Journal of Algebraic Combinatorics **11** (2000), 241–268

On Four-Weight Spin Models and their Gauge Transformations

FRANÇOIS JAEGER†
Laboratoire Leibniz, IMAG, Grenoble, France

Received July 16, 1996; Revised March 11, 1999

Abstract. We study the four-weight spin models (W_1, W_2, W_3, W_4) introduced by Eiichi and Etsuko Bannai (Pacific J. of Math, to appear). We start with the observation, based on the concept of special link diagram, that two such spin models yield the same link invariant whenever they have the same pair (W_1, W_3), or the same pair (W_2, W_4). As a consequence, we show that the link invariant associated with a four-weight spin model is not sensitive to the full reversal of orientation of a link. We also show in a similar way that such a link invariant is invariant under mutation of links.

Next, we give an algebraic characterization of the transformations of four-weight spin models which preserve W_1, W_3 or preserve W_2, W_4. Such "gauge transformations" correspond to multiplication of W_2, W_4 by permutation matrices representing certain symmetries of the spin model, and to conjugation of W_1, W_3 by diagonal matrices. We show for instance that up to gauge transformations, we can assume that W_1, W_3 are symmetric.

Finally we apply these results to two-weight spin models obtained as solutions of the modular invariance equation for a given Bose-Mesner algebra **B** and a given duality of **B**. We show that the set of such spin models is invariant under certain gauge transformations associated with the permutation matrices in **B**. In the case where **B** is the Bose-Mesner algebra of some Abelian group association scheme, we also show that any two such spin models (which generalize those introduced by Eiichi and Etsuko Bannai in *J. Alg. Combin.* **3** (1994), 243–259) are related by a gauge transformation. As a consequence, the link invariant associated with such a spin model depends only trivially on the link orientation.

Keywords: spin model, link invariant, association scheme, Bose-Mesner algebra

1. Introduction

Spin models are basic data for a certain construction of invariants of oriented links in 3-space. They are given in terms of matrices (satisfying certain equations) which are used to compute the link invariant on link diagrams. The original construction by Jones [18] involved a pair of symmetric matrices, which we call here a symmetric two-weight spin model. This was generalized by the two-weight spin models of Kawagoe, Munemasa, Watatani [21] which consist of a pair of not necessarily symmetric matrices. Finally Eiichi and Etsuko Bannai [2] introduced the much more general four-weight spin models which involve four matrices W_1, W_2, W_3, W_4.

So far, research on spin models has been mostly devoted to two-weight spin models, which exhibit nice connections with association schemes: the spin model matrices belong to some Bose-Mesner algebra and define a duality on this algebra via the so-called modular invariance equation (see [3, 14, 16, 24]). We study here four-weight spin models, also obtaining some new results on two-weight spin models whose proof apparently involves the concept of four-weight spin model in an essential way.

The content of this paper can be summed up as follows.

In Section 2 we give the necessary preliminaries on link diagrams, link invariants and spin models.

Then Section 3 deals with special link diagrams. Our starting point is the observation that for such link diagrams, the computation of the link invariant associated with a four-weight spin model can be done using only the matrices W_1, W_3, or only the matrices W_2, W_4. Every link can be represented by a special diagram, and hence if two four-weight spin models have the same matrices W_1, W_3, or the same matrices W_2, W_4, they yield the same link invariant. This is used to show that the link invariant associated with any four-weight spin model is invariant under simultaneous orientation reversal of all components. We also use the concept of special link diagram to show that link invariants associated with four-weight spin models are invariant under mutation of oriented links. This is a strong restriction on such invariants, since for instance many quantum group invariants (as defined for instance in [25]) can distinguish mutant links [23].

In Section 4 we describe algebraically the transformations of four-weight spin models which preserve W_1, W_3 or preserve W_2, W_4. We call them gauge transformations to point out their similarity with transformations of spin models as considered in statistical mechanics (they have also been considered independently in [10] under the same name). These gauge transformations belong to the following two types: multiplication of W_2, W_4 by permutation matrices representing certain symmetries of the spin model, and conjugation of W_1, W_3 by diagonal matrices. We show for instance that up to gauge transformations, we can assume that W_1, W_3 are symmetric. We have no similar result for W_2, W_4. However, some power of W_2 must be symmetric (and likewise for W_4).

In Section 5 we consider two-weight spin models obtained as solutions of the modular invariance equation for a given Bose-Mesner algebra **B** and a given duality of **B**. We show that the set of such spin models is invariant under certain gauge transformations associated with the permutation matrices in **B**. In the case where **B** is the Bose-Mesner algebra of some Abelian group association scheme, we also show that any two such spin models (which generalize those introduced in [1]) are related by a gauge transformation. As a consequence, since one of these spin models is symmetric, the link invariant associated with such a spin model depends only trivially on the link orientation.

We conclude in Section 6 with some directions for future research.

2. Spin models for link invariants

For a more complete survey on this topic, see [12].

2.1. Links and link diagrams

For more details on this section the reader can refer to [7, 8, 20].

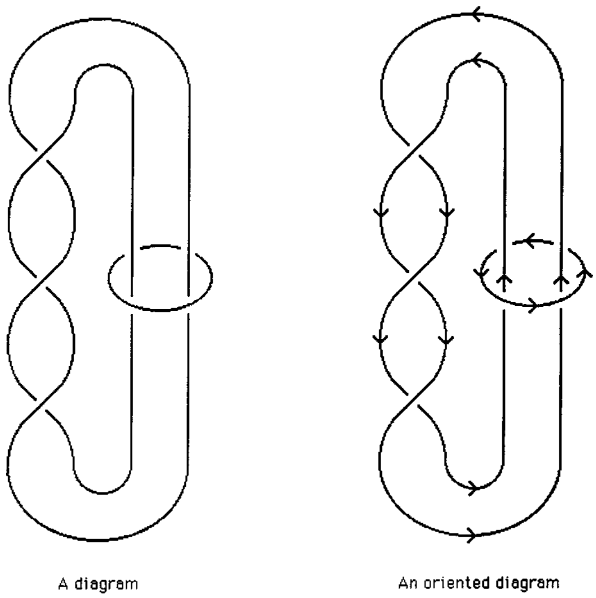

Figure 1.

A *link* consists of a finite collection of disjoint simple closed curves smoothly embedded in $\mathbb{R}^3$ (these curves are the *components* of the link). If each component has received an orientation, the link is said to be *oriented*. (Oriented) links can be represented by (*oriented*) *diagrams*. A diagram of a link is a generic plane projection (there is only a finite number of multiple points, each of which is a simple crossing), together with an indication at each crossing of which part of the link goes over the other. For oriented diagrams, the orientations of the components are indicated by arrows. See figure 1 for examples. An oriented diagram L has two kinds of crossings, characterized by a *sign* as shown on figure 2. The *Tait number* (*or writhe*) of L, denoted by $T(L)$, is the sum of signs of its crossings.

A diagram L will be considered as a graph embedded in the plane $\mathbb{R}^2$, with sets of vertices and faces denoted by $V(L)$, $F(L)$ respectively. The vertices of L correspond to the crossings, the edges are the connected components of $L - V(L)$, and the faces are the connected components of $\mathbb{R}^2 - L$. We allow a special kind of edge called a *free loop*

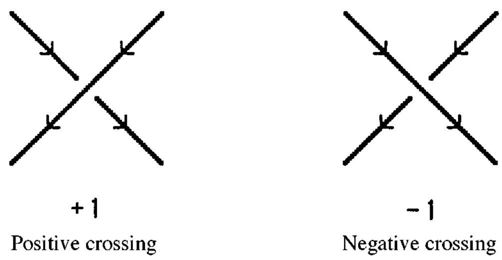

Figure 2.

which is embedded as a simple closed curve disjoint from the remaining part of the graph. All edges of an oriented diagram will be directed in agreement with the orientation of the corresponding link component.

(Oriented) diagrams are considered up to isomorphism of (directed) plane graphs (preserving the crossing information at each vertex).

2.2. Link invariants and Reidemeister moves

Two links are *ambient isotopic* if there exists an isotopy of the ambient 3-space which carries one onto the other (for oriented links, this isotopy must preserve the orientations). This natural equivalence of links is described in terms of diagrams by Reidemeister's Theorem, which asserts that two diagrams represent ambient isotopic links if and only if one can be obtained from the other by a finite sequence of elementary local transformations, the *Reidemeister moves*. These moves belong to three basic types described for the unoriented case in figure 3.

A move is performed by replacing a part of diagram which is one of the configurations of figure 3 by an equivalent configuration without modifying the remaining part of the diagram. For the oriented case, all local orientations of these pairs of equivalent configurations must be considered.

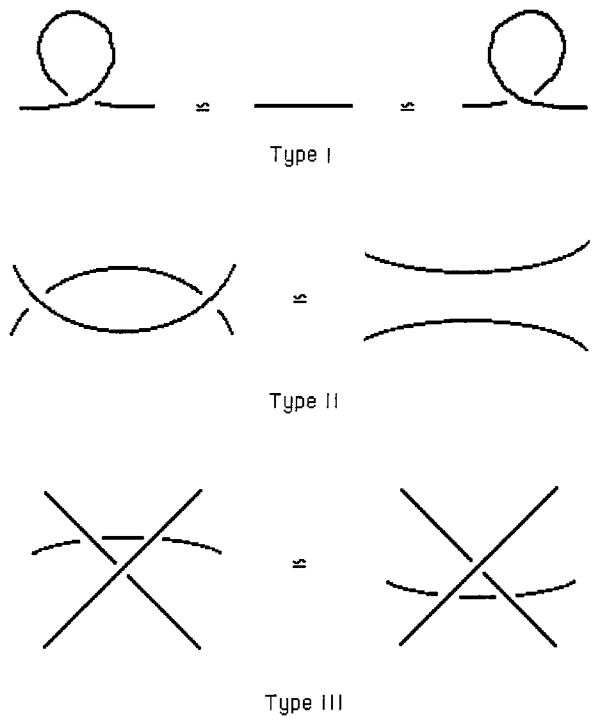

Figure 3. Reidemeister moves.

Reidemeister's Theorem allows the combinatorial definition of a *link invariant* as an assignment of values to diagrams such that the value of any diagram is preserved by Reidemeister moves. As shown in [18], one may use partition functions of statistical mechanical models, and in particular of spin models, to define such assignments.

2.3. Spin models: Generalities

Link invariants associated with spin models are defined as follows. Given a link diagram L, we first color its faces with two colors, black and white, in such a way that adjacent faces receive different colors and the unbounded face is colored white. Let X be a finite non-empty set of *spins*. Let $B(L)$ be the set of faces of L colored black. A state of L is a mapping from $B(L)$ to X. Loosely speaking, a *spin model* will be a certain prescription for associating with every state σ and vertex v of L a complex number $\langle v, \sigma\rangle$ called the *local weight of* σ *at* v. Then the *weight* of a state will be the product of local weights over all vertices (this product will be set to 1 if there are no vertices). Finally the *partition function* $Z(L)$ will be the sum of weights of all states. Thus

$$Z(L) = \sum_{\sigma : B(L)\to X} \prod_{v\in V(L)} \langle v, \sigma\rangle. \tag{1}$$

One can write down natural conditions on the spin model (called *invariance equations*) which will guarantee that the partition function, multiplied by a suitable *normalization factor*, is not modified by Reidemeister moves and hence defines a link invariant. This normalization factor consists of two terms. The first one (needed to accomodate Reidemeister moves of type I) necessarily involves an orientation of L, and is equal to $\mu^{-T(L)}$, where μ is some non-zero complex number called the *modulus* of the spin model. The second one is, assuming that L is connected as a graph, $D^{-|B(L)|}$, where D is some square root of the number of spins (this connectivity restriction is not significant since every link can be represented by a connected diagram).

To sum up, if a spin model satisfies the invariance equations, the assignment of the quantity $\mu^{-T(L)}D^{-|B(L)|}Z(L)$ to every connected oriented diagram L defines an invariant of oriented links.

Clearly the link invariant $\mu^{-T(L)}D^{-|B(L)|}Z(L)$ takes the value D if L consists of one free loop, and D is called the *loop variable* of the spin model.

2.4. Spin models: Definitions

Let us now describe spin models more precisely.

The initial definition of [18] was given in terms of a pair of symmetric matrices. Then it was generalized to non symmetric matrices in [21]. Finally it was further generalized in [2] under the name of *four-weight spin models* by using four matrices. To simplify the exposition we shall begin with this last generalization, which will be the main topic of the present paper.

A four-weight spin model is given by four matrices W_1, W_2, W_3, W_4 with rows and columns indexed by the set of spins X. Given a state σ and vertex v of the oriented diagram

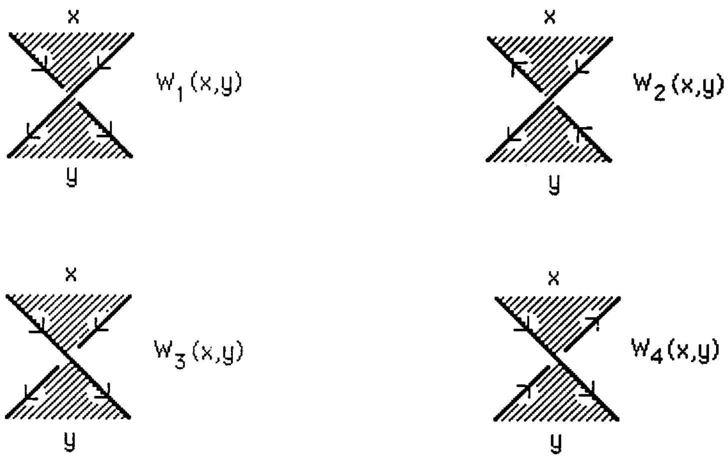

Figure 4. Local weights of a four-weight spin model.

L, the local weight $\langle v, \sigma \rangle$ is defined on figure 4, where x and y denote the values of σ on the black faces incident to v and $W_i(x, y)$ denotes the entry of the matrix W_i $(i = 1..4)$ corresponding to row x and column y. Note that x and y are distinguished by reference to the orientation of the upper part of the link at v. We shall say that the vertex v is *of type* $W_i (i = 1..4)$ if figure 4 prescribes the use of W_i to compute $\langle v, \sigma \rangle$.

Let us now present briefly the invariance equations (see [2]).

The study of Reidemeister moves of type II (there are four oriented versions of the unoriented move depicted on figure 3, and two local black and white face-colorings for each of these) forces the introduction of the normalization term $D^{-|B(L)|}$ (where $D^2 = |X|$) and leads to the equations (to be satisfied for all a, b in X):

$$\sum_{x \in X} W_1(a, x) W_3(x, b) = |X| \, \delta(a, b), \tag{2}$$

$$W_1(a, b) W_3(b, a) = 1, \tag{3}$$

$$\sum_{x \in X} W_2(a, x) W_4(x, b) = |X| \, \delta(a, b), \tag{4}$$

$$W_2(a, b) W_4(b, a) = 1, \tag{5}$$

where δ is the Kronecker symbol.

Note that (3), (5) imply that the matrices $W_i (i = 1..4)$ have non-zero entries.

A similar study of Reidemeister moves of type III leads in [2] to sixteen equations (in that paper the mirror image of the Reidemeister move of type III depicted on figure 3 is also considered, but this is unnecessary since both versions are equivalent under Reidemeister moves of type II). However it is shown in Theorem 1 of [2] that, assuming (2), (3), (4), (5), these sixteen equations can be separated into two groups of eight in such a way that all equations in one group are mutually equivalent. Thus, to obtain the invariance of $D^{-|B(L)|} Z(L)$

under oriented Reidemeister moves of types II and III, it is enough to impose one equation in each group, together with (2), (3), (4), (5). There is another way to see this: it is shown in [26] that any oriented version of the Reidemeister move of type III depicted on figure 3 (there are eight of them) can be replaced by any other one combined with suitable oriented Reidemeister moves of type II. Thus, assuming invariance under oriented moves of type II, we can reduce the invariance under oriented moves of type III to the case of one arbitrarily chosen such move. For reasons which will become clear later, we shall choose the following invariance equations for Reidemeister moves of type III, to be satisfied for all a, b, c in X:

$$\sum_{x\in X} W_2(a,x)W_2(b,x)W_4(x,c) = DW_1(b,a)W_3(a,c)W_3(c,b) \tag{6}$$

$$\sum_{x\in X} W_2(x,a)W_2(x,b)W_4(c,x) = DW_1(a,b)W_3(b,c)W_3(c,a). \tag{7}$$

These correspond to the two local black and white face-colorings of the move of type III depicted on figure 3, oriented in such a way that the triangle in the left-hand side becomes an anticlockwise circuit. In the terminology of [2], (6) and (7) are equations III_6 and III_{12} respectively.

We observe that the exchange of a and b in (6) does not modify the left-hand side and transforms the right-hand side into the right-hand side of (7). Thus we may replace (6), (7) by

$$\begin{aligned}\sum_{x\in X} W_2(a,x)W_2(b,x)W_4(x,c) &= DW_1(b,a)W_3(a,c)W_3(c,b)\\ &= \sum_{x\in X} W_2(x,a)W_2(x,b)W_4(c,x)\\ &= DW_1(a,b)W_3(b,c)W_3(c,a).\end{aligned} \tag{8}$$

Finally, taking $c = b$ in (6), (7) and using (3), (5), we obtain:

$$\sum_{x\in X} W_2(a,x) = \sum_{x\in X} W_2(x,a) = DW_3(b,b) \quad \text{for all } a, b \text{ in } X.$$

Hence there exists a non-zero complex number μ such that

$$W_3(a,a) = \mu^{-1}, \qquad \sum_{x\in X} W_2(a,x) = \sum_{x\in X} W_2(x,a) = D\mu^{-1} \quad \text{for all } a \text{ in } X. \tag{9}$$

It then easily follows from (3) and (4) that

$$W_1(a,a) = \mu, \qquad \sum_{x\in X} W_4(a,x) = \sum_{x\in X} W_4(x,a) = D\mu \quad \text{for all } a \text{ in } X. \tag{10}$$

Eqs. (9) and (10) imply the invariance of $\mu^{-T(L)}D^{-|B(L)|}\,Z(L)$ under oriented Reidemeister moves of type I. Moreover, assuming (2), (3), (4), (5), (8), this quantity is still invariant under oriented Reidemeister moves of type II and III, since the Tait number is also invariant under these moves.

We now sum up the above discussion in the following definition.

Definition 1 A *four-weight spin model* on a finite non-empty set X is a 5-tuple (W_1, W_2, W_3, W_4, D), where $D^2 = |X|$ and W_1, W_2, W_3, W_4 are complex matrices with rows and columns indexed by X which satisfy the following equations for all a, b, c in X:

$$\sum_{x \in X} W_1(a,x) W_3(x,b) = |X|\, \delta(a,b), \tag{2}$$

$$W_1(a,b) W_3(b,a) = 1, \tag{3}$$

$$\sum_{x \in X} W_2(a,x) W_4(x,b) = |X|\, \delta(a,b), \tag{4}$$

$$W_2(a,b) W_4(b,a) = 1, \tag{5}$$

$$\begin{aligned} \sum_{x \in X} W_2(a,x) W_2(b,x) W_4(x,c) &= D W_1(b,a) W_3(a,c) W_3(c,b) \\ &= \sum_{x \in X} W_2(x,a) W_2(x,b) W_4(c,x) \\ &= D W_1(a,b) W_3(b,c) W_3(c,a). \end{aligned} \tag{8}$$

These equations imply that there exists a non-zero complex number μ, called the *modulus* of the spin model, such that, for all a in X,

$$W_3(a,a) = \mu^{-1}, \qquad \sum_{x \in X} W_2(a,x) = \sum_{x \in X} W_2(x,a) = D\mu^{-1}, \tag{9}$$

$$W_1(a,a) = \mu, \qquad \sum_{x \in X} W_4(a,x) = \sum_{x \in X} W_4(x,a) = D\mu. \tag{10}$$

The *associated link invariant* is defined for every connected oriented diagram L by $\mu^{-T(L)}$ $D^{-|B(L)|}$ $Z(L)$, where

$$Z(L) = \sum_{\sigma : B(L) \to X} \prod_{v \in V(L)} \langle v, \sigma \rangle$$

and the local weights $\langle v, \sigma \rangle$ are defined on figure 4.

We shall be interested in the following special cases.

Definition 2 A *two-weight spin model* on a finite non-empty set X is a triple (W_+, W_-, D), where $D^2 = |X|$ and W_+, W_- are complex matrices with rows and columns indexed by X, such that (W_+, W_+, W_-, W_-, D) is a four-weight spin model. It is called *symmetric* if the matrices W_+, W_- are symmetric. The modulus of (W_+, W_-, D) is the modulus of (W_+, W_+, W_-, W_-, D), and similarly for the associated link invariants.

It can be shown that the above definitions are equivalent to the definitions given for "generalized spin models" in [21] or for "two-weight spin models of Jones type" in [2]. Moreover, symmetric two-weight spin models are exactly those introduced in [18]. These symmetric models have the property that the corresponding partition function can be computed on unoriented diagrams (see figure 4) and hence the associated link invariant depends only trivially on the link orientation via the normalization term $\mu^{-T(L)}$.

We shall make use of the following immediate consequences of (2), (3), (9), (10). For every two-weight spin model (W_+, W_-, D) on X of modulus μ, and for all a, b in X:

$$\sum_{x\in X} W_+(a,x)W_-(x,b) = |X|\,\delta(a,b), \tag{11}$$

$$W_+(a,b)W_-(b,a) = 1, \tag{12}$$

$$W_-(a,a) = \mu^{-1}, \qquad \sum_{x\in X} W_+(a,x) = \sum_{x\in X} W_+(x,a) = D\mu^{-1}, \tag{13}$$

$$W_+(a,a) = \mu, \qquad \sum_{x\in X} W_-(a,x) = \sum_{x\in X} W_-(x,a) = D\mu. \tag{14}$$

2.5. *Exchanging black and white*

Recall that to define the link invariant associated with a spin model we have chosen arbitrarily to color the unbounded face of every connected oriented diagram white (and then the color of every face is determined). If we had chosen to color the unbounded face black (then black and white are exchanged for all faces), we would also have obtained a link invariant. We shall need the basic (and not surprising) fact that this link invariant is the same as the first one. The proof for four-weight spin models is an immediate extension of the one given for symmetric two-weight spin models in [18] (Proposition 2.14). So from now on we shall choose freely the black and white face-coloring to evaluate the link invariant associated with a four-weight spin model.

2.6. *A remark on normalization*

It is clear from Definition 1 that if (W_1, W_2, W_3, W_4, D) is a four-weight spin model, the same holds for $(\lambda W_1, \lambda^{-1}W_2, \lambda^{-1}W_3, \lambda W_4, D)$, where λ is any non-zero complex number. We shall say that these spin models are *proportional*. Given a connected oriented diagram L, replacing the first spin model by the second one multiplies the weight of each state, and hence the partition function, by $\lambda^{T(L)}$ (see figures 2 and 4). On the other hand, if the first spin model has modulus μ, the second one has modulus $\lambda\mu$. Hence proportional four-weight spin models yield the same link invariant.

Thus the concept of modulus for four-weight spin models might appear to be redundant. However it is essential for the study of two-weight spin models.

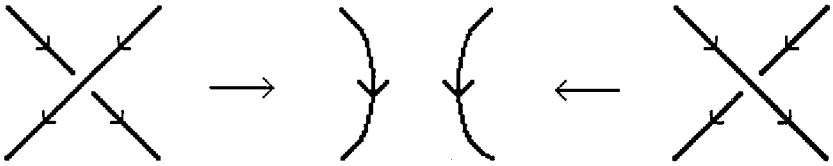

Figure 5.

3. Special diagrams and applications

3.1. Special diagrams

Let L be an oriented link diagram. If we “smoothe out” each crossing as shown on figure 5, we obtain an oriented link diagram consisting only of free loops. These free loops are called the *Seifert circles* of L and can be identified with some directed circuits of the directed graph L. Clearly every vertex of L is incident to exactly two distinct Seifert circles. The diagram L will be called *special* if no Seifert circle lies in the interior of another.

We shall give the proof of the following Proposition 13.15 in [7] since we shall need later some extension of it.

Proposition 1 *Every oriented link can be represented by a connected special diagram.*

Proof: Let us start with some connected oriented diagram L of the given link $\mathbf{L}$. For every Seifert circle C of L we introduce a disk with boundary $\mathbf{C}$, and for every vertex incident with the two Seifert circles C_1, C_2 we introduce an appropriately twisted band with ends attached to the disks corresponding to C_1 and C_2, thus obtaining a connected surface S with boundary $\mathbf{L}$. This surface is easily seen to be orientable. We embed S in $\mathbb{R}^3$ in such a way that the disks are disjointly embedded in the plane $\mathbb{R}^2$, the twisted bands are pairwise disjoint, and the projection of each band onto $\mathbb{R}^2$ is disjoint from all disks except for its two ends. We may assume that the projection of the boundary $\mathbf{L}$ of S onto $\mathbb{R}^2$ is generic and hence yields a diagram L' of $\mathbf{L}$. We may arrange so that all crossings of L' occur either between two opposite sides of the same band, or in groups of four according to the two situations depicted on figure 6 or to the oppositely oriented ones (these situations correspond to the crossing of two distinct bands, or of two separate sections of the same band). We replace all configurations depicted on figure 6(i) by the configuration of figure 7, and perform similar replacements for the oppositely oriented configurations. It is then easy to see that the resulting diagram of $\mathbf{L}$ is special. $\square$

3.2. Four-weight spin models with the same associated link invariant

Let us call a special diagram *even* (respectively: *odd*) if it is endowed with a black and white face-coloring such that the unbounded face is white (respectively: black). Clearly in an even special diagram L the interior of every Seifert circle is a black face, and consequently

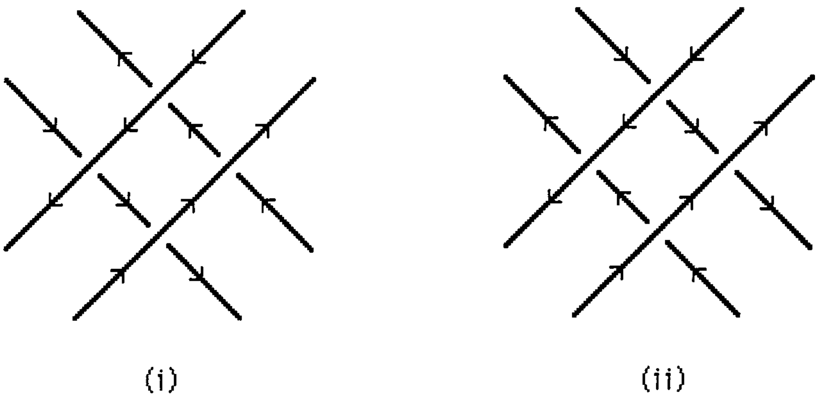

Figure 6. Crossings of L' as in Proposition 1.

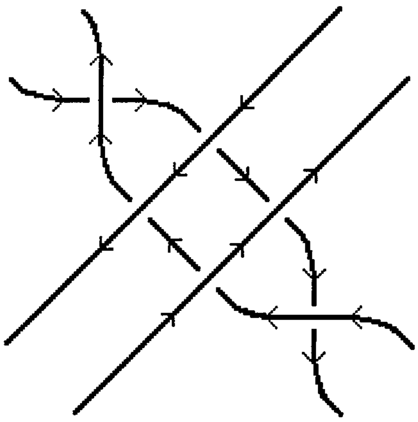

Figure 7. Link diagram modification as in Proposition 1.

to compute on L the partition function of a four-weight spin model (W_1, W_2, W_3, W_4, D) only the matrices W_2, W_4 are used (see figure 4). Similarly, in an odd special diagram L only the matrices W_1, W_3 are used.

Proposition 2 *Let* (W_1, W_2, W_3, W_4, D) *and* $(W_1', W_2', W_3', W_4', D)$ *be four-weight spin models on the same set of spins X. If* $W_2 = W_2'$, $W_4 = W_4'$, *these two spin models have the same associated link invariant, and similarly if* $W_1 = W_1'$, $W_3 = W_3'$.

Proof: By Proposition 1, we may represent any oriented link by a connected even (respectively: odd) special diagram. The partition functions of (W_1, W_2, W_3, W_4, D) and $(W_1', W_2, W_3', W_4, D)$ (respectively: $(W_1, W_2', W_3, W_4', D)$) will coincide on this diagram. The same holds for the associated link invariants since by (9) the modulus of a four-weight spin model (W_1, W_2, W_3, W_4, D) is determined either by W_3 or by W_2 and D. □

Remark By (3), (5) the two equalities $W_2 = W_2'$, $W_4 = W_4'$ are equivalent, and similarly for $W_1 = W_1'$, $W_3 = W_3'$.

3.3. *Invariance under orientation reversal*

The *reverse* of an oriented link is obtained by reversing the orientations of all its components.

So far the following result was only known for the trivial case of symmetric two-weight spin models.

Proposition 3 *A link invariant associated with a four-weight spin model does not distinguish between an oriented link and its reverse.*

Proof: We denote by ${}^tM$ the transpose of a matrix M. Let (W_1, W_2, W_3, W_4, D) be a four-weight spin model. We claim that $({}^tW_1, W_2, {}^tW_3, W_4, D)$ and $(W_1, {}^tW_2, W_3, {}^tW_4, D)$ are also four-weight spin models. Indeed Eqs. (3), (5), (8) are obviously invariant under transposition of W_1, W_3 or of W_2, W_4. The same holds for Eqs. (2) and (4) since they can be written as $W_1W_3 = |X|I$ and $W_2W_4 = |X|I$ respectively, where I denotes the identity matrix. Hence $({}^tW_1, {}^tW_2, {}^tW_3, {}^tW_4, D)$ is a four-weight spin model which by Proposition 2 has the same associated link invariant as (W_1, W_2, W_3, W_4, D). But on the other hand it is clear from figures 2 and 4 that computing the link invariant associated with $({}^tW_1, {}^tW_2, {}^tW_3, {}^tW_4, D)$ on some oriented link amounts to compute the link invariant associated with (W_1, W_2, W_3, W_4, D) on the reverse of that link. □

3.4. *Changing signs*

Proposition 4 *Let (W_1, W_2, W_3, W_4, D) be a four-weight spin model. Then $(-W_1, W_2, -W_3, W_4, -D)$ and $(W_1, -W_2, W_3, -W_4, -D)$ are also four-weight spin models. For any oriented link $\mathbf{L}$, the link invariant associated with any of these two models differs from the link invariant associated with (W_1, W_2, W_3, W_4, D) by a sign factor $(-1)^{c(\mathbf{L})}$, where $c(\mathbf{L})$ denotes the number of components of $\mathbf{L}$.*

Proof: It is clear from Definition 1 that $(-W_1, W_2, -W_3, W_4, -D)$ and $(W_1, -W_2, W_3, -W_4, -D)$ are four-weight spin models (see also Proposition 3 of [2]). Since they are proportional, they have the same associated link invariant. Let us represent the oriented link $\mathbf{L}$ by the connected even special diagram $\mathbf{L}$. The partition functions of (W_1, W_2, W_3, W_4, D) and $(-W_1, W_2, -W_3, W_4, -D)$ on L are equal. Since these two spin models have opposite moduli and opposite loop variables, the associated link invariants differ by a factor $(-1)^{-T(L)-|B(L)|} = (-1)^{|V(L)|+|B(L)|}$. It is well known (and easy to prove by induction using the smoothing operation of figure 5) that, for any oriented diagram, the numbers of components of the corresponding link, of crossings, and of Seifert circles, add up to an even number. Since L is an even special diagram, it has $|B(L)|$ Seifert circles, and this shows that $(-1)^{|V(L)|+|B(L)|} = (-1)^{c(\mathbf{L})}$. □

Remarks

(i) The above result shows that in the definition of spin models we could ask without loss of generality that D be positive.
(ii) Let $i^2 = -1$ and let (W_1, W_2, W_3, W_4, D) be a four-weight spin model. Then $(iW_1, iW_2, -iW_3, -iW_4, -D)$ is also a four-weight spin model proportional to $(W_1, -W_2, W_3, -W_4, -D)$. Hence we obtain a new proof of Proposition 12 of [15]. Note as a special case that if (W_+, W_-, D) is a two-weight spin model, $(iW_+, -iW_-, -D)$ is also a two-weight spin model, and the two associated link invariants differ by the sign factor $(-1)^{c(\mathbf{L})}$ for every oriented link $\mathbf{L}$.

3.5. Invariance under mutation

The following definitions are essentially taken from [22].

Let $\mathbf{L}$ be a link in $\mathbb{R}^3$ such that there is a 2-sphere which meets $\mathbf{L}$ transversely in exactly four points. Up to ambient isotopy, we may assume that this 2-sphere is $\{(x, y, z) \in \mathbb{R}^3/x^2 + y^2 + z^2 = 2\}$, that its intersection with $\mathbf{L}$ consists of the four points $(1, 1, 0)$, $(-1, 1, 0)$, $(-1, -1, 0)$, $(1, -1, 0)$, and that $\mathbf{L}$ is orthogonal to the sphere at each of these points. We now apply to the interior of the sphere a rotation through angle π about one of the coordinate axes. The union of the part of $\mathbf{L}$ situated outside the sphere with the rotated part of $\mathbf{L}$ situated inside yields a new link $\mathbf{L}'$. If $\mathbf{L}$ is oriented, we keep its orientation outside the sphere, and inside the sphere either we keep it or we reverse it in order to obtain a consistent orientation for $\mathbf{L}'$. The new link $\mathbf{L}'$ is said to be obtained from $\mathbf{L}$ by a *mutation*. If we assume that the projection of $\mathbf{L}$ onto the (x, y)-plane is generic, the same holds for $\mathbf{L}'$, and we can easily describe mutations in terms of the corresponding diagrams: rotations through angle π about the x- or y-axis are replaced by plane reflections with respect to the same axis, rotation about the z-axis becomes rotation about the origin $(0, 0)$, and the indications of crossing structure at vertices are transferred in the obvious way. See figure 8 for an (unoriented) example.

The following result generalizes Proposition 5 of [14].

Proposition 5 *Any link invariant associated with a four-weight spin model is invariant under mutation.*

Proof: We consider a four-weight spin model (W_1, W_2, W_3, W_4, D) and we want to show that the associated link invariant takes the same value on two oriented links $\mathbf{L}$, $\mathbf{L}'$ related as above by a mutation. We may assume that this mutation corresponds to a rotation about the x- or y-axis since the composition of these two mutations will yield a mutation corresponding to a rotation about the z-axis. We consider diagrams L, L' which we may assume to be connected, obtained from generic projections of $\mathbf{L}$, $\mathbf{L}'$ onto the (x, y)-plane. The plane reflection (restricted to the disk $\{(x, y) \in \mathbb{R}^2/x^2 + y^2 \leq 2\}$) which transforms L into L' defines a bijection from the set of vertices (respectively: faces) of L to the set of vertices (respectively: faces) of L' and we denote by v' (respectively: f') the image of the vertex v (respectively: face f) of L under this bijection. We choose a black and white

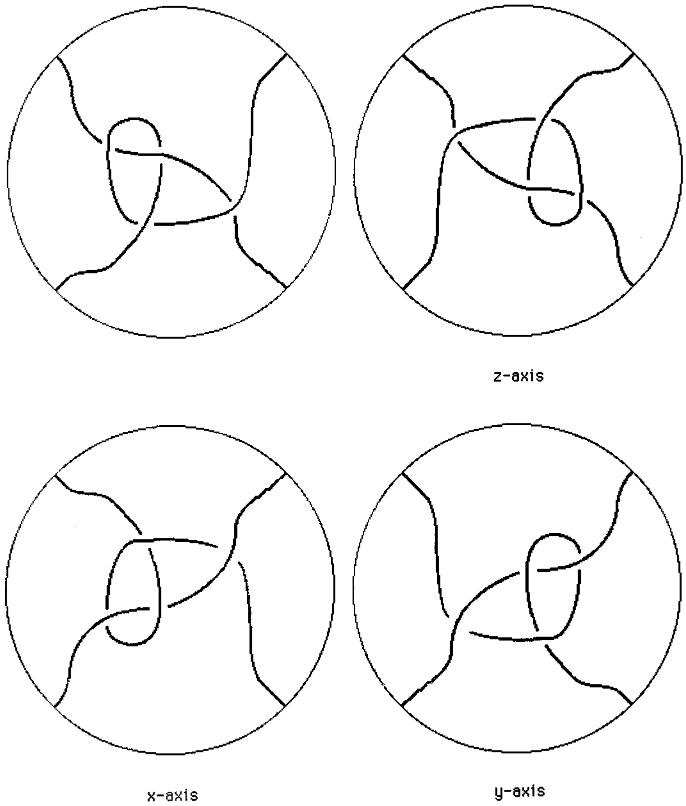

Figure 8. Examples of mutation.

face-coloring for L such that the points of intersection of the reflection axis with the circle $C = \{(x, y) \in \mathbb{R}^2/x^2 + y^2 = 2\}$ lie inside black faces which we denote by f_1, f_2. We have a similar black and white face-coloring for L' such that for every face f of L, f and f' have the same color. Drawing the reflection axis as vertical and exchanging the roles of **L**, **L**$'$ if necessary we can restrict our attention to one of the three situations depicted in figure 9(i) and (ii) or (iii) (this figure describes symbolically the effect of the plane reflection on the relevant part of the diagram L; to obtain L' all orientations must be reversed in the right-hand sides of (ii) and (iii)).

We define a bijection between the states of L and the states of L' by associating with every state σ of L the state σ' of L' such that $\sigma'(f') = \sigma(f)$ for every black face f of L. Let us consider a state σ and a vertex v of L. Clearly, if v lies outside the circle C, $\langle v', \sigma' \rangle = \langle v, \sigma \rangle$. On the other hand, if v lies inside the circle C, examination of figure 4 shows that (i) if v is of type W_i for $i = 1$ or $i = 3$, and if $\langle v, \sigma \rangle = W_i(x, y)$, then $\langle v', \sigma' \rangle = W_i(y, x)$ if the orientation of the part of L inside C is reversed to obtain L' from L, and $\langle v', \sigma' \rangle = W_i(x, y)$ otherwise; (ii) if v is of type W_i for $i = 2$ or $i = 4$, and if $\langle v, \sigma \rangle = W_i(x, y)$, then $\langle v', \sigma' \rangle = W_i(x, y)$ if the orientation of the part of L inside C is reversed to obtain L' from L, and $\langle v', \sigma' \rangle = W_i(y, x)$ otherwise.

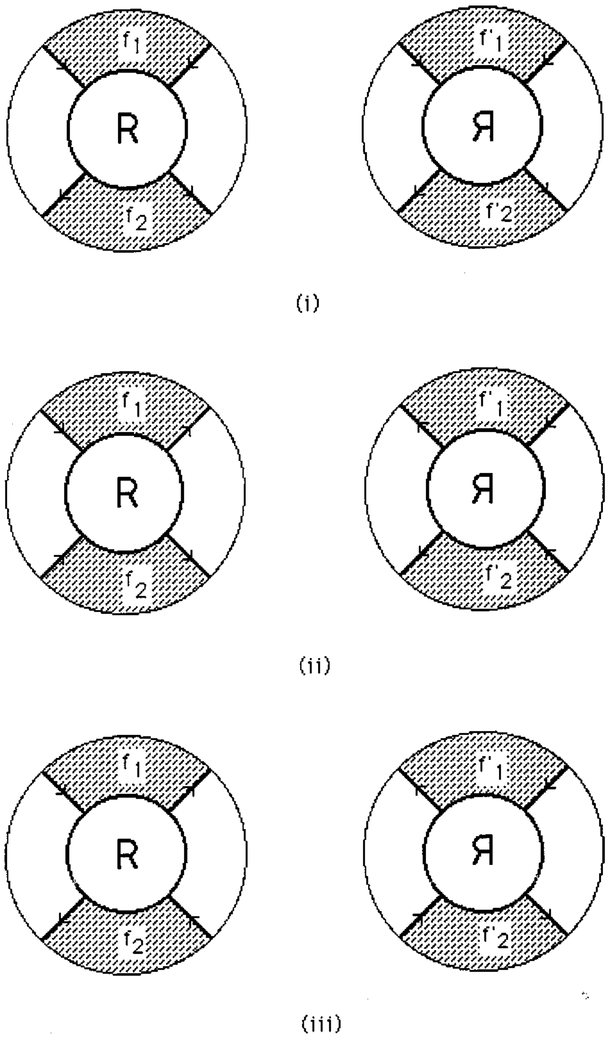

Figure 9. Reflections in the proof of Proposition 5.

Hence if in the situation of figure 9(i) all vertices v of L lying inside C are of type W_1 or W_3, then $Z(L) = Z(L')$. The same holds in the situations of figure 9(ii) and (iii) if all vertices v of L lying inside C are of type W_2 or W_4. Since clearly L and L' have the same number of black faces and the same Tait number, this will imply that the link invariant associated with (W_1, W_2, W_3, W_4, D) takes the same value on $\mathbf{L}$ and $\mathbf{L}'$.

Thus to finish the proof it is enough to show that if we complete the configurations appearing in the left-hand sides of figures 9(i)–(iii) into link diagrams by the addition of two edges in the exterior of C as shown in figures 10(i)–(iii) respectively, we can obtain special diagrams representing the same links without modifying the exterior of C. This is

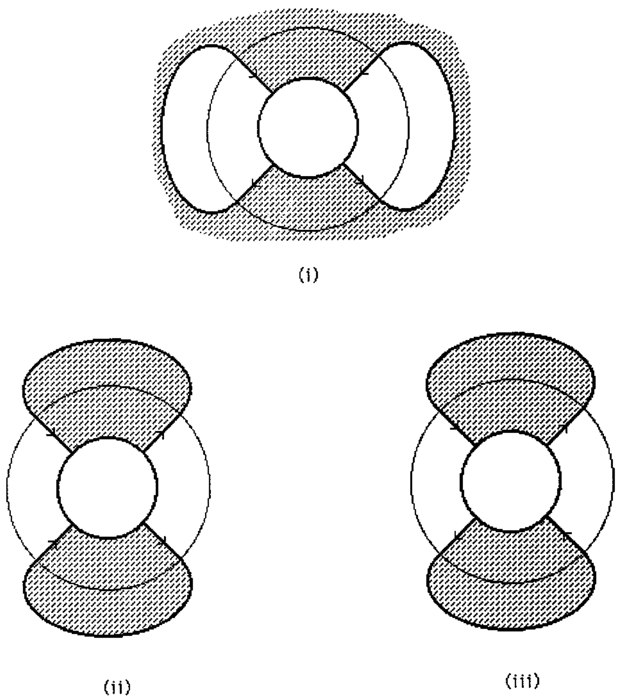

Figure 10. Complexing the proof of Proposition 5.

clear from the following adjustment of the proof of Proposition 1. Let us call a Seifert circle *external* if it meets the exterior of C and let us call it *internal* otherwise. Clearly there are at most two external Seifert circles and if there are two of them, no one lies in the interior of the other. Thus we may embed disjointly in the plane some disks whose boundaries are the Seifert circles without modifying the external Seifert circle(s) and in such a way that internal Seifert circles lie in the interior of C. □

4. Gauge transformations

In this section we give a more intrinsic description of transformations of four-weight spin models (W_1, W_2, W_3, W_4, D) which preserve (W_1, W_3, D) or (W_2, W_4, D). It turns out that these transformations are analogous to some *gauge transformations* considered in statistical mechanics and also introduced independenty for 4-weight spin models in [10].

In this section all spin models are defined on a given set X, and M_X denotes the set of complex matrices with rows and columns indexed by X.

4.1. Odd gauge transformations

Proposition 6 *Let (W_1, W_2, W_3, W_4, D) be a four-weight spin model and let W_1', W_3' be matrices in M_X. The following properties are equivalent*:

(i) $(W_1', W_2, W_3', W_4, D)$ *is a four-weight spin model*,
(ii) *there exists an invertible diagonal matrix* Δ *in* M_X *such that* $W_1' = \Delta W_1 \Delta^{-1}$, $W_3' = \Delta W_3 \Delta^{-1}$.

Proof: Assume first that (ii) holds. As already observed, Eq. (2) can be written as $W_1 W_3 = |X| I$ and hence is preserved by the replacement of W_i by $W_i' = \Delta W_i \Delta^{-1} (i = 1, 3)$. Eq. (3) is also preserved by this replacement since $W_1'(a, b) = \Delta(a, a) W_1(a, b)(\Delta(b, b))^{-1}$, $W_3'(b, a) = \Delta(b, b) W_3(b, a)(\Delta(a, a))^{-1}$. One sees similarly that $W_1'(b, a) W_3'(a, c) W_3'(c, b) = W_1(b, a) W_3(a, c) W_3(c, b)$ and $W_1'(a, b) W_3'(b, c) W_3'(c, a) = W_1(a, b) W_3(b, c) W_3(c, a)$, so that Eq. (8) is preserved. Hence $(W_1', W_2, W_3', W_4, D)$ is a four-weight spin model.

Conversely, assume that $(W_1', W_2, W_3', W_4, D)$ is a four-weight spin model. Then by (8), $W_1'(a, b) W_3'(b, c) W_3'(c, a) = W_1(a, b) W_3(b, c) W_3(c, a)$ for every a, b, c in X, or equivalently

$$W_1'(a, b) = W_3(c, a)(W_3'(c, a))^{-1} W_1(a, b) W_3(b, c)(W_3'(b, c))^{-1}. \tag{15}$$

Let us fix the element c of X and define the diagonal matrices Δ, Δ' in M_X by $\Delta(x, x) = W_3(c, x)(W_3'(c, x))^{-1}$, $\Delta'(x, x) = W_3(x, c)(W_3'(x, c))^{-1}$ for every x in X. Then (15) can be written as the equality $W_1' = \Delta W_1 \Delta'$.

For every a in X we have $W_1'(a, a) = \Delta(a, a) W_1(a, a) \Delta'(a, a)$ and this together with (10) shows that Δ and Δ' are inverse matrices. Thus $W_1' = \Delta W_1 \Delta^{-1}$ and the equality $W_3' = \Delta W_3 \Delta^{-1}$ follows from (2). □

When the equivalent properties (i), (ii) of Proposition 6 hold we shall say that the two four-weight spin models (W_1, W_2, W_3, W_4, D) and $(W_1', W_2, W_3', W_4, D)$ are related by an *odd gauge transformation*. In this case they have the same associated link invariant by Proposition 2. Actually one can easily see directly that a stronger property holds (see [10]). Consider a state σ of an oriented link diagram L. Replacing (W_1, W_2, W_3, W_4, D) by $(W_1', W_2, W_3', W_4, D)$ multiplies the weight of σ by a product of terms of the form $\Delta(\sigma(f_1), \sigma(f_1))\Delta(\sigma(f_2), \sigma(f_2))^{-1}$ (with Δ as in (ii) of Proposition 6). There is one such term for each vertex of type W_1 or W_3 where the two incoming edges are incident with the black face f_1 and the two outcoming edges are incident with the black face f_2 (see figure 4). Then clearly for every black face f the total exponent of $\Delta(\sigma(f), \sigma(f))$ in this product of terms is zero. Thus an odd gauge transformation preserves the weight of each state.

Proposition 7 *Let* (W_1, W_2, W_3, W_4, D) *be a four-weight spin model.*
(i) *There exists an invertible diagonal matrix* Δ *in* M_X *such that* ${}^tW_1 = \Delta W_1 \Delta^{-1}$, ${}^tW_3 = \Delta W_3 \Delta^{-1}$.
(ii) *There exists a four-weight spin model* $(W_1', W_2, W_3', W_4, D)$ *such that* W_1', W_3' *are symmetric.*

Proof: We have seen in the proof of Proposition 3 that $({}^tW_1, W_2, {}^tW_3, W_4, D)$ is a four-weight spin model, and hence (i) is an immediate consequence of Proposition 6.

To prove (ii), we look for an invertible diagonal matrix Δ' in M_X such that $W_1' = \Delta' W_1 \Delta'^{-1}$ is symmeric (then by (2) $W_3' = \Delta' W_3 \Delta'^{-1}$ will also be symmetric). This can be written as $\Delta'^{-1t}W_1\Delta' = \Delta' W_1 \Delta'^{-1}$, or equivalently ${}^tW_1 = \Delta'^2 W_1 \Delta'^{-2}$, which will hold by (i) whenever $\Delta'^2 = \Delta$. □

Thus to classify four-weight spin models (W_1, W_2, W_3, W_4, D) up to odd gauge transformations we may restrict our attention to the case where W_1, W_3 are symmetric. Then, by Theorem 1 of [28], the problem can be reformulated in terms of symmetric two-weight models, for which some general classification results are known (see [14, 16, 24] and Section 5.1 of the present paper).

We also note that the number of four-weight spin models $(W_1', W_2, W_3', W_4, D)$ which appear in Proposition 7(ii) is finite. Indeed, since W_1 has non-zero entries, any diagonal matrix which commutes with W_1 is a scalar multiple of the identity. Hence the equation ${}^tW_1 = \Delta' 2 W_1 \Delta'^{-2}$ (with notations as in the proof of Proposition 7) defines Δ'^2 up to scalar multiplication.

4.2. Even gauge transformations

Proposition 8 *Let (W_1, W_2, W_3, W_4, D) be a four-weight spin model and let W_2', W_4' be matrices in M_X. The following properties are equivalent*:

(i) *$(W_1, W_2', W_3, W_4', D)$ is a four-weight spin model*,

(ii) *there exist permutation matrices P, Q in M_X such that $W_2' = PW_2 = W_2Q$, $W_4' = W_4{}^tP = {}^tQW_4$*,

(iii) *there exist a permutation matrix P and an invertible diagonal matrix Δ in M_X such that $PW_1P^{-1} = \Delta W_1 \Delta^{-1}$ and $W_2' = PW_2$, $W_4' = W_4{}^tP$*,

(iv) *there exist a permutation matrix Q and an invertible diagonal matrix Δ in M_X such that $Q^{-1}W_1Q = \Delta W_1 \Delta^{-1}$ and $W_2' = W_2Q$, $W_4' = {}^tQW_4$*.

Proof: (i) implies (ii): Four every a, b, c in X let $p_{ab}^c = \sum_{x\in X} W_2(x, a)W_2(x, b)W_4(c, x) = \sum_{x\in X} W_2'(x, a)W_2'(x, b)W_4'(c, x)$ (this is well defined by (8)). Let $\mathbf{A}$ be a complex vector space with basis $\{A_x, x \in X\}$ indexed by X. Define a bilinear product on $\mathbf{A}$ by the following rule for basis elements: for every a, b in X, $A_aA_b = \sum_{c\in X} P_{ab}^c A_c$. We introduce for every i in X the element $E_i = |X|^{-2} \sum_{a\in X} W_4(a, i)A_a$ of $\mathbf{A}$. Since W_4 is invertible by (4) (which can be written as $W_2W_4 = |X|I$), $\{E_i, i \in X\}$ is a basis of $\mathbf{A}$. Then

$$
\begin{aligned}
E_iE_j &= |X|^{-4}\left(\sum_{a\in X} W_4(a, i)A_a\right)\left(\sum_{b\in X} W_4(b, j)A_b\right)\\
&= |X|^{-4}\sum_{a\in X}\sum_{b\in X} W_4(a, i)W_4(b, j)\left(\sum_{c\in X} P_{ab}^c A_c\right)\\
&= |X|^{-4}\sum_{c\in X}\sum_{a\in X}\sum_{b\in X} W_4(a, i)W_4(b, j)\left(\sum_{x\in X} W_2(x, a)W_2(x, b)W_4(c, x)\right)A_c\\
&= |X|^{-4}\sum_{c\in X}\sum_{x\in X}\sum_{a\in X}\sum_{b\in X} W_2(x, a)W_4(a, i)W_2(x, b)W_4(b, j)W_4(c, x)A_c
\end{aligned}
$$

$$= |X|^{-2} \sum_{c \in X} \sum_{x \in X} \delta(x, i)\, \delta(x, j) W_4(c, x) A_c \quad \text{(by (4))}$$
$$= \delta(i, j) |X|^{-2} \sum_{c \in X} W_4(c, i) A_c = \delta(i, j) E_i.$$

Hence **A** is a semisimple commutative associative algebra with basis of orthogonal idempotents $\{E_i, i \in X\}$. Let $E_i' = |X|^{-2} \sum_{a \in X} W_4'(a, i) A_a$ for every i in X. The same proof shows that $\{E_i', i \in X\}$ is a basis of orthogonal idempotents of **A**. But such a basis is unique and hence there exists a permutation π of X such that $E_i' = E_{\pi(i)}$ or equivalently $\sum_{a \in X} W_4'(a, i) A_a = \sum_{a \in X} W_4(a, \pi(i)) A_a$ for every i in X. Then $W_4'(a, i) = W_4(a, \pi(i))$ for every a, i in X. Let P be the permutation matrix in M_X defined by $P(x, y) = \delta(y, \pi(x))$ for every x, y in X. We obtain the equation $W_4' = W_4{}^t P$, and the equation $W_2' = P W_2$ is then obtained from (4). To establish the existence of the permutation matrix Q such that $W_2' = W_2 Q$, $W_4' = {}^t Q W_4$, we transpose the matrices W_2, W_2', W_4, W_4' in the above proof.

(ii) implies (iii): With the same notations as in the above proof, we have $W_2'(x, y) = W_2(\pi(x), y)$ and $W_4'(x, y) = W_4(x, \pi(y))$ for all x, y in X. Let θ be the permutation of X such that $Q(x, y) = \delta(x, \theta(y))$ for all x, y in X. We also have $W_2'(x, y) = W_2(x, \theta(y))$, $W_4'(x, y) = W_4(\theta(x), y)$. Then, by (8),for every a, b, c in X,

$$\begin{aligned} &D W_1(\pi(a), \pi(b)) W_3(\pi(b), \pi(c)) W_3(\pi(c), \pi(a)) \\ &\quad = \sum_{x \in X} W_2(\pi(a), x) W_2(\pi(b), x) W_4(x, \pi(c)) \\ &\quad = \sum_{x \in X} W_2'(a, x) W_2'(b, x) W_4'(x, c) \\ &\quad = \sum_{x \in X} W_2(a, \theta(x)) W_2(b, \theta(x)) W_4(\theta(x), c) \\ &\quad = \sum_{x \in X} W_2(a, x) W_2(b, x) W_4(x, c). \end{aligned}$$

Let $W_1' = P W_1 P^{-1}$, $W_3' = P W_3 P^{-1}$. The above equations, together with obvious verifications of Eqs. (2) and (3), show that $(W_1', W_2, W_3', W_4, D)$ is a four-weight spin model. The result now follows from Proposition 6.

(ii) implies (iv): The proof is exactly similar to the previous one.

(iii) implies (i): It is easy to check that Eqs. (4) and (5) are preserved by the replacement of W_2 by $W_2' = P W_2$ and of W_4 by $W_4' = W_4{}^t P$. Now let $W_1' = P W_1 P^{-1} = \Delta W_1 \Delta^{-1}$, $W_3' = P W_3 P^{-1}$, so that $W_3' = \Delta W_3 \Delta^{-1}$ by (2). By Proposition 6, $(W_1', W_2, W_3', W_4, D)$ is a four-weight spin model. Introducing again the permutation π of X such that $P(x, y) = \delta(y, \pi(x))$ for every x, y in X, we get from Eq. (8):

$$\begin{aligned} &\sum_{x \in X} W_2'(a, x) W_2'(b, x) W_4'(x, c) \\ &\quad = \sum_{x \in X} W_2(\pi(a), x) W_2(\pi(b), x) W_4(x, \pi(c)) \\ &\quad = D W_1(\pi(a), \pi(b)) W_3(\pi(b), \pi(c)) W_3(\pi(c), \pi(a)) \end{aligned}$$

$$= DW_1'(a, b)W_3'(b, c)W_3'(c, a)$$
$$= \sum_{x \in X} W_2(a, x)W_2(b, x)W_4(x, c).$$

Moreover

$$\sum_{x \in X} W_2'(x, a)W_2'(x, b)W_4'(c, x) = \sum_{x \in X} W_2(\pi(x), a)W_2(\pi(x), b)W_4(c, \pi(x))$$
$$= \sum_{x \in X} W_2(x, a)W_2(x, b)W_4(c, x).$$

Thus (8) is preserved by the replacement of (W_2, W_4) by (W_2', W_4') and $(W_1, W_2', W_3, W_4', D)$ is a four-weight spin model.

(iv) implies (i): The proof is exactly similar to the previous one. □

When the equivalent properties (i) to (iv) of Proposition 8 hold we shall say that the two four-weight spin models (W_1, W_2, W_3, W_4, D) and $(W_1, W_2', W_3, W_4', D)$ are related by an *even gauge transformation*. Then they have the same associated link invariant by Proposition 2. In contrast with the case of odd gauge transformations, we have no direct proof of this which would work for arbitrary link diagrams. See however [10] for even special diagrams (a special case of even gauge transformations is considered there, but the proof is easily generalized).

Note that the number of even gauge tranformations which can be performed on a given four-weight spin model is finite. More precisely, such transformations involve certain symmetries of the spin model which we consider now in detail.

We denote by S_X the group of permutation matrices in M_X.

Proposition 9 *Let (W_1, W_2, W_3, W_4, D) be a four-weight spin model. The groups $S_X \cap (W_2 S_X W_2^{-1})$ and $S_X \cap (W_2^{-1} S_X W_2)$ are both equal to the set of matrices P in S_X such that $PW_1P^{-1} = \Delta W_1 \Delta^{-1}$ for some invertible diagonal matrix Δ in M_X.*

Proof: The set of matrices P in S_X such that $PW_1P^{-1} = \Delta W_1 \Delta^{-1}$ for some invertible diagonal matrix Δ in M_X forms a group. Indeed if $PW_1P^{-1} = \Delta W_1 \Delta^{-1}$ and $P'W_1P'^{-1} = \Delta' W_1 \Delta'^{-1}$, then $(PP')W_1(PP')^{-1} = (P\Delta' P^{-1})\Delta W_1 \Delta^{-1}(P\Delta' P^{-1})^{-1}$. By Proposition 8, the matrix P in S_X belongs to this group iff $(W_1, PW_2, W_3, W_4{}^tP, D)$ is a four-weight spin model iff $W_2^{-1}PW_2$ is some permutation matrix Q. Similarly, the matrix Q in S_X belongs to this group iff Q^{-1} belongs to this group iff $(W_1, W_2Q, W_3, {}^tQW_4, D)$ is a four-weight spin model iff $W_2QW_2^{-1}$ is some permutation matrix P. □

Thus an even gauge transformation of the four-weight spin model (W_1, W_2, W_3, W_4, D) corresponds to a left (or equivalently right) multiplication of W_2 by some element of the group introduced in Proposition 9, together with the corresponding similar transformation for W_4. The "gauge transformations 2" of [10] are an interesting special case.

Note that when (W_+, W_-, D) is a two-weight spin model, Proposition 9 states an interesting property of the matrix $W_+ = W_1 = W_2$.

For every M in M_X, the *automorphism group of M*, denoted by Aut(M), is the group of matrices in S_X which commute with M. We observe that both Aut(W_1) and Aut(W_2) are subgroups of the group of Proposition 9.

Proposition 10 *Let (W_1, W_2, W_3, W_4, D) be a four-weight spin model.*

(i) *There exists a matrix P in* Aut(W_2) *such that ${}^tW_2 = PW_2$. Hence $W_2{}^tW_2 = {}^tW_2W_2$, some power of W_2 is symmetric, and if* Aut(W_2) *is trivial, W_2 is symmetric.*
(ii) *If P is a square in* Aut(W_2), *there exists a four-weight spin model $(W_1, W_2', W_3, W_4', D)$ such that W_2', W_4' are symmetric.*

Proof:

(i) We have seen in the proof of Proposition 3 that $(W_1, {}^tW_2, W_3, {}^tW_4, D)$ is a four-weight spin model. By Proposition 8, there is a permutation matrix P in M_X such that ${}^tW_2 = PW_2$. Transposing and multiplying by P on the right, we obtain $W_2P = {}^tW_2$. The remaining statements are clear.
(ii) If $P = Q^2$ with Q in Aut(W_2), ${}^tW_2 = Q^2W_2 = QW_2Q$ and hence ${}^t(W_2Q) = W_2Q$. We can take $W_2' = W_2Q$, $W_4' = {}^tQW_4$. □

4.3. Gauge equivalence of spin models

The following result is clear from Propositions 2, 6, 8.

Proposition 11 *Let (W_1, W_2, W_3, W_4, D) be a four-weight spin model. Let P a permutation matrix in M_X such that $W_2^{-1}PW_2$ is also a permutation matrix, Δ be an invertible diagonal matrix in M_X, and λ be a non-zero complex number. Then $(\lambda\Delta W_1\Delta^{-1}, \lambda^{-1}PW_2, \lambda^{-1}\Delta W_3\Delta^{-1}, \lambda W_4{}^tP, D)$ is also a four-weight spin model which has the same associated link invariant as (W_1, W_2, W_3, W_4, D).*

The two four-weight spin models appearing in Proposition 11 will be said to be *gauge equivalent*. Thus two four-weight spin models are gauge equivalent if, up to proportionality, one can be obtained from the other by a sequence of even and odd gauge transformations. Similarly, we shall say that the two-weight spin models (W_+, W_-, D) and (W_+', W_-', D) are gauge equivalent if the corresponding four-weight spin models (W_+, W_+, W_-, W_-, D) and $(W_+', W_+', W_-', W_-', D)$ are gauge equivalent.

Let us illustrate Proposition 11 with a simple example. It is easy to check (see [18]) that when $D = -\alpha^2 - \alpha^{-2}$, $W_+ = \alpha DI + \alpha^{-1}J$, $W_- = \alpha^{-1}DI + \alpha J$, where all entries of J are equal to 1, (W_+, W_-, D) is a (symmetric) 2-weight spin model with modulus $-\alpha^3$. Up to a change of variables, the associated link invariant is the Jones polynomial introduced in [17]. Since Aut$(W_+) = S_X$, for every permutation matrix P and invertible diagonal matrix Δ in M_X we obtain a four-weight spin model $(-\alpha^{-3}\Delta W_+\Delta^{-1}, -\alpha^3PW_+, -\alpha^3\Delta W_-\Delta^{-1}, -\alpha^{-3} W_-{}^tP, D) = (-\alpha^{-2}DI - \alpha^{-4}\Delta J\Delta^{-1}, -\alpha^4DP - \alpha^2J, -\alpha^2DI - \alpha^4\Delta J\Delta^{-1}, -\alpha^{-4}D{}^tP - \alpha^{-2}J, D) = (W_1, W_2, W_3, W_4, D)$ of modulus 1 with the same associated link invariant. The identification of this link invariant can be obtained directly as follows. Let L_+, L_-

and L_0 be three oriented link diagrams such that L_- is obtained from L_+ by changing the sign of one crossing from positive to negative (see figure 2), and L_0 is obtained from L_+ by smoothing out the same crossing (see figure 5). Then it is easy to show, using the equations $\alpha^4 W_1 - \alpha^{-4} W_3 = (\alpha^{-2} - \alpha^2) DI, \alpha^{4t} W_4 - \alpha^{-4} W_2 = (\alpha^{-2} - \alpha^2) J$, that, assuming L_0 connected, $\alpha^4 D^{-|B(L_+)|} Z(L_+) - \alpha^{-4} D^{-|B(L_-)|} Z(L_-) = (\alpha^{-2} - \alpha^2) D^{-|B(L_0)|} Z(L_0)$. This is precisely (for a suitable choice of variable) the defining relation for the Jones polynomial.

5. Gauge equivalence of some two-weight spin models

5.1. Two-weight spin models and Bose-Mesner algebras

Let X be a finite non-empty set. The set M_X of complex matrices with rows and columns indexed by X is considered as usual as a vector space over the complex numbers. The *Hadamard product* of two matrices A, B in M_X, denoted by $A \circ B$, is defined by $(A \circ B)(x, y) = A(x, y)B(x, y)$ for every x, y in X. The identity element for this associative and commutative product is the matrix J with all entries equal to 1. We shall call a *Bose-Mesner algebra* on X any vector subspace **B** of of M_X containing I and J which is closed under transposition, Hadamard product, and ordinary matrix product, this second product being commutative on **B**.

Every Bose-Mesner algebra **B** has a basis $\{A_i, i = 0, \ldots, d\}$ such that $A_i \circ A_j = \delta(i, j)\, A_i$ and $A_0 = I$. The matrices A_i, called the *primitive Hadamard idempotents* of **B**, can be viewed as the adjacency matrices of some relations on X which form a (*commutative*) *association scheme* (see [4, 6] for definitions). The notions of (commutative) association schemes and Bose-Mesner algebras are completely equivalent (the proof given in Theorem 2.6.1 of [6] for the symmetric case can easily be extended), but it is more convenient for our purposes to work in the framework of Bose-Mesner algebras.

A *duality* of a Bose-Mesner algebra **B** on X is a linear map Ψ from **B** to itself which satisfies the following properties:

For every matrix M in **B**, $\quad \Psi(\Psi(M)) = |X|\,{}^tM, \qquad (16)$

For any two matrices M, N in **B**, $\quad \Psi(MN) = \Psi(M) \circ \Psi(N). \qquad (17)$

It easily follows that

For any two matrices M, N in **B**, $\quad \Psi(M \circ N) = |X|^{-1} \Psi(M)\Psi(N), \qquad (18)$

$$\Psi(I) = J, \qquad (19)$$

$$\Psi(J) = |X| I. \qquad (20)$$

It is shown in [16] (see also [14, 24]) that if (W_+, W_-, D) is a two-weight spin model on X with modulus μ, there exists a Bose-Mesner algebra **B** on X which contains W_+, W_- and admits a duality Ψ given by the expression

$\Psi(M) = \mu\, {}^tW_- \circ (W_+(W_- \circ M)) \quad$ for every M in **B** $\qquad (21)$

Remarks

(i) To obtain the expression (21), which we prefer to that given in Theorem 11 of [16] since it is more convenient for the proof of Proposition 12 below and appears previously in [3], we consider the two-weight spin model $({}^tW_-, {}^tW_+, D)$ instead of (W_+, W_-, D) (this is allowed by Theorem 9 of [2] or Proposition 2 of [21]).

(ii) (21) implies that μ is the modulus of (W_+, W_-, D). To see this, we apply (21) to $M = I$, use (13), (12) and compare with (19).

When W_+, W_- belong to **B** and (21) holds we shall say that (W_+, W_-, D) satisfies the *modular invariance property with* respect to the pair $(\mathbf{B}, \Psi)$ (see [3]).

5.2. A general equivalence result

Proposition 12 *Let* **B** *be a Bose-Mesner algebra on* X *which admits a duality* Ψ, *let* (W_+, W_-, D) *be a two-weight spin model which satisfies the modular invariance property with respect to* $(\mathbf{B}, \Psi)$, *and let* P *be a permutation matrix in* **B**. *There is a non-zero complex number* λ *such that* $(\lambda^{-1}PW_+, \lambda W_-{}^tP, D)$ *is a two-weight spin model which is gauge equivalent to* (W_+, W_-, D) *and satisfies the modular invariance property with respect to* $(\mathbf{B}, \Psi)$.

Proof: A four-weight spin model which is gauge equivalent to (W_+, W_+, W_-, D) is of the form $(\lambda\Delta W_+\Delta^{-1}, \lambda^{-1}PW_+, \lambda^{-1}\Delta W_-\Delta^{-1}, \lambda W_-{}^tP, D)$,where P is a permutation matrix in M_X such that $W_+^{-1}PW_+$ is also a permutation matrix, Δ is an invertible diagonal matrix in M_X and λ is a non-zero complex number. We fix P in **B**, so that $W_+^{-1}PW_+ = P$ is also a permutation matrix, and we look for Δ and λ as above such that $\lambda\Delta W_+\Delta^{-1} = \lambda^{-1}PW_+$ (the other equality $\lambda^{-1}\Delta W_-\Delta^{-1} = \lambda W_-{}^tP$ will then follow from (11)).

By (18), $\Psi(P)^2 = |X|\Psi(P \circ P) = |X|\Psi(P)$. Hence $E = |X|^{-1}\Psi(P)$ is an idempotent (for the ordinary matrix product). Moreover by (20) and (17), $|X|I = \Psi(J) = \Psi(JP) = \Psi(J) \circ \Psi(P) = |X|I \circ \Psi(P)$, hence, Trace $E = |X|^{-1}$ Trace $\Psi(P) = 1$ and E has rank 1. The fact that $\Psi(P)$ has rank 1 implies that there exist diagonal matrices Δ, Δ' in M_X such that $\Psi(P)(x, y) = \Delta(x, x)\Delta'(y, y)$ for all x, y in X. Moreover since $I \circ \Psi(P) = I$, Δ and Δ' are inverse matrices. Hence Δ is invertible and $\Delta W_+\Delta^{-1} = \Psi(P) \circ W_+$, $\Delta W_-\Delta^{-1} = \Psi(P) \circ W_-$.

If we express P in the basis of primitive Hadamard idempotents of **B**, we see that exactly one of the coefficients is equal to 1 while the others are zero. In other words, P belongs to this basis. Let β be the coefficient of P in the expression of W_- in the same basis. Then $W_- \circ P = \beta P$. Now by (21), $\Psi(P) = \mu {}^tW_- \circ (W_+(W_- \circ P)) = \beta\mu {}^tW_- \circ (W_+P)$. Hence, by (12), $\Delta W_+\Delta^{-1} = \Psi(P) \circ W_+ = \beta\mu W_+P$ and the required equality $\lambda\Delta W_+\Delta^{-1} = \lambda^{-1}PW_+$ is satisfied when $\lambda^2 = (\beta\mu)^{-1}$.

Thus we have shown that, writing $W'_+ = \lambda^{-1}PW_+ = \lambda\Psi(P) \circ W_+$ and $W'_- = \lambda W_-{}^tP = \lambda^{-1}\Psi(P) \circ W_-$, (W'_+, W'_-, D) is a two-weight spin model which is gauge equivalent to

(W_+, W_-, D). Let M be a matrix in **B**. Then

$$\begin{aligned}
{}^tW'_- \circ (W'_+(W'_- \circ M)) &= (\lambda P\,{}^tW_-) \circ ((\lambda^{-1}PW_+)(W'_- \circ M)) \\
&= (P\,{}^tW_-) \circ (P(W_+(W'_- \circ M))) \\
&= P({}^tW_- \circ (W_+(W'_- \circ M))) \\
&= P({}^tW_- \circ (W_+(\lambda^{-1}\Psi(P) \circ W_- \circ M))) \\
&= \lambda^{-1}P({}^tW_- \circ (W_+(W_- \circ (\Psi(P) \circ M)))) \\
&= \lambda^{-1}P\mu^{-1}\Psi(\Psi(P) \circ M)) \quad \text{(by (21))} \\
&= (\lambda\mu)^{-1}P|X|^{-1}\Psi^2(P)\Psi(M) \quad \text{(by (18))} \\
&= (\lambda\mu)^{-1}P\,{}^tP\Psi(M) = (\lambda\mu)^{-1}\Psi(M) \quad \text{(by (16))}.
\end{aligned}$$

Hence (W'_+, W'_-, D) satisfies the modular invariance property with respect to $(\mathbf{B}, \Psi)$. □

The permutation matrices in **B** form an Abelian group of size at most dim **B**, $\leq |X|$. The extremal case where this group is of size $|X|$ is of special interest and is studied below.

5.3. Equivalence of some two-weight spin models on Abelian groups

In this section we assume that X is an Abelian group written additively. For every i in X define the matrix A_i in M_X by $A_i(x, y) = \delta(i, y - x)$ for every x, y in X. The complex linear span **B** of the matrices A_i is easily seen to be a Bose-Mesner algebra on X, with basis of primitive Hadamard idempotents $\{A_i, i \in X\}$. It is also easy to show (see for instance [3]) that **B** admits dualities, all of which can be described as follows (a more explicit classification is given in [5]). It is possible to index the characters of X with the elements of X in such a way that, denoting by χ_i the character indexed by i, the equality $\chi_i(j) = \chi_j(i)$ holds for all i, j in X. Then one defines the linear map Ψ from **B** to itself by the equalities $\Psi(A_i) = \sum_{j \in X} \chi_i(j) A_j$. It is easy to check that Ψ is a duality. We now assume that such a duality Ψ is given.

The spin models studied in the next result have appeared in several work: [11, 18] (symmetric models in the cyclic group case), [1] (general cyclic group case), [3, 13]. See also [19] for a related construction and [9, 10, 27] for some connections with physics.

The following result was motivated by information received from Eiichi and Etsuko Bannai and Takashi Takamuki. It is related (this follows from [3]) with the result stated at the end of [10] on the spin models of [19]. It is also related with the result of [27] that the link invariant introduced there depends only trivially on link orientation.

Proposition 13 *There are exactly* $2|X|$ *two-weight spin models which satisfy the modular invariance property with respect to* $(\mathbf{B}, \Psi)$. *They are mutually gauge equivalent. One of them is symmetric and consequently the link invariant associated with these two-weight spin models depends only trivially* (*i.e. via the normalization factor* $\mu^{-T(L)}$, *where* μ *is the modulus*) *on link orientation.*

Proof: Two-weight spin models which satisfy the modular invariance property with respect to ($\mathbf{B}$, Ψ) are described explicitly in [3]. To be as self-contained as possible we shall only use here the fact that there exists at least one such spin model, say (W_+, W_-, D). Let us write $W_+ = \sum_{j\in X} t_j A_j$. Note that the t_j are non-zero by (12) and that (W_+, W_-, D) has modulus t_0 by (14). By (21), for every i in X, $\Psi({}^tW_+ \circ {}^tA_i) = t_0 {}^tW_- \circ (W_+(W_- \circ {}^tW_+ \circ {}^tA_i))$. Using (12) we get

$$W_+ \circ \Psi\left({}^tW_+ \circ {}^tA_i\right) = t_0 W_+ {}^tA_i.$$

Now

$$\Psi\left({}^tW_+ \circ {}^tA_i\right) = \Psi\left(t_i {}^tA_i\right) = t_i \Psi\left({}^tA_i\right) = t_i \Psi(A_{-i}) = t_i \sum_{j\in X} \chi_{-i}(j) A_j$$

and thus

$$W_+ \circ \Psi\left(tW_+ \circ {}^tA_i\right) = \sum_{j\in X} t_i t_j \chi_{-i}(j) A_j.$$

On the other hand

$$W_+ {}^tA_i = W_+ A_{-i} = \sum_{j\in X} t_j A_{j-i} = \sum_{j\in X} t_{i+j} A_j.$$

It follows that

$$t_i t_j \chi_{-i}(j) = t_0 t_{i+j} \quad \text{for every } i, j \text{ in } X. \tag{22}$$

Writing $s_i = t_i t_0^{-1}$ this becomes

$$s_i s_j \chi_{-i}(j) = s_{i+j} \quad \text{for every } i, j \text{ in } X. \tag{23}$$

Let now (W'_+, W'_-, D) be another two-weight spin model which satisfies the modular invaiance property with respect to ($\mathbf{B}$, Ψ). Let us write $W'_+ = \sum_{j\in X} t'_j A_j$, where the t'_j are non-zero, and $s'_i = t'_i t_0'^{-1}$.

We define a map ζ from X to the complex numbers by $\zeta(i) = s'_i s_i^{-1}$ for every i in X. We get from (23) the identity $\zeta(i)\zeta(j) = \zeta(i+j)$. Hence ζ is a character of X and there exists k in X such that $\zeta = \chi_k$. Then

$$\begin{aligned} t_0'^{-1} W'_+ &= \sum_{j\in X} S'_j A_j = \sum_{j\in X} \zeta(j) s_j A_j = \sum_{j\in X} \chi_k(j) S_j A_j \\ &= \left(\sum_{j\in X} \chi_k(j) A_j\right) \circ \left(\sum_{j\in X} s_j A_j\right) = \Psi(A_k) \circ \left(t_0^{-1} W_+\right). \end{aligned}$$

Thus

$$W'_+ = t'_0 t_0^{-1} \Psi(A_k) \circ W_+.$$

From the proof of Proposition 12, $\Psi(A_k) \circ W_+ = \beta\mu W_+ A_k$, where β is the coefficient of A_k in the expression of W_- in the basis $\{A_i, i \in X\}$ and μ is the modulus of (W_+, W_-, D). We have seen that $\mu = t_0$, and by (12) $\beta = t_{-k}^{-1}$. Hence

$$W'_+ = t'_0 t_0^{-1} \Psi(A_k) \circ W_+ = t'_0 t_{-k}^{-1} A_k W_+.$$

If follows that

$$\sum_{i \in X} t'_i = t'_0 t_{-k}^{-1} \sum_{i \in X} t_{i+k} = t'_0 t_{-k}^{-1} \sum_{i \in X} t_i.$$

On ther other hand by (13),

$$\sum_{i \in X} t'_i = D {t'_0}^{-1} \quad \text{and} \quad \sum_{i \in X} t_i = D t_0^{-1}.$$

Hence ${t'_0}^{-1} = t'_0 t_{-k}^{-1} t_0^{-1}$. It follows that t'_0 is one of the two square roots of $t_{-k} t_0$ and that we may write $W'_+ = \lambda_k \Psi(A_k) \circ W_+ = \lambda_k^{-1} A_k W_+$ with $\lambda_k = t'_0 t_0^{-1} = {t'_0}^{-1} t_{-k}$. Hence there are at most $2|X|$ possibilities for W'_+. Since W_+ is invertible by (11), $A_{K'} W_+$ is not a scalar multiple of $A_K W_+$ when $k' \neq k$, and we have just exhibited $2|X|$ distinct possibilities for W'_+. Finally it is clear from the proof of Proposition 12 that these possibilities actually correspond to two weight spin models which satisfy the modular invariance property with respect to $(\mathbf{B}, \Psi)$ and which are gauge equivalent to (W_+, W_-, D).

Finally, let us show that one of the above possibilities for W'_+ is symmetric. We want to find k in X such that $A_K W_+ = \sum_{j \in X} t_j A_{j+k} = \sum_{j \in X} t_{j-k} A_j$ is symmetric, or equivalently $t_{j-k} = t_{-j-k}$ for every j in X. Using (22) this can be written: $t_0^{-1} t_j t_{-k} \chi_{-j}(-k) = t_0^{-1} t_{-j} t_{-k} \chi_j(-k)$, or equivalently $t_j \chi_{-j}(-k) = t_{-j} \chi_j(-k)$ for every j in X.

Thus we want to find k in X such that $t_j t_{-j}^{-1} = \chi_j(-k)\chi_{-j}(-k)^{-1} = \chi_j(-k)^2 = \chi_{-k}(j)^2$ for every j in X. In other words, defining the map η from X to the complex numbers by $\eta(j) = t_j t_{-j}^{-1}$ for every j in X, we want to show that η is a square in the group $X^\wedge$ of characters of X.

By (22), $\eta(i)\eta(j) = t_i t_j (t_{-i} t_{-j})^{-1} = \chi_{-1}(j)^{-1} t_0 t_{i+j} (\chi_i(-j)^{-1} t_0 t_{-i-j})^{-1} = \eta(i+j)$ and hence η is a character. Let φ be the endomorphism of X defined by $\varphi(i) = 2i$ for every i in X, let G be the subgroup of squares in $X^\wedge$ and let H be the subgroup of characters of X which take the value 1 on Kerφ. Since $\chi^2(i) = \chi(\varphi(i))$ for every character χ, $G \subseteq H$. Moreover G is isomorphic to Im φ, and H is isomorphic to the group of characters of $X/\text{Ker}\varphi$, so that $|G| = