. For an arbitrary G of rank two, this has been proved by one of the authors [4].

Let us describe our results more precisely. Fix $n \in \mathbb{N}^*$ and K an algebraically closed field of characteristic 0. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathrm{sl}_{n+1}(K)$ and $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra. Let α_i, $i = 1, \dots, n$, be the corresponding set of positive simple roots so that $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$ where $(a_{ij})_{i,j}$ is the Cartan matrix, and let ω_i be the corresponding fundamental weights. Denote by P, P^+, W, $\ell(-)$ and $\preceq$ respectively the weight lattice, the set of dominant weights, the Weyl group which is just the symmetric group of $n+1$ letters, the length function and the Bruhat order on W. Let $\lambda \in P^+$ and $w \in W$. Set V_λ to be the finite-dimensional irreducible representation of highest weight λ, $v_{w\lambda}$ to be a non-zero weight vector of weight $w\lambda$ and $E_w(\lambda)$ to be the $\mathfrak{b}$-module $U(\mathfrak{b})v_{w\lambda}$ which is called the Demazure module [5] associated to w. Set W^i to be the stabilizer of ω_i in W and W_i the quotient W/W^i. Endow W_i with the induced Bruhat order that we shall denote equally by $\preceq$ and if $\sigma \in W_i$, then we shall denote by $\ell(\sigma)$ the induced length of σ, which is the minimum of the lengths of representatives of σ.

The representation theory of a semisimple algebraic group G is closely related to the geometry of Schubert varieties (in particular G/B) since the Demazure modules can be realized as the global sections of line bundles over Schubert varieties. Degenerations of Schubert varieties into toric varieties will allow us to study the geometry of the former via toric varieties which are combinatorial.

Let $\lambda = \sum_i a_i\omega_i$ be a dominant weight, then the dimension of $E_w(\lambda)$ is a polynomial in the variables a_i of degree $\ell(w)$ because the dimension of its dual $E_w(\lambda)^*$ can be described as the Euler characteristic of the ample line bundle $\bigotimes_i \mathcal{L}_{\omega_i}^{\otimes a_i}$ over the Schubert variety associated to w in G/P_λ ([7, 18.3.6] or [2, 2.3]). Whereas, given convex lattice polytopes Δ_i in $\mathbb{R}^{\ell(w)}$, a theorem of Ehrhart [6] implies that under the condition that a lattice point in the Minkowski sum $\Delta := \sum_i a_i\Delta_i = \{\sum_i a_i v_i$ where $v_i \in \Delta_i\}$ is the sum over i of a_i lattice points of Δ_i, the number of lattice points in Δ is a polynomial of degree $\ell(w)$ in the variables a_i. On the other hand, suppose that we have a degeneration of the Schubert variety S_w equipped with line bundles $\mathcal{L}_{\omega_1}, \dots, \mathcal{L}_{\omega_n}$ into the toric variety X equipped with line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$. Then $\dim H^0(S_w, \bigotimes_i \mathcal{L}_{\omega_i}^{\otimes a_i}) = \dim H^0(X, \bigotimes_i \mathcal{L}_i^{\otimes a_i})$. But to say that X is equipped with line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ is equivalent to having n lattice polytopes $\Delta_1^w, \dots, \Delta_n^w$ in $\mathbb{R}^{\ell(w)}$ such that $\dim H^0(X, \bigotimes_i \mathcal{L}_i^{\otimes a_i})$ is the number of lattice points in the Minkowski sum $\sum_{i=1}^n a_i\Delta_i^w$ (for example, see properties B3, B4 of Section 2.3 in [19]).

These facts lead us to construct a polytope Δ_i^w for each fundamental weight ω_i and then we form the appropriate Minkowski sum.

We prove first in this paper the case where $w = w_0$, the longest element of the Weyl group W.

Theorem 1.1 *There exist lattice polytopes $\Delta_i \subset \mathbb{R}^{\ell(w_0)}$, $i = 1, \dots, n$, such that for any $\lambda = \sum_{i=1}^n a_i\omega_i \in P^+$, the number of lattice points in the Minkowski sum $\Delta_\lambda := \sum_{i=1}^n a_i\Delta_i$ is the dimension of the irreducible representation V_λ.*

Polytopes satisfying Theorem 1.1 (although there was no mention of the Minkowski sum decomposition, they do have a Minkowski sum decomposition) have been constructed using Gelfand-Tsetlin patterns in [9, 12], by Berenstein and Zelevinsky [1] and by Littelmann [16] via the combinatorics of Lakshmibai-Seshadri paths.

Our polytope Δ_λ is different and it turns out that the toric variety associated to this polytope is the same as the one constructed by Gonciulea and Lakshmibai in [10]. In fact the Minkowski sum decomposition gives a direct link between lattice points and the standard monomial basis (see [14, 17]) of the irreducible representation since we can prove that a lattice point of Δ_λ can be written as a sum over i of a_i lattice points of Δ_i.

Furthermore, since standard monomial theory exists also for Demazure modules (and for other simple algebraic groups), we believe that our construction can be generalized to any simple algebraic group G as follows.

Conjecture 1.2 *Let $w \in W$. There exist lattice polytopes $\Delta_i^w \subset \mathbb{R}^{\ell(w)}$, $i = 1, \dots, n$, such that for any $\lambda = \sum_{i=1}^n a_i \omega_i \in P^+$, the number of lattice points in the Minkowski sum $\Delta_\lambda^w := \sum_{i=1}^n a_i \Delta_i^w$ is the dimension of the Demazure module $E_w(\lambda)$.*

As a matter of fact, the polytopes Δ_i constructed in Theorem 1.1 are such that the vertices $\{v_\tau\}_{\tau \in W_i}$ are indexed by W_i. We believe that Δ_i^w of the conjecture can be chosen as the convex hull of $\{v_\tau\}_{\tau \preceq w}$ embedded (by a permutation of coordinates) in $\mathbb{R}^{\ell(w)}$.

Indeed, we prove that this is true when w can be written in a certain way (see Section 7 for details). Unfortunately, this does not cover all the elements of the Weyl group except in the case where $G = SL(2)$ or $SL(3)$. By weakening to a notion called polytopes with integral structure, one of the authors proved in [3] that one can construct a polytope with integral structure for any $w \in W$ such that the number of lattice points in the polytope is the dimension of the associated Demazure module. However, there is no Minkowski sum decomposition and these polytopes do not provide directly toric varieties.

This paper is organised as follows. In Section 2, we construct for each fundamental weight ω_i a lattice convex polytope Δ_i whose vertices are indexed by the set W_i. We shall prove later in Section 5 that Δ_i is triangulable by primitive simplices parametrized by maximal chains. We then present an example in Section 3. In Sections 4–6, we show how, in the case where $w = w_0$, a lattice point in the Minkowski sum $\sum_{i=1}^n a_i \Delta_i$ can be written as a sum over i of a_i lattice points of Δ_i, and that these points exhaust the dimension of the irreducible representation V_λ where $\lambda = \sum_{i=1}^n a_i \omega_i$. Sections 7 and 8 contain a discussion of the case of Demazure modules where we specify and prove the cases where the conjecture is true. We give another example in Section 9 and finally, in Section 10, we present applications of our results concerning combinatorial descriptions of weight multiplicities as lattice points of a polytope with rational vertices.

We shall use the above notations throughout this paper. Furthermore, let $s_1, \dots, s_n$ be the reflections associated to the positive simple roots. For any $N \in \mathbb{N}$, we shall endow $\mathbb{R}^N$ with the following partial-ordering: let $X, Y \in \mathbb{R}^N$ be such that $X \neq Y$, then

$$X < Y \text{ if and only if } Y - X \in \mathbb{R}_+^N$$

2. Construction of the polytope Δ_i for each fundamental weight w_i

Let $1 \le i \le n$ be fixed in this section. Recall that W_i can be identified with the subset of W consisting of elements w such that $ws_j \succeq w$ for all $j \neq i$. It is also well known that W_i is in bijection with the set of i-tuples $(r_1, \dots, r_i)$ such that $0 \le r_1 < r_2 < \cdots < r_i \le n$. Namely, we can think of $W = S_{n+1}$ as the group of permutations on the set $\{0, 1, \dots, n\}$. Then the bijection $w \mapsto (r_1, \dots, r_i)$ is given by $\{r_1, \dots, r_i\} = w(\{0, 1, \dots, i-1\})$.

The induced Bruhat order on W_i is then given by:

$$(r_1, \dots, r_i) \prec (s_1, \dots, s_i) \Leftrightarrow (r_1, \dots, r_i) < (s_1, \dots, s_i)$$

where on the right hand side, the i-tuples are considered as elements of $\mathbb{R}^i$.

Note that in this notation, the smallest element is $(0, 1, 2, \ldots, i-1)$ that we shall denote sometimes simply by 1 when there is no confusion, and the biggest element is $(n-i+1, n-i+2, \ldots, n)$, and that the length of the latter is $(n-i+1)i$. In fact, the minimal representative of $(r_1, \ldots, r_i)$ is

$$s_{r_1}s_{r_1-1}\cdots s_1 s_{r_2}s_{r_2-1}\cdots s_2 s_{r_3}\cdots s_{r_i}s_{r_i-1}\cdots s_i$$

where $s_{r_j}\cdots s_j = 1$ if $r_j < j$ and its length is the sum over j of $r_j - j + 1$.

We shall fix a particular reduced decomposition of w_0. Namely, we use the lexicographic minimal expression $w_0 = s_1s_2s_1s_3s_2s_1\cdots s_ns_{n-1}\cdots s_1$. Notice that each minimal representative of W_i can be written as a subexpression of this reduced decomposition.

Remark 2.1 We shall think of this as n blocks where block 1 is s_1, block 2 is $s_2s_1, \ldots,$ block n is $s_ns_{n-1}\cdots s_1$.

Let us write the standard basis vectors in $\mathbb{R}^{\ell(w_0)}$ as e_{pq} with $1 \le q \le p \le n$. Let $1 \le i \le n$, and $(r_1, \ldots, r_i)$ be an element of W_i, we then define

$$\varphi(r_1, \ldots, r_i) = \sum_{p=n-i+1}^{n} \sum_{q=p+i-n}^{r_{p+i-n}} e_{pq} \in \mathbb{R}^{\ell(w_0)}$$

Definition 2.2 Let $\mathbf{c}$: $\tau_1 \succ \cdots \succ \tau_m$ be a chain in W_i. We define $S_\mathbf{c}$ to be the convex hull of the points $\{\varphi(\tau_j)\}_{j=1}^m$ and we define Δ_i to be the convex hull of the points $\{\varphi(\tau)\}_{\tau\in W_i}$.

Lemma 2.3
(a) *The vertices of Δ_i are the only lattice points in Δ_i and they are indexed by the elements of W_i.*
(b) *The map φ is order-preserving.*
(c) *Let* $\mathbf{c}$: $\tau_1 \succ \cdots \succ \tau_{(n-i+1)i} \succ 1$ *be a maximal chain in W_i. The polytope $S_\mathbf{c}$ is a simplex of dimension $(n-i+1)i$ and its volume is $1/((n-i+1)i)!$.*

Proof: The first two assertions are direct consequences of the definition of φ_i. For part (c), notice that the points $\varphi(\tau_1), \ldots, \varphi(\tau_{(n-i+1)i})$ are linearly independent and $\varphi(1)$ is zero in $\mathbb{R}^{\ell(w_0)}$. So $S_\mathbf{c}$ is a simplex. Since $\varphi(\tau_1), \ldots, \varphi(\tau_{(n-i+1)i})$ can be obtained from the canonical basis via a matrix (with integer entries) of determinant 1 or -1, the volume of $S_\mathbf{c}$ is $1/((n-i+1)i)!$. □

We deduce from our definition the following properties between the polytopes Δ_i.

Proposition 2.4
(a) *The intersection of Δ_i and Δ_j is $\{0\}$ whenever $i \neq j$.*
(b) *Let $x = \sum_{p,q} x_{pq}e_{pq} \in \Delta_i$, then $x_{pq} = 0$ if $p < n-i+1$.*
(c) *If $x = \sum_{p,q} x_{pq}e_{pq} \in \Delta_i$ is such that $x_{st} \neq 0$, then $x_{st'} \neq 0$ for all $t' = t, t+1, \ldots, r_{s+i-n}$.*

Proof: Assertions (b) and (c) are straightforward. So let us prove (a). We can assume that $i < j$. Notice that the coefficient of e_{ni} for any non-zero element of Δ_i is non-zero while it is zero for any element of Δ_j. Thus (a) follows. □

Let $\lambda = \sum_{i=1}^{n} a_i \omega_i$ be a dominant weight (thus each $a_i \in \mathbb{N}$) and V_λ be the irreducible sl_{n+1}-module of highest weight λ.

Definition 2.5 We define the polytope Δ_λ to be the Minkowski sum $\sum_{i=1}^{n} a_i \Delta_i$.

Since the Δ_i's are lattice convex polytopes, the polytope Δ_λ is also a lattice convex polytope. We can now state our theorem in the case where $w = w_0$.

Theorem 2.6 *The number of lattice points in Δ_λ is equal to the dimension of V_λ.*

3. Example

The first interesting example is sl_4. We write $w_0 = s_1 s_2 s_1 s_3 s_2 s_1$ and we have, in terms of minimal representatives,

$$W_1 = \{1, s_1, s_2 s_1, s_3 s_2 s_1\}, \quad W_2 = \{1, s_2, s_3 s_2, s_1 s_2, s_1 s_3 s_2, s_2 s_1 s_3 s_2\}$$
$$W_3 = \{1, s_3, s_2 s_3, s_1 s_2 s_3\}$$

We then obtain via φ the following table where each row contains the coefficients of a $\varphi(\tau)$:

	s_1	s_2	s_1	s_3	s_2	s_1	
	e_{11}	e_{22}	e_{21}	e_{33}	e_{32}	e_{31}	
s_1	0	0	0	0	0	1	(1)
$s_2 s_1$	0	0	0	0	1	1	(2)
$s_3 s_2 s_1$	0	0	0	1	1	1	(3)
s_2	0	0	0	0	1	0	(0, 2)
$s_3 s_2$	0	0	0	1	1	0	(0, 3)
$s_1 s_2$	0	0	1	0	1	0	(1, 2)
$s_1 s_3 s_2$	0	0	1	1	1	0	(1, 3)
$s_2 s_1 s_3 s_2$	0	1	1	1	1	0	(2, 3)
s_3	0	0	0	1	0	0	(0, 1, 3)
$s_2 s_3$	0	1	0	1	0	0	(0, 2, 3)
$s_1 s_2 s_3$	1	1	0	1	0	0	(1, 2, 3)

The images of (0), (0, 1), (0, 1, 2) are all (0, 0, 0, 0, 0, 0).

Let us now consider the adjoint representation. The highest weight is $\omega_1 + \omega_3$. One then verifies easily by hand that a lattice point of $\Delta_1 + \Delta_3$ is the sum of a lattice point of Δ_1 and a lattice point of Δ_3. Hence a quick computation shows that the lattice points are the ones in Δ_1 and Δ_3 together with 8 other points:

$$\begin{gathered} e_{31}+e_{33}, \quad e_{31}+e_{33}+e_{22}, \quad e_{31}+e_{33}+e_{22}+e_{11} \\ e_{32}+e_{31}+e_{33}+e_{22}, \quad e_{32}+e_{31}+e_{33}+e_{22}+e_{11} \\ e_{31}+e_{32}+2e_{33}, \quad e_{31}+e_{32}+2e_{33}+e_{22}, \quad e_{31}+e_{32}+2e_{33}+e_{22}+e_{11} \end{gathered}$$

Thus there are 15 lattice points in $\Delta_1+\Delta_3$ which is the dimension of sl_4.

Remark that $\varphi(2)+\varphi(0,1,3)=\varphi(3)+\varphi(0,1,2)$ is the only sum repeated here. This can be seen to correspond to the tensor product decomposition

$$V_{\omega_1}\otimes V_{\omega_3}\cong V_{\omega_1}\otimes\left(V_{\omega_1}\right)^*\cong \mathrm{gl}_4=\mathrm{sl}_4\oplus V_0$$

4. Correspondence with semi-standard Young tableaux

Let $\lambda=\sum_{i=1}^n a_i\omega_i$ be a dominant weight. Set $\mathcal{W}$ be the disjoint union of the W_i and $\mathcal{W}(\lambda)=\prod_{i=1}^n\prod_{j=1}^{a_i}W_i$. We can associate to an element of $\mathcal{W}(\lambda)$ via φ a lattice point of Δ_λ. Namely, an element $(w_{ij})_{i,j}$ of $\mathcal{W}(\lambda)$ is sent to $\sum_{i,j}\varphi(w_{ij})$ in Δ_λ.

However this association is not necessarily injective (that is, a lattice point can be the image of another element in $\mathcal{W}(\lambda)$). We claim that with respect to a certain partial ordering of $\mathcal{W}$, there is a unique such element which is decreasing. At the end of this section, we shall show that the set of lattice points corresponding to the elements in $\mathcal{W}(\lambda)$ is in bijection with the set of semi-standard Young tableaux of type λ.

Let us first define our partial order in $\mathcal{W}$, denoted by $\prec$, which extends the induced Bruhat ordering in W_i. Let $(r_1,\ldots,r_i)$ and $(s_1,\ldots,s_j)$ be two elements of $\mathcal{W}$, then

$$(r_1,\ldots,r_i)\prec(s_1,\ldots,s_j)\Leftrightarrow(\underbrace{-1,\ldots,-1}_{n-i},r_1,\ldots,r_i)<(\underbrace{-1,\ldots,-1}_{n-j},s_1,\ldots,s_j)$$

where the elements on the right hand side are in $\mathbb{R}^n$.

Remark 4.1 Using the notations above, if we have $r\prec s$ then $i\le j$. Furthermore, there is a unique maximal element $(1,2,\ldots,n)$ and a unique minimal element (0).

Lemma 4.2

(a) *The set $\mathcal{W}$ is a lattice, that is, every pair of elements of $\mathcal{W}$ have a well defined* max *and* min.

(b) *If $r\in W_i$, $s\in W_j$ and $i\le j$, then we have* $\min(r,s)\in W_i$ *and* $\max(r,s)\in W_j$.

(c) (MAX–MIN) *Let $r,s\in\mathcal{W}$, then* $\varphi(r)+\varphi(s)=\varphi(\max(r,s))+\varphi(\min(r,s))$.

Proof: Let $r=(r_1,\ldots,r_i)$ be an element of $\mathcal{W}$. By adding -1's on the left as above, we can associate to r, an element $R=(R_1,\ldots,R_n)$ of $\mathbb{R}^n$.

Definition 4.3 Let r, s be elements of $\mathcal{W}$ and R, S the corresponding associated elements in $\mathbb{R}^n$. We define $\min(R, S) = (T_1, \dots, T_n)$ where $T_i = \min(R_i, S_i)$ and $\min(r, s)$ the element of $\mathcal{W}$ associated to $\min(R, S)$ by taking away all the -1's.

We define $\max(r, s)$ similarly.

One verifies easily assertions (a) and (b) from this definition. It suffices therefore to check max–min.

Let $r = (r_1, \dots, r_i)$ and $s = (s_1, \dots, s_j)$ where $i \leq j$. Then

$$\begin{aligned}
\varphi(r) + \varphi(s) &= \sum_{p=n-i+1}^{n} \sum_{q=p+i-n}^{r_{p+i-n}} e_{pq} + \sum_{p=n-j+1}^{n} \sum_{q=p+j-n}^{s_{p+j-n}} e_{pq} \\
&= \sum_{p=n-i+1}^{n} \sum_{q=p+i-n}^{r_{p+i-n}} e_{pq} + \sum_{p=n-j+1}^{n-i} \sum_{q=p+j-n}^{s_{p+j-n}} e_{pq} + \sum_{p=n-i+1}^{n} \sum_{q=p+j-n}^{s_{p+j-n}} e_{pq} \\
&= \sum_{p=n-i+1}^{n} \left(\sum_{q=p+i-n}^{r_{p+i-n}} e_{pq} + \sum_{q=p+j-n}^{s_{p+j-n}} e_{pq} \right) + \sum_{p=n-j+1}^{n-i} \sum_{q=p+j-n}^{s_{p+j-n}} e_{pq} \\
&= \sum_{p=n-i+1}^{n} \sum_{q=p+i-n}^{\min(r_{p+i-n}, s_{p+j-n})} e_{pq} + \sum_{p=n-j+1}^{n} \sum_{q=p+j-n}^{\max(r_{p+i-n}, s_{p+j-n})} e_{pq} \\
&= \varphi(\min(r, s)) + \varphi(\max(r, s)) \qquad \square
\end{aligned}$$

We shall now state and prove our claim.

Proposition 4.4 *Let $\theta = \{\theta_{ij}\}_{i=1,\dots,n;\ j=1,\dots,a_i}$ be an element of $\mathcal{W}(\lambda)$. Then there exists a unique element $\psi = \{\psi_{ij}\}$ of $\mathcal{W}(\lambda)$ such that*
(i) *$\psi_{ij} \preceq \psi_{k\ell}$ if $i < k$ or if $i = k$ and $j \leq \ell$;*
(ii) *$\sum_{i,j} \varphi(\theta_{ij}) = \sum_{i,j} \varphi(\psi_{ij})$.*

Before proving this proposition, let us remark that condition (i) says that

$$\psi_{11} \preceq \psi_{12} \preceq \cdots \preceq \psi_{1a_1} \preceq \psi_{21} \preceq \cdots \preceq \psi_{2a_2} \preceq \cdots \preceq \psi_{(n-1)a_{n-1}} \preceq \psi_{n1} \preceq \cdots \preceq \psi_{na_n}$$

This is similar to the definition for a Young tableaux of Lakshmibai and Seshadri of type λ modulo liftings to the Weyl group W, see [14]. As we shall see, our theorem says that this is exactly the same definition.

Proof: We shall prove the existence by induction on $a = \sum_{i=1}^{n} a_i$. It is clear that the induction hypothesis holds for $a = 1$. (In fact, by max–min of Lemma 4.2, it holds equally for $a = 2$).

Let us now suppose that the induction hypothesis holds for $a - 1$. Let r be maximal such that $a_r \neq 0$. By the induction hypothesis, we can suppose that $\theta' = \theta \setminus \{\theta_{ra_r}\}$ satisfies the conditions (i) and (ii) of the proposition.

We shall now divide θ' into three disjoint totally-ordered sets. Let

$$E_{\prec} = \{\theta_{ij} \mid \theta_{ij} \prec \theta_{ra_r}\}, \quad E_{\succeq} = \{\theta_{ij} \mid \theta_{ij} \succeq \theta_{ra_r}\}$$

and

$$E_0 = \{\theta_{ij} \mid \theta_{ij} \text{ and } \theta_{ra_r} \text{ are not comparable}\}.$$

Note that the elements of $E_{\succeq}$ are all in W_r.

If E_0 is empty, then we can insert θ_{ra_r} in the sequence to obtain a totally-ordered sequence and hence by rearranging the subscripts, we obtain an element of $\mathcal{W}(\lambda)$ satisfying the required conditions.

Suppose now that E_0 is not empty. Then θ_{ra_r} is in neither $E_{\prec}$, $E_{\succeq}$ nor E_0. Let ϕ be the maximal element in E_0. By max–min of Lemma 4.2, replacing ϕ and θ_{ra_r} by $\max(\phi, \theta_{ra_r})$ and $\min(\phi, \theta_{ra_r})$ does not alter the sum via φ. Furthermore, if we let $E'_{\prec}$, $E'_{\succeq}$ and E'_0 be the new partition as defined above relative to $\theta'_{ra_r} = \max(\phi, \theta_{ra_r})$, then the cardinal of E'_0 is strictly less than E_0 since $\min(\phi, \theta_{ra_r})$ will belong to $E'_{\prec}$.

Now repeat the same procedure until E_0 is empty and we have the existence since E_0 is a finite set.

Let us turn to the uniqueness which is a consequence of the following lemma.

Lemma 4.5 *Let r and s be two distinct elements of W_i. Then there exist k, m_k such that one of the following conditions is satisfied*:

(i) *the e_{km_k}-coordinate is* 1 *for $\varphi(r)$ and the $e_{k\ell}$-coordinate for $\varphi(s)$ is* 0 *for all $m_k \le \ell \le k$.*

(ii) *the e_{km_k}-coordinate is* 1 *for $\varphi(s)$ and the $e_{k\ell}$-coordinate for $\varphi(r)$ is* 0 *for all $m_k \le \ell \le k$.*

Furthermore let us suppose that (i) *is satisfied* (*we have obviously the same statement with the roles of r and s exchanged when* (ii) *is satisfied*). *Then we can choose k and m_k such that for all $t \in W_j$ satisfying $t \preceq s$, the e_{km_k}-coordinate is* 1 *for $\varphi(r)$ and the $e_{k\ell}$-coordinate for $\varphi(t)$ is* 0 *for all $m_k \le \ell \le k$.*

Proof: Let us denote $r = (r_1, \ldots, r_i)$ and $s = (s_1, \ldots, s_i)$. Since r and s are distinct, there exists k such that either $r_k > s_k$ or $s_k > r_k$. Suppose we have $r_k > s_k$ (resp. $s_k > r_k$). Since $r_k > s_k \ge k-1$ (resp. $s_k > r_k \ge k-1$), r (resp. s) has non-zero entries in the $(n-i+k)^{\text{th}}$ block. By the definition of our embedding, it is clear that if we put $m_k = r_k$ (resp. $m_k = s_k$), then the conditions of (i) (resp. (ii)) are satisfied.

To prove the last statement, let us suppose that (i) is satisfied. Then, there exists k such that $r_k > s_k$. Now let $t \in W_j$ be such that $t \preceq s$. By Remark 4.1, we must have $i \ge j$ and hence we can write $t = (t_1, \ldots, t_i)$ by adding -1's on the left. Since $t \preceq s$, we have $t_k \le s_k < r_k$. It follows again from our embedding that we have our result by letting $m_k = r_k$. □

We can now finish our proof. Let θ and $\theta' = \{\theta'_{ij}\}$ be two elements in $\mathcal{W}(\lambda)$ satisfying the conditions of the proposition. Let r be maximal such that $a_r \ne 0$. Then θ_{ra_r} and θ'_{ra_r} are maximal in θ and θ' respectively. If $\theta_{ra_r} \ne \theta'_{ra_r}$, then by applying the previous lemma, we can suppose that there exists k, m_k such that the entry e_{km_k} is 1 for $\varphi(\theta_{ra_r})$ and the entries $e_{k\ell}$ for $\varphi(\theta'_{ra_r})$ is 0 for all $m_k \le \ell \le k$. Hence by the same lemma, the same entries for $\varphi(\theta'_{ij})$

are 0 for all i, j since θ'_{ra_r} is maximal in θ'. It follows that $\sum_{i,j} \varphi(\theta_{ij}) - \sum_{i,j} \varphi(\theta'_{ij}) \neq 0$ which contradicts the fact that θ and θ' satisfy the second condition of the proposition.

Thus $\theta_{ra_r} = \theta'_{ra_r}$. Now by induction on a, the sum of the a_i's, the elements θ, θ' must be the same (the case $a = 1$ is equivalent to the fact that φ is an embedding). □

Thus we have proved what we claimed at the start of this section. Let us denote by $\mathcal{W}(\lambda)_d$ the set of elements in $\mathcal{W}(\lambda)$ satisfying property (i) of the proposition. Now given an element θ in $\mathcal{W}(\lambda)_d$, using the notations $(r_1, \ldots, r_i)$ for elements in W_i, we can arrange each θ_{ij} as a row of numbers flushright, and stack them in order with the largest row on top, the smallest row on the bottom, what we obtain then is a semi-standard Young tableau of type λ. For example, the sequence (1), (0, 1), (0, 2), (0, 2, 3) corresponds to the semi-standard Young tableau

0	2	3
	0	2
	0	1
		1

By the uniqueness proved in the proposition, we obtain a well-defined map from $\mathcal{W}(\lambda)_d$ to the set of semi-standard Young tableaux of type λ, which is obviously injective. On the other hand, given a semi-standard Young tableau of type λ, we obtain an element of $\mathcal{W}(\lambda)_d$ by reading off the rows. It is clear that this is the inverse of the former map. Now by Lemma 2.3, lattice points in Δ_i are in bijection with elements of W_i, thus Proposition 4.4 says that $\mathcal{W}(\lambda)_d$ is in bijection with the set of lattice points in Δ_λ which can be written as a sum over i of a_i lattice points of Δ_i, we can hence state

Theorem 4.6 *The set of lattice points of Δ_λ which can be written as a sum over i of a_i lattice points of Δ_i is in bijection with the set of semi-standard Young tableaux of type λ.*

Remark 4.7 In fact, the existence part of Proposition 4.4 can be proved with semi-standard Young tableaux since it involves only max–min of Lemma 4.2 and not the explicit embedding. The idea is to put the maximal entry of each column at the top row and then use induction which is roughly what we have done.

5. Characterization of lattice points in Δ_λ

Let $\lambda = \sum_{i=1}^{n} a_i \omega_i$ be a dominant weight. Recall from Definition 3.2 that Δ_λ is the Minkowski sum $\sum_{i=1}^{n} a_i \Delta_i$, where $\Delta_1, \ldots, \Delta_n$ are the polytopes associated to the fundamental weights $\omega_1, \ldots, \omega_n$ which were defined in Section 2. In this section we shall prove the following theorem:

Theorem 5.1 *A lattice point in the Minkowski sum $\sum_{i=1}^{n} a_i \Delta_i$ can be written as the sum of a_1 lattice points in Δ_1, a_2 lattice points of Δ_2 and so on.*

As in the previous section, denote by $\mathcal{W}$ the union over all i of W_i equipped with the partial order defined in the same section, and for any dominant weight μ, denote by $\mathcal{W}(\mu)_d$ the set of elements in $\mathcal{W}(\mu)$ satisfying property (i) of Proposition 4.4. Let $\theta = \{\theta_{ij}\}_{i,j}$ be an element of $\mathcal{W}(\mu)_d$, we shall denote by $C_\mu(\theta)$ the convex cone generated by $\{\varphi(\theta_{ij})\}_{i,j}$.

Theorem 5.2 *Let $x \in \Delta_\lambda$, then there exist a dominant weight μ and a $\theta \in \mathcal{W}(\mu)_d$ such that $x \in C_\mu(\theta)$.*

This theorem is a direct consequence of the following technical lemma.

Lemma 5.3 *Let $\{\sigma_{ij}\}_{i=1,\dots,n;\ j=1,\dots,a_i}$ be a sequence of elements of $\mathcal{W}$ such that $\sigma_{ij} \in W_i$. Let $p_{ij} \in \mathbb{R}_+$. Then there exists $\{\sigma'_{ij}\}_{i=1,\dots,n;\ j=1,\dots,a'_i}$ a sequence of elements of $\mathcal{W}$ and $p'_{ij} \in \mathbb{R}_+$ such that*

(i) $\sigma'_{ij} \in W_i$.

(ii) $\sigma'_{ij} \prec \sigma'_{kl}$ *if* $i < k$ *or if* $i = k$ *and* $j < l$.

(iii) $\sum_{j=1}^{a_i} p_{ij} = \sum_{j=1}^{a'_i} p'_{ij}$.

(iv) $\sum_{i=1}^{n}\sum_{j=1}^{a_i} p_{ij}\varphi(\sigma_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{a'_i} p'_{ij}\varphi(\sigma'_{ij})$.

Proof: We shall prove the lemma by induction on the $\sum_{i=1}^n a_i$.

The assertion is obvious when the sum is 1. So let us suppose that the sum is strictly bigger than one. Let l be maximal such that $a_l > 0$. By the induction hypothesis, we can assume that $\{\sigma_{ij}\}_{i=1,\dots,l;\ j=1,\dots,a_i} \backslash \{\sigma_{la_l}\}$ satisfies (i) and (ii) of the lemma. For simplicity we shall denote σ_{la_l} by σ and $q = p_{la_l}$.

If $\sigma \succeq \sigma_{ij}$ for all $i = 1, \dots, l$, $j = 1, \dots, a_i$ or if $\sigma = \sigma_{lj}$ for some $j \le a_l - 1$, then we are done.

So let us suppose the contrary. Then there exists $\tau = \sigma_{cd}$ minimal such that $\sigma \not\succ \tau$. Let $\kappa = \sigma_{rs}$ be maximal such that $P := \sum_{\tau \preceq \sigma_{ij} \prec \kappa} p_{ij} \le q$. Denote by $m_{ij} = \min(\sigma, \sigma_{ij}) \in W_i$ and by $M_l^{ij} = \max(\sigma, \sigma_{ij}) \in W_l$. Note that we have

$$M_l^{ij} \succeq M_l^{i,j-1} \succeq \cdots \succeq M_l^{cd} \succ \sigma \succ m_{ij} \succeq \cdots \succeq m_{cd}$$

Now using repeatedly max–min of Lemma 4.2, we obtain:

$$\begin{aligned}\sum_{i=1}^{l}\sum_{j=1}^{a_i} p_{ij}\varphi(\sigma_{ij}) &= \sum_{\sigma_{ij}\prec\tau} p_{ij}\varphi(\sigma_{ij}) + \sum_{\sigma_{ij}\succeq\kappa} p_{ij}\varphi(\sigma_{ij})\\ &\quad + \sum_{\tau\preceq\sigma_{ij}\prec\kappa} p_{ij}(\varphi(\sigma_{ij}) + \varphi(\sigma)) + (q-P)\varphi(\sigma)\\ &= \sum_{\sigma_{ij}\prec\tau} p_{ij}\varphi(\sigma_{ij}) + \sum_{\sigma_{ij}\succ\kappa} p_{ij}\varphi(\sigma_{ij})\\ &\quad + \sum_{\tau\preceq\sigma_{ij}\prec\kappa} p_{ij}\big(\varphi(m_{ij}) + \varphi\big(M_l^{ij}\big)\big) + p_{rs}\varphi(\kappa) + (q-P)\varphi(\sigma)\end{aligned}$$

Now if $p_{rs} \leq q - P$, then we must have $\kappa = \sigma_{l,a_l-1}$. Consequently, we have

$$\sum_{i=1}^{l}\sum_{j=1}^{a_i} p_{ij}\varphi(\sigma_{ij}) = \sum_{\sigma_{ij}\prec\tau} p_{ij}\varphi(\sigma_{ij}) + \sum_{\tau\preceq\sigma_{ij}\preceq\kappa} p_{ij}\big(\varphi(m_{ij}) + \varphi\big(M_l^{ij}\big)\big) + (q - P - p_{rs})\varphi(\sigma)$$

Thus we obtain a chain

$$M_l^{rs} \succeq \cdots \succeq M_l^{cd} \succ \sigma \succ m_{rs} \succeq m_{r,s-1} \succeq \cdots \succeq m_{c,d} \succ \sigma_{c,d-1} \succ \cdots \succ \sigma_{11} \succ 1$$

from which we can compress into a chain $\{\sigma'_{ij}\}$ where $i = 1, \ldots, l$ and $j = 1, \ldots, a'_i$ satisfying the required properties of the lemma.

If $p_{rs} \geq q - P$, then:

$$\begin{aligned}\sum_{i=1}^{l}\sum_{j=1}^{a_i} p_{ij}\varphi(\sigma_{ij}) &= \sum_{\sigma_{ij}\prec\tau} p_{ij}\varphi(\sigma_{ij}) + \sum_{\sigma_{ij}\succ\kappa} p_{ij}\varphi(\sigma_{ij})\\ &\quad + \sum_{\tau\preceq\sigma_{ij}\prec\kappa} p_{ij}\big(\varphi(m_{ij}) + \varphi\big(M_l^{ij}\big)\big)\\ &\quad + (q-P)(\varphi(\kappa) + \varphi(\sigma)) + (p_{rs} - (q-P))\varphi(\kappa)\\ &= \sum_{\sigma_{ij}\prec\tau} p_{ij}\varphi(\sigma_{ij}) + \sum_{\sigma_{ij}\succ\kappa} p_{ij}\varphi(\sigma_{ij})\\ &\quad + \sum_{\tau\preceq\sigma_{ij}\prec\kappa} p_{ij}\big(\varphi(m_{ij}) + \varphi\big(M_l^{ij}\big)\big)\\ &\quad + (q-P)\big(\varphi(m_{rs}) + \varphi\big(M_l^{rs}\big)\big) + (p_{rs} - (q-P))\varphi(\kappa)\end{aligned}$$

Thus we obtain a chain

$$\sigma_{l,a_l-1} \succ \cdots \succ \kappa \succ m_{rs} \succeq m_{r,s-1} \succeq \cdots \succeq m_{cd} \succ \sigma_{c,d-1} \cdots \succ \sigma_{11} \succ 1$$

from which we can compress into a chain $\{\sigma'_{ij}\}$ where $i = 1, \ldots, l$ and $j = 1, \ldots, a'_i$ satisfying (i), (ii) and (iii) of the lemma (look at the coefficents). Therefore we have

$$\sum_{i=1}^{l}\sum_{j=1}^{a_i} p_{ij}\varphi(\sigma_{ij}) = \sum_{i=1}^{l}\sum_{j=1}^{a'_i} p'_{ij}\varphi(\sigma'_{ij}) + \sum_{\tau\preceq\sigma_{ij}\prec\kappa} p_{ij}\varphi\big(M_l^{ij}\big) + (q-P)\varphi\big(M_i^{rs}\big)$$

We now observe that the length of the remaining elements $M_l^{cd}, \ldots, M_l^{rs}$ are strictly greater than that of σ. Thus we can repeat the same reasoning and the lemma is proved because there is a maximal element in W_l. □

Corollary 5.4 *The polytope Δ_i is triangulable by primitive simplices of dimension $(n - i + 1)i$.*

Proof: Recall from [11] that a simplex is called primitive of dimension d if its vertices are lattice points and its volume is $1/d!$.

It is clear from the proof of the preceding lemma applied to the sequence $\{\sigma_{ij}\}_{j=1,\dots,a_i}$ that Δ_i is the union of all the $S_{\mathbf{c}}$ where $\mathbf{c}$ is a chain in W_i (see Definition 2.2). Moreover, if $\mathbf{c}'$ is a (strict) subchain of $\mathbf{c}$, then $S_{\mathbf{c}'}$ lies in the boundary of $S_{\mathbf{c}}$. Since by Lemma 2.3, $S_{\mathbf{c}}$ is a primitive simplex of dimension $(n-i+1)i$ when $\mathbf{c}$ is a maximal chain, to show that Δ_i is triangulable by primitve simplices, it suffices to show that the interior of any two distinct simplices $S_{\mathbf{c}}$ and $S_{\mathbf{c}'}$ do not meet.

Consider two chains $\mathbf{c}$: $\sigma_1 \succ \cdots \succ \sigma_\ell \succ 1$ and $\mathbf{c}'$: $\tau_1 \succ \cdots \succ \tau_m \succ 1$. Suppose that the intersection of the interiors of $S_{\mathbf{c}}$ and $S_{\mathbf{c}'}$ is not empty and that Q belongs to this intersection. We can therefore write Q as (recall that $\varphi(1) = (0, \dots, 0) \in \mathbb{R}^{\ell(w_0)}$)

$$\sum_{j=1}^{\ell} p_j \varphi(\sigma_j) = Q = \sum_{k=1}^{m} q_k \varphi(\tau_k) \quad (*)$$

where $p_j, q_k \in]0, 1[$ and $p_1 + \cdots + p_\ell \leq 1$, $q_1 + \cdots + q_m \leq 1$.

Assume that $\sigma_1 \neq \tau_1$. Writing $\sigma_1 = (s_1, \dots, s_i)$ and $\tau_1 = (t_1, \dots, t_i)$. By Lemma 4.5, there exists a coordinate e_{pq} which is non-zero on the left hand side of $(*)$, whereas it is zero on the right hand side (because τ_1 is maximal in the chain $\mathbf{c}'$). So we have a contradiction and therefore $\sigma_1 = \tau_1$.

Without loss of generality, we can suppose $p_1 \geq q_1$. We can then rewrite $(*)$ as follows:

$$(p_1 - q_1)\varphi(\sigma_1) + \sum_{j=2}^{\ell} p_j \varphi(\sigma_j) = Q' = \sum_{k=2}^{m} q_k \varphi(\tau_k) \quad (**)$$

Consequently, we must have $p_1 = q_1$. Now by repeating the same argument (or use induction on $\ell + m$) on $(**)$, we conclude that $\ell = m$, $p_j = q_j$ and $\sigma_j = \tau_j$ for all $j = 1, \dots, \ell$. That is $\mathbf{c} = \mathbf{c}'$. Thus the corollary is proved. □

Proof of Theorem 5.1: Suppose that x is a lattice point of $\sum_{i=1}^{l} a_i \Delta_i$. Without loss of generality we can assume that $a_l \neq 0$. We can write $x = x_1 + \cdots + x_l$ where

$$x_i = p_{i1}\varphi(\sigma_{i1}) + \cdots + p_{ir_i}\varphi\big(\sigma_{ir_i}\big) \quad \text{with } p_{ij} > 0 \text{ and } \sum_{j=1}^{r_i} p_{ij} = a_i$$

where $\sigma_{ij} \in W_i$. By Theorem 5.2, we can assume that σ_{ij} is a strictly increasing sequence of elements of $\mathcal{W}$.

If $r_l = 1$, then $p_{lr_l} = a_l$ and so $x_l = a_l \varphi(\sigma_{lr_l})$ which implies that $x - x_l$ is a lattice point of $\sum_{i=1}^{l-1} a_i \Delta_i$.

If $r_l > 1$, then by Lemma 4.5, there exists a coordinate $e_{\alpha\beta}$ such that the $e_{\alpha\beta}$ coordinate of x is equal to p_{lr_l}. So p_{lr_l} is a positive integer and $x - p_{lr_l}\varphi(\sigma_{lr_l})$ is a lattice point of $\sum_{i=1}^{l-1} a_i \Delta_i + (a_l - p_{lr_l})\Delta_l$.

Thus, in both cases the assertion follows by induction on $\sum_{i=1}^{l} a_i$. □

6. End of proof of Theorem 2.6

By Theorem 5.1, an integral point in Δ_λ is a sum over i of a_i lattice points of Δ_i. Hence by Theorem 4.6, the set of lattice points of Δ_λ is in bijection with the set of semi-standard Young tableaux of type λ. Now by a classical result from the theory of invariants (see for example [8] or [18]) that the number of semi-standard Young tableaux of type λ is exactly the dimension of the sl_{n+1}-module V_λ. Thus Theorem 2.6 is proved.

7. The case of Demazure modules

In the previous sections, we explained how to construct for each fundamental weight a polytope Δ_i whose vertices are indexed by the set W_i. Let W_i^w be the set $\{\sigma \in W_i \mid \sigma \preceq \bar{w}\}$ where $\bar{w}$ is the class of w in W_i. Then we can define the polytope Δ_i^w to be the convex hull of the set of vertices of Δ_i corresponding to the set W_i^w. It is clear that the vertices of Δ_i^w are indexed by the set W_i^w.

We would like to embed Δ_i^w in $\mathbb{R}^{\ell(w)}$ in such a way that given a dominant weight $\lambda = \sum_i c_i \omega_i$, the number of lattice points in $\sum_i c_i \Delta_i^w$ is the dimension of the Demazure module $E_w(\lambda)$. For some w we can construct such an embedding. In this section we shall describe this embedding and explain why it works.

Recall that W is considered as the permutations of the set $\{0, 1, \ldots, n\}$, with simple transpositions $s_i = (i-1, i)$. Consider the unique factorization of a permutation $w \in W$ in the form

$$w = s(1, c_1)s(2, c_2) \cdots s(n, c_n)$$

where we denote $s(a, b) = s_a s_{a-1} \cdots s_b$ and $s(a, a+1) = 1$. Then c_j is the cardinal of the set $\{d$ such that $d \leq w^{-1}(j),\ w(d) \leq j\}$. It follows that there exist $k \in \mathbb{N}^*$, $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ and $1 \leq b_j \leq a_j$ for all $j = 1, \ldots, k$ such that

$$w = s(a_1, b_1)s(a_2, b_2) \cdots s(a_k, b_k)$$

Note that we have $w_0 = s(1, 1)s(2, 1) \cdots s(n, 1)$. We shall use this notation in this section.

Definition 7.1 Let $d \leq e$ be positive integers. We shall call the subexpression $s(a_d, b_d) \cdots s(a_e, b_e)$ of w, a part if the following conditions are satisfied:

(i) $a_e + 1 < b_m$ for all $m > e$,
(ii) $a_{d-1} + 1 < b_m$ for all $d \leq m \leq e$.

A part of w is connected if it is not the product of two distinct parts of w.

It is clear that w is the product of connected parts, say $w = \mathcal{P}_1 \cdots \mathcal{P}_l$, and that when $1 \leq i \neq j \leq l$, then $\mathcal{P}_i$ commutes with $\mathcal{P}_j$.

Theorem 7.2 *Suppose that $w \in W$ is either the identity or else each connected part $s(a_d, b_d) \cdots s(a_e, b_e)$ of w satisfies one of the following conditions*:

(i) $b_d \geq b_{d+1} \geq \cdots \geq b_e$.
(ii) $a_d = b_d < a_{d+1} = b_{d+1} < \cdots < a_e = b_e$,

Then for each i, there exists an embedding φ_i^w of Δ_i^w in $\mathbb{R}^{l(w)}$ such that for any dominant weight $\lambda = \sum_i c_i\omega_i$, we have $\mathrm{Card}(\Delta_\lambda^w \cap \mathbb{Z}^{\ell(w)}) = \dim E_w(\lambda)$ *where* $\Delta_\lambda^w = \sum_i c_i\varphi_i^w(\Delta_i^w)$.

Remark 7.3

(1) As remarked by one of the referees, there is no obvious relation between the set of Kempf elements (for the definition of Kempf elements, see [13]) and the set of w whose connected parts satisfy condition (i) or (ii) of the theorem. Note that in the case of sl_3, all the elements of the Weyl group satisfy the conditions of the theorem, while s_2s_1 is not a Kempf element.
(2) In the case of sl_4, there are exactly 7 elements in W which satisfy neither of the 2 conditions. Namely they are $s(1,1)s(3,2)$, $s(2,1)s(3,3)$, $s(2,1)s(3,2)$, $s(1,1)s(2,2)s(3,2)$, $s(1,1)s(2,1)s(3,3)$, $s(1,1)s(2,1)s(3,2)$ and $s(1,1)s(2,2)s(3,1)$. However, a case by case analysis shows that, by using the same construction, the theorem is true in these cases.
(3) Let λ be a dominant weight. If $w' \equiv w$ modulo W_λ, the stabiliser of λ, then $E_{w'}(\lambda) = E_w(\lambda)$. Therefore, if we can find a w satisfying the conditions of Theorem 7.2, then the number of lattice points in $m\Delta_\lambda^w$ is equal to the dimension of $E_{w'}(m\lambda)$ for any $m \in \mathbb{N}$. In particular, such elements can always be found in the case of fundamental weights (that is, when the stabiliser is W_i for some i).

Let us fix $w = s(a_1, b_1)s(a_2, b_2)\cdots s(a_k, b_k)$.

Let us make the idea behind our embedding more precise. Since Δ_i^w is the convex hull of its vertices and that the vertices are in one-to-one correspondence with the elements of W_i^w, we simply need to specify the image of the vertex corresponding to an element $\sigma \in W_i^w$, denoted by $\varphi_i^w(\sigma)$. We have

$$\sigma = (r_1, \ldots, r_i) = s(r_1, n-i+1)s(r_2, n-i+2)\cdots s(r_i, i).$$

The following description of $\varphi_i^w(\sigma)$ may seem vague, but with the example that follows it will become more transparent. We shall index the standard basis of $\mathbb{R}^{\ell(w)}$ using the expression of w, that is, we write the standard basis as e_{pq} where $p = 1, \ldots, k$ and $q = b_p, \ldots, a_p$. Consider the rightmost subexpression of w identical to the above expression of σ. Then we define the coefficient of e_{pq} of $\varphi_i^w(\sigma)$ to be 1 if the index belongs to this subexpression, and zero otherwise.

Let us clarify all this with an example. Let $w = s_2s_1s_3s_2s_1 = s(2,1)s(3,1)$ be an element of the Weyl group of sl_4. It satisfies condition (i) of Theorem 7.2. We have

$$W_1^w = \{1, s_1, s_2s_1, s_3s_2s_1\}, \quad W_2^w = \{1, s_2, s_3s_2, s_1s_2, s_1s_3s_2, s_2s_1s_3s_2\},$$
$$W_3^w = \{1, s_3, s_2s_3\}$$

According to the discussion above, $\varphi_i^w(1)$ is the zero vector for any i. To specify $\varphi_1^w(s_1)$, we “embed” s_1 as right as possible in the expression $s_2s_1s_3s_2s_1$, so we get $\varphi_1^w(s_1) =$

$(0, 0, 0, 0, 1)$. Similarly, we "embed" s_2s_1 as right as possible in $s_2s_1s_3s_2s_1$, and we get $\varphi_1^w(s_2s_1) = (0, 0, 0, 1, 1)$ and so forth. Hence, we obtain the following table.

	s_2	s_1	s_3	s_2	s_1
s_1	0	0	0	0	1
s_2s_1	0	0	0	1	1
$s_3s_2s_1$	0	0	1	1	1
s_2	0	0	0	1	0
s_3s_2	0	0	1	1	0
s_1s_2	0	1	0	1	0
$s_1s_3s_2$	0	1	1	1	0
$s_2s_1s_3s_2$	1	1	1	1	0
s_3	0	0	1	0	0
s_2s_3	1	0	1	0	0

Although it is easy to describe the image of σ in this way, this description needs to be formalised so that we can prove that it works.

Definition 7.4 Let $w = s(a_1, b_1) \cdots s(a_k, b_k)$. Define $\mathbf{p}_i = (p_1^i, \dots, p_i^i)$ by reverse induction (i.e., starting from i and going down to 1). Set $p_{i+1}^i = +\infty$, then

$$p_j^i = \begin{cases} \max\left\{l \mid b_l \le j \le a_l,\ l < p_{j+1}^i\right\} & \text{if it exists} \\ -1 & \text{otherwise} \end{cases}$$

In other words, if we write $u_j = s(a_j, b_j)$, then p_i^i is the biggest integer l such that s_i occurs in u_l (or in $u_l u_{l+1} \cdots u_k$). And p_{i-1}^i is the biggest integer l' such that $s_{i-1}s_i$ appears as a subexpression of $u_{l'}u_{l'+1} \cdots u_k$. And so on.

For instance, let us look at the example above where we let $w = s_2s_1s_3s_2s_1 = s(2, 1)s(3, 1)$. Here we have $a_1 = 2$, $b_1 = 1$, $a_2 = 3$, $b_2 = 1$. Thus according to the definition $\mathbf{p}_1 = (2)$, $\mathbf{p}_2 = (1, 2)$, $\mathbf{p}_3 = (-1, 1, 2)$. Note that if $w = w_0$, then $\mathbf{p}_i = (n - i + 1, \dots, n)$.

Remark 7.5 Note that the class of w in W_i^w is $\bar{w} = (0, 1, 2, \dots, l - 2, a_{p_l^i}, \dots, a_{p_i^i})$, where l is maximal such that $p_1^i = \cdots = p_{l-1}^i = -1$. Therefore an element $(r_1, \dots, r_i)$ of W_i is in W_i^w if and only if $r_{l-1} = l - 2$ and $r_j \le a_{p_j^i}$ for $l \le j \le i$.

Now we can formalise the description of φ_i^w given above. Let $w = s(a_1, b_1) \cdots s(a_k, b_k)$ and let us write the standard basis of $\mathbb{R}^{\ell(w)}$ as e_{pq} with $p = 1, \dots, k$ and $q = b_p, b_p + 1, \dots, a_p$. We define the map $\varphi_i^w : W_i^w \longrightarrow \mathbb{R}^{\ell(w)}$ by sending

$$(r_1, \dots, r_i) \longmapsto \sum_{p=p_l^i} \sum_{l \le q \le r_l} e_{pq}$$

Definition 7.6

(i) We define, by abuse of notation, Δ_i^w to be the convex hull of the image of W_i^w via φ_i^w.
(ii) Let $\lambda = \sum_{i=1}^n c_i\omega_i$ be a dominant weight, then we define Δ_λ^w to be the Minkowski sum $\sum_{i=1}^n c_i\Delta_i^w$.

8. Proof of Theorem 7.2

We shall first prove that the conditions (i) and (ii) of Theorem 7.2 give nice properties on $\mathbf{p}_i$. Then we shall define a partial order on the union of W_i^w similar to the one given in Section 4. Finally, we prove Theorem 7.2 by showing that there is a one-to-one correspondence between lattice points in Δ_λ and the standard monomial basis of the Demazure module $E_w(\lambda)$.

In this section, we shall fix an element $w = s(a_1, b_1)\cdots s(a_k, b_k)$ of W which satisfies the conditions of Theorem 7.2. By definition, non-negative entries of $\mathbf{p}_i$ are distinct. We shall denote by $B(\mathbf{p}_i)$ the set of non-negative entries of $\mathbf{p}_i$.

Lemma 8.1 *Let us suppose that $B(\mathbf{p}_i)$ is not empty and let u_i (resp. v_i) be minimal (resp. maximal) in $B(\mathbf{p}_i)$.*

(i) *The element $s(a_{u_i}, b_{u_i})s(a_{u_i+1}, b_{u_i+1})\cdots s(a_{v_i}, b_{v_i})$ occurs (as a subexpression) in a connected part of w.*
(ii) *The set $B(\mathbf{p}_i)$ is a set of consecutive integers, that is, $B(\mathbf{p}_i) = \{m \in \mathbb{N}^* \mid u_i \le m \le v_i\}$ where u_i and v_i are as defined in* (i).
(iii) *If $i < j$ and $B(\mathbf{p}_j)$ is non-empty, then $v_i \le v_j$.*

Proof: Assertion (i) is a direct consequence of the definition of a connected part.

By definition, $B(\mathbf{p}_i) \subset \{1, \ldots, k\}$. Let us suppose that there exists $r > 1$ such that $r \in B(\mathbf{p}_i)$ and $r - 1 \notin B(\mathbf{p}_i)$. To prove (ii), il suffices to show that r is minimal in $B(\mathbf{p}_i)$.

There exists j such that $r = p_j^i$. Hence, by the definition of $\mathbf{p}_i$, we have $b_r \le j \le a_r$, and that either $j-1 < b_{r-1}$ or $a_{r-1} < j-1$.

If $b_r - 1 \le j - 1 < b_{r-1}$, then $b_{r-1} \ge b_r$. We are therefore in a connected part of w satisfying condition (i) of Theorem 7.2. It follows that $p_{j-1}^i = -1$.

If $a_{r-1} < j-1$, then $j-1 > a_l$ for all $l = 1, \ldots, r-1$. It follows again that $p_{j-1}^i = -1$.

Consequently, $p_{j-1}^i = -1$ in both cases and therefore r is minimal in $B(\mathbf{p}_i)$.

Finally, for (iii). Suppose that $v_j < v_i$, then $a_{v_j} < a_{v_i}$. Since $v_i = p_i^i$ and $v_j = p_j^j$, we would have

$$b_{v_i} \le i \le j \le a_{v_j} < a_{v_i}$$

Therefore $v_j = p_j^j \ge v_i$, contradiction. □

In view of the lemma, we define $s(\mathbf{p}_i)$ to be $s(a_{u_i}, b_{u_i})s(a_{u_i+1}, b_{u_i+1})\cdots s(a_{v_i}, b_{v_i})$. By part (i), $s(\mathbf{p}_i)$ occurs in a connected part of w.

Corollary 8.2 *Let $i \leq j$ be such that $B(\mathbf{p}_i)$ and $B(\mathbf{p}_j)$ are not empty. Let u_i, v_i, u_j, v_j be as defined in Lemma* 8.1. *Then we have one of the following three possibilities*:

(i) $s(\mathbf{p}_i)$ *and* $s(\mathbf{p}_j)$ *both occur in a connected part of w which satisfies condition* (i) *of Theorem* 7.2. *Furthermore $v_i = v_j$.*

(ii) $s(\mathbf{p}_i)$ *and* $s(\mathbf{p}_j)$ *both occur in a connected part of w which satisfies condition* (ii) *of Theorem* 7.2. *Furthermore*, $v_i + j - i = v_j$, $u_i = u_j$.

(iii) $s(\mathbf{p}_i)$ *and* $s(\mathbf{p}_j)$ *occur in different connected parts of w. Furthermore*, *this implies that $i < j$ and $a_{v_i} + 1 < b_l$ for all $u_j \leq l \leq v_j$.*

Example 8.3 Let us look at some examples:

(1) Let $w = w_0$, then $B(\mathbf{p}_i) = \{m \in \mathbb{Z} \mid n - i + 1 \leq m \leq n\}$ and so $u_i = n - i + 1$, $v_i = n$.

(2) Let $n = 3$ and $w = s(1,1)s(3,1)$, then $B(\mathbf{p}_1) = \{2\}$, $u_1 = v_1 = 2$, $B(\mathbf{p}_2) = \{1,2\}$, $u_2 = 1$, $v_2 = 2$, and $B(\mathbf{p}_3) = \{2\}$, $u_2 = v_2 = 2$.

We shall now define a partial order on $\mathcal{W}^w := \coprod_{i=1}^n W_i^w$.

Let $\mathbf{r} = (r_1, \ldots, r_i) \in W_i^w$, $\mathbf{r}' = (r'_1, \ldots, r'_j) \in W_j^w$ with $i \leq j$. And let u_i, v_i, u_j, v_j be as in Lemma 8.1.

Lemma 8.4 *Suppose that $s(\mathbf{p}_i)$ and $s(\mathbf{p}_j)$ both occur in a connected part of w which satisfies condition* (i) *of Theorem* 7.2, *we define* $\max(\mathbf{r}, \mathbf{r}')$, $\min(\mathbf{r}, \mathbf{r}')$ *as in Section* 4. *Then* $\max(\mathbf{r}, \mathbf{r}') \in W_j^w$ *and* $\min(\mathbf{r}, \mathbf{r}') \in W_i^w$.

Proof: From Section 4, we know that $\max(\mathbf{r}, \mathbf{r}') \in W_j$ and $\min(\mathbf{r}, \mathbf{r}') \in W_i$. So we only need to show that they are in W_j^w and W_i^w respectively.

Let $d(i) = v_i - u_i + 1$. By Remark 7.5, we can write

$$\mathbf{r} = (0, 1, \ldots, i - d(i) - 1, r_{i-d(i)+1}, \ldots, r_i)$$

and

$$\mathbf{r}' = (0, 1, \ldots, j - d(j) - 1, r'_{j-d(j)+1}, \ldots, r'_j).$$

By Corollary 8.2, $I := B(\mathbf{p}_i) \cap B(\mathbf{p}_j)$ is the set of integers between $\max(u_i, u_j)$ and $v_i = v_j$.

Let $l \in I$, then since $p_i^i = v_i = v_j = p_j^j$, there exists m such that $l = p_{i-m}^i = p_{j-m}^j$. It follows from Remark 7.5 that $i - m - 1 \leq r_{i-m} \leq a_l$ and $i - m - 1 \leq j - m - 1 \leq r'_{j-m} \leq a_l$.

To finish the proof, we shall show that $r'_{m+j-i} \geq r_m$ for all $m \leq \max(i - d(i), j - d(j))$.

If $j - d(j) \leq i - d(i)$, then $I = B(\mathbf{p}_i) \subset B(\mathbf{p}_j)$. Since $j \geq i$, we obtain by inspection that $r'_{m+j-i} \geq m + j - i - 1 \geq m - 1 = r_m$ for all $m \leq i - d(i)$.

If $i - d(i) < j - d(j)$, then $i < j$ and $I = B(\mathbf{p}_j)$ is a strict subset of $B(\mathbf{p}_i)$. Note that $p_{j-d(j)}^j = -1$. We claim that $j - d(j) > a_{u_j - 1}$.

Let us prove our claim. If we have $i - d(i) < j - d(j) < b_{u_j - 1}$, then $p_{i-d(i)+1}^i = -1$ since we are working in a connected part satisfying condition (i) of Theorem 7.2. But this will imply that $B(\mathbf{p}_i)$ has at most $d(i) - 1$ elements which is absurd.

So we have $j - d(j) > a_{u_j-1}$. It follows from the above expressions for $\mathbf{r}$, and $\mathbf{r}'$ that

$$r'_m = m - 1 \geq a_{u_j-1} - (j - d(j) - m) \geq a_{m+v_i-j} \geq r_{m+i-j}$$

for all $j - i + u_i \leq m \leq j - d(j)$.

Lastly, if $j - i < m < j - i + u_i$, then $r_{m+i-j} = m + i - j - 1 < m - 1 = r'_m$. Hence our proof is complete. □

Lemma 8.5 *Suppose that* $s(\mathbf{p}_i)$ *and* $s(\mathbf{p}_j)$ *both occur in a connected part of* w *which satisfies condition* (ii) *of Theorem* 7.2, *we define* $\min(\mathbf{r}, \mathbf{r}') := (m_1, \dots, m_j)$ *and* $\max(\mathbf{r}, \mathbf{r}') := (M_1, \dots, M_i)$ *as follows*:

(i) $m_l := \min(r_l, r'_l)$ *and* $M_l := \max(r_l, r'_l)$ *si* $u_i \leq l \leq v_i$;

(ii) $m_l := r'_l$ *and* $M_l := r_l$ *otherwise.*

Then $\max(\mathbf{r}, \mathbf{r}') \in W_i^w$ *and* $\min(\mathbf{r}, \mathbf{r}') \in W_j^w$.

Proof: By Corollary 8.2, $I := B(\mathbf{p}_i) \cap B(\mathbf{p}_j)$ is the set of integers between $u_i = u_j$ and $v_i = v_j + i - j$. Since we are working in a connected part satisfying condition (ii) of Theorem 7.2, we can write as in the previous lemma:

$$\mathbf{r} = (0, 1, \dots, i - d(i) - 1, r_{i-d(i)+1}, \dots, r_i)$$

and

$$\mathbf{r}' = (0, 1, \dots, i - d(i) - 1, r'_{i-d(i)+1}, \dots, r'_i, \dots, r'_j)$$

where $m - 1 \leq r_m, r'_m \leq m$. It is now clear that $\max(\mathbf{r}, \mathbf{r}') \in W_i^w$ and $\min(\mathbf{r}, \mathbf{r}') \in W_j^w$. □

Definition 8.6 If $B(\mathbf{p}_i) \cap B(\mathbf{p}_j)$ is not empty, then we define $\max(\mathbf{r}, \mathbf{r}')$ and $\min(\mathbf{r}, \mathbf{r}')$ according to the lemmas above. On the other hand, if $i < j$ and $B(\mathbf{p}_i) \cap B(\mathbf{p}_j)$ is empty, then we define $\max(\mathbf{r}, \mathbf{r}') = \mathbf{r}'$ and $\min(\mathbf{r}, \mathbf{r}') = \mathbf{r}$. Notice that if $B(\mathbf{p}_i)$ is empty, then W_i^w has only one element and therefore this definition is well-defined.

Moreover, we define a binary relation $\preceq_w$ on $\mathcal{W}^w$ by:

$$\mathbf{r} \preceq_w \mathbf{r}' \text{ if and only if } \max(\mathbf{r}, \mathbf{r}') = \mathbf{r}' \text{ and } \min(\mathbf{r}, \mathbf{r}') = \mathbf{r}$$

where here, we do not assume that $i \leq j$.

Remark 8.7 When $i < j$ and $B(\mathbf{p}_i) \cap B(\mathbf{p}_j)$ is empty, the binary relation $\mathbf{r} \preceq_w \mathbf{r}'$ coincides with the partial ordering defined in Section 4.

Example 8.8

(1) Let $w = w_0$, then this definition is the same as the one defined in Section 4.

(2) Let $n = 2$ and $w = s(1, 1)s(2, 2)$, then $W_1^w = \{1 = (0), s_1 = (1)\}$ and $W_2^w = \{1 = (0, 1), s_2 = (0, 2), s_1 s_2 = (1, 2)\}$. We have therefore,

$$(0, 1) \prec (0, 2) \prec (1, 2) \prec (1), (0, 1) \prec (0, 2) \prec (0), (0) \prec (1)$$

and $\max((0),(1,2)) = (1)$, $\min((0),(1,2)) = (0,2)$. Note that the maximal element is (1).

Lemma 8.9 *The binary relation $\preceq_w$ defines a partial ordering on $\mathcal{W}^w$. Furthermore, together with the operations* max *and* min, *it defines a lattice structure on $\mathcal{W}^w$ (see Lemma* 4.2).

Proof: The only point which is unclear is transitivity. But by Corollary 8.2, if $s(\mathbf{p}_i)$, $s(\mathbf{p}_j)$ and $s(\mathbf{p}_l)$ are in the same connected part, then either $v_i = v_j = v_k$ or $u_i = u_j = u_k$. The former is just an analogue of the partial ordering in Section 4. The latter can be verified easily using Lemma 8.4. Finally, the fact that the operations max and min induces a lattice structure on $\mathcal{W}^w$ is clear from the definition. □

Theorem 8.10 *Let $\lambda = \sum_{i=1}^{n} c_i\omega_i$ be a dominant weight. Then a lattice point in Δ_λ can be written as the sum of c_1 lattice points in Δ_1^w, c_2 lattice points in Δ_2^w and so on.*

Proof: This theorem is the analogue of Theorem 5.1, and it can be proved similarly. The key point is that the proofs of Theorem 5.1 and Theorem 5.2 only require a partial ordering equipped with a max–min operation satisfing Lemma 4.3 and 4.5. The analogues of Lemma 4.3 and 4.5 can be easily shown. □

Example 8.11 Let us first look at example 2 of 8.8. Consider $w = s_1s_2 = s(1,1)s(2,2)$. It satisfies condition (i) of Theorem 7.2.

We obtain immediately that $\mathbf{p}_1 = (1)$ and $\mathbf{p}_2 = (1,2)$. As above, we have in terms of minimal representatives,

$$W_1^w = \{1, s_1\}$$
$$W_2^w = \{1, s_2, s_1s_2\}$$

We then obtain via φ_i^w the following table:

	s_1	s_2	
	e_{11}	e_{22}	
s_1	1	0	(1)
s_2	0	1	(0, 2)
s_1s_2	1	1	(1, 2)

The right most column corresponds to the notations of the elements in $\mathcal{W}^w$. The images of $(0,1)$ and (0) are both 0.

If we consider the adjoint representation, then the highest weight is $\omega_1 + \omega_2$ and one verifies easily by hand that the lattice points in $\Delta_1^w + \Delta_2^w$ are the ones in Δ_1^w and Δ_2^w

together with the point $2e_{11} + e_{22}$. Thus the number of lattice points is 5 which is exactly the dimension of the Demazure module $E_w(\omega_1 + \omega_2)$.

Note that $\varphi_1^w(1) + \varphi_2^w(0, 2) = \varphi_1^w(0) + \varphi_2^w(1, 2)$. Again this can be seen to correspond to the tensor product decomposition of **b**-modules.

We prove the following key lemmas. As before, we let $\mathbf{r} = (r_1, \ldots, r_i) \in W_i^w$ and $\mathbf{r}' = (r_1', \ldots, r_j') \in W_j^w$ with $i \le j$. Recall that $w = s(a_1, b_1) \cdots s(a_k, b_k)$.

Lemma 8.12 *Suppose that $s(\mathbf{p}_i)$ and $s(\mathbf{p}_j)$ both occur in a connected part of w which satisfies condition* (i) *of Theorem* 7.2. *Then $\mathbf{r} \preceq_w \mathbf{r}'$ if and only if there exist $r \preceq r' \preceq w$ in W such that the class of r (resp. r') in W_i^w (resp. W_j^w) is $\mathbf{r}$ (resp. $\mathbf{r}'$).*

Proof: By Lemma 8.4, the partial ordering $\preceq_w$ coincides with the one defined in Section 4. It follows that the "if" part has already been proved using semi-standard Young tableaux.

Now suppose that $\mathbf{r} \preceq_w \mathbf{r}'$, we define

$$r := s\big(r_{i-v_i+u_i}, i - v_i + u_i\big)s\big(r_{i-v_i+u_i+1}, i - v_i + u_i + 1\big) \cdots s(r_i, i) \in W$$

where we let $s(a-1, a)$ to be the identity in W, and

$$r' := s\big(r'_{j-v_j+u_j}, j - v_j + u_j\big)s\big(r'_{j-v_j+u_j+1}, j - v_j + u_j + 1\big) \cdots s(r'_j, j) \in W \quad \text{if } u_i \ge u_j$$

and if $u_i < u_j$, we set

$$r' := s\big(r_{i-v_i+u_i}, i - v_i + u_i\big)s\big(r_{i-v_i+u_i+1}, i - v_i + u_i + 1\big) \cdots s\big(r'_{j-v_j+u_j}, \\ j - v_j + u_j\big) \cdots s(r'_j, j)$$

Note that $v_i = v_j$. In the first case, it is easy to see that the class of r' in W_j is $\mathbf{r}'$. For the second, we use the fact that $r_{i-v_j+u_j-1} < j - v_j + u_j - 1$ (see the proof of Lemma 8.4). Moreover, these are clearly reduced expressions. Since $\mathbf{r} \preceq_w \mathbf{r}'$, $r_{i-m} \le r'_{i-m}$ for all m and we have $r \preceq r' \preceq w$ in W as required. □

Lemma 8.13 *Suppose that $s(\mathbf{p}_i)$ and $s(\mathbf{p}_j)$ both occur in a connected part of w which satisfies condition* (ii) *of Theorem* 7.2. *Then $\mathbf{r}' \preceq_w \mathbf{r}$ if and only if there exist $r' \preceq r \preceq w$ in W such that the class of r (resp. r') in W_i^w (resp. W_j^w) is $\mathbf{r}$ (resp. $\mathbf{r}'$).*

Proof: Suppose that $\mathbf{r}' \preceq_w \mathbf{r}$. This implies that $l-1 \le r_l' \le r_l \le l$ for $l = i - v_i + u_i, \ldots, i$ (see the proof of Lemma 8.5). Note that here, $\mathbf{r}' = s_{t'}s_{t'+1} \cdots s_j$ where $t' \le j$ is minimal such that $r'_{t'} = t'$ and $\mathbf{r} = s_t s_{t+1} \cdots s_i$ where $t \le i$ is minimal such that $r_t = t$. In particular $t \le t'$. We shall simply define $r' := \mathbf{r}' = s_{t'}s_{t'+1} \cdots s_j$, $r := s_t s_{t+1} \cdots s_i s_{i+1} \cdots s_j$ and $r' \preceq r \preceq w$ as required.

On the other hand suppose that $r' \preceq r \preceq w$ in W. As above, we have $\mathbf{r}' = s_{t'}s_{t'+1} \cdots s_j$ where $t' \le j$ is minimal such that $r'_{t'} = t'$. Recall that we are working in a connected part satisfying condition (ii) of Theorem 7.2.

If $i < t'$, then $l - 1 = r'_{l-1}$ for $l \le i$, and so $r'_l \le r_l$. We then have $\mathbf{r}' \preceq_w \mathbf{r}$.

If $t' \leq i$, then since $\mathbf{r}' \preceq r' \preceq r$, we have $s_{t'}s_{t'+1}\cdots s_j$, and consequently $s_{t'}s_{t'+1}\cdots s_i$ can be written as subexpressions of a reduced expression for r. It follows that $\mathbf{r}' \preceq \mathbf{r}$. □

Lemma 8.14 *Suppose that $s(\mathbf{p}_i)$ and $s(\mathbf{p}_j)$ occur in different connected parts. Then $\mathbf{r} \preceq_w \mathbf{r}'$ if and only if there exist $r \preceq r' \preceq w$ in W such that the class of r (resp. r') in W_i^w (resp. W_j^w) is $\mathbf{r}$ (resp. $\mathbf{r}'$).*

Proof: By Remark 8.7, the partial ordering $\preceq_w$ coincides with with the one defined in Section 4. It follows that the "if" part has already been proved using semi-standard Young tableaux.

Since we are working in different connected parts, and distinct connect parts commute, $\mathbf{r}$ and $\mathbf{r}'$ commute also. Therefore if we define $r := \mathbf{r}$ and $r' = \mathbf{r}\mathbf{r}'$ in W. By definition, we always have $\mathbf{r} \preceq \mathbf{r}' \preceq w$. □

Corollary 8.15 *Let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ be elements of $\mathcal{W}^w$ such that $\mathbf{r}_1 \preceq_w \cdots \preceq_w \mathbf{r}_m$. Then there exist liftings $r_1, \ldots, r_m$ in W^w such that the class of r_j is $\mathbf{r}_j$ (in the appropriate W_l^w) and $r_1 \preceq \cdots \preceq r_m$.*

Corollary 8.16 *Let λ be a dominant weight. There exists a bijection between the set of lattice points of Δ_λ^w and the standard monomial basis of the Demazure module $E_w(\lambda)$ (see* [14, 17]*).*

Proof: We can prove an analogue of Proposition 4.4. Using this and Theorem 8.10, we can write a lattice point in Δ_λ^w as the sum $\sum_{i=1}^m \mathbf{r}_i$ such that $\mathbf{r}_i \in W^w$ and $\mathbf{r}_1 \preceq_w \mathbf{r}_2 \preceq_w \cdots \preceq_w \mathbf{r}_m$. Corollary 8.15 then implies that the set of lattice points of Δ_λ^w is in bijection with the standard monomial basis of $E_w(\lambda)$. □

Example 8.17 In the case of example 2 of 8.8, the reader can verify that the three chains can be lifted as follows:

$$1 \preceq s_2 \preceq s_1s_2 \preceq s_1s_2,\ 1 \preceq s_2 \preceq s_2,\ s_2 \preceq s_1s_2$$

and there are no liftings r of (0) and r' of (1, 2) such that $r' \preceq r$.

9. An Example

Let us consider a slightly more complicated case. Let $w = s_1s_3s_2s_1 = s(1,1)s(3,1)$ be an element of the Weyl group of sl_4. It satisfies condition (i) of Theorem 7.2. In this case, we have $\mathbf{p}_1 = (3)$, $\mathbf{p}_2 = (1, 3)$ and $\mathbf{p}_3 = (-1, -1, 3)$. Thus,

$$W_1^w = \{1, s_1, s_2s_1, s_3s_2s_1\}$$

$$W_2^w = \{1, s_2, s_3s_2, s_1s_2, s_1s_3s_2\}$$

$$W_3^w = \{1, s_3\}$$

As before, we compute the following table:

	s_1	s_3	s_2	s_1	
	e_{11}	e_{33}	e_{32}	e_{31}	
s_1	0	0	0	1	(1)
s_2s_1	0	0	1	1	(2)
$s_3s_2s_1$	0	1	1	1	(3)
s_2	0	0	1	0	(0, 2)
s_3s_2	0	1	1	0	(0, 3)
s_1s_2	1	0	1	0	(1, 2)
$s_1s_3s_2$	1	1	1	0	(1, 3)
s_3	0	1	0	0	(0, 1, 3)

The reader can verify for example for the adjoint representation as in Section 4 that the number of lattice points in $\Delta_1^w + \Delta_3^w$ is the dimension of the Demazure module $E_w(\omega_1 + \omega_3)$ which is 7.

10. Some applications

We shall apply our results in this section to obtain a combinatorial description of the weight multiplicities of a Demazure module, and we also present a polytope which is closely connected to the weight.

Let $\lambda = \sum_{i=1}^n a_i\omega_i$ be a dominant weight, $w = s(a_1, b_1) \cdots s(a_k, b_k)$ be an element of W satisfying the conditions of Theorem 7.2 and denote by $m_\lambda^w(\mu)$ the multiplicity of the weight μ in $E_w(\lambda)$.

Let e_{pq} be the standard basis as in Section 7 and α_i, $i = 1, \ldots, n$ be the set of simple roots as in the introduction.

Definition 10.1 We define a linear map

$$A^w : \mathbb{R}^{\ell(w)} \longrightarrow P \otimes_{\mathbb{Z}} \mathbb{R} =: P_{\mathbb{R}}$$

by sending e_{pq} to α_q.

We shall denote by A_λ^w the affine map $\lambda - A^w$ from $\mathbb{R}^{\ell(w)}$ to $P_{\mathbb{R}}$.

Theorem 10.2 *The character of the Demazure module $E_w(\lambda)$ is given by*

$$\text{char}\, E_w(\lambda) = e^\lambda \sum_x e^{-A^w(x)}$$

where the sum runs through the lattice points x of Δ_λ^w.

Proof: Let $\mu \in P$. To prove the theorem, it suffices to prove that

$$m_\lambda^w(\mu) = \mathrm{Card}\big(\mathbb{Z}^{\ell(w)} \cap \Delta_\lambda^w \cap \big(A_\lambda^w\big)^{-1}(\mu)\big)$$

Recall that the weight of a standard monomial of type λ is the weight of the corresponding weight vector in $E_w(\lambda)$ (see [17, 14]). Therefore it suffices to show that for a lattice point x of Δ_λ^w, $A_\lambda^w(x)$ is the weight of the standard monomial $T(x)$ corresponding to x as in the proof of Theorem 7.2.

Let x be a lattice point of Δ_λ^w. According to Proposition 6.2, we can find $\{\sigma_{ij}\}$, $i = 1, \dots, n$ and $j = 1, \dots, a_i$ satisfying the conclusions of the proposition and such that $x = \sum_{i=1}^n \sum_{j=1}^{a_i} \varphi_i^w(\sigma_{ij})$. It follows that the weight of $T(x)$ is the sum $\sum_{i,j} \sigma_{ij}(\omega_i)$. Since $A_\lambda^w(x) = \sum_{i=1}^n \sum_{j=1}^{a_i} A_{\omega_i}^w(\varphi_i^w(\sigma_{ij}))$, we are reduced to the case where λ is a fundamental weight.

According to Lemma 2.3 and Theorem 3.3, there is a bijection between the weights of V_{ω_i} and the vertices of Δ_i, which in turn are indexed by the elements of W_i. Explicitly, the vertex $(r_1, \dots, r_i)$ corresponds to the weight

$$s_{r_1} \cdots s_1 s_{r_2} \cdots s_2 \cdots s_{r_i} \cdots s_i(\omega_i) = \omega_i - \sum_{p=1}^{i} \sum_{q=p}^{r_p} \alpha_q = \omega_i - \sum_{a_p = p_j^i} \sum_{q=j}^{r_j} \alpha_q$$

On the other hand, $(r_1, \dots, r_i)$ corresponds to the point $\sum_{a_p = p_j^i} \sum_{q=j}^{r_j} e_{pq}$ (see Section 7). Therefore

$$A_{\omega_i}^w \left(\sum_{a_p = p_j^i} \sum_{q=j}^{r_j} e_{pq} \right) = \omega_i - \sum_{a_p = p_j^i} \sum_{q=j}^{r_j} \alpha_q$$

and we are done since the multiplicity of any weight of V_{ω_i} is 1. □

Corollary 10.3 *The image of Δ_λ^w via A_λ^w is the convex hull of $\{\sigma(\lambda) \mid \sigma \in W$ with $\sigma \preceq w\}$. In particular it is the Minkowski sum $\sum_i a_i A_{\omega_i}^w(\Delta_i^w)$.*

Corollary 10.4 *Let μ be a weight of $E_w(\lambda)$, then $\Delta_\lambda^w(\mu) := (A_\lambda^w)^{-1}(\mu) \cap \Delta_\lambda^w$ is a convex polytope with rational vertices containing all the points which correspond to the weight μ. Moreover $k\Delta_\lambda^w(\mu) = \Delta_{k\lambda}^w(k\mu)$.*

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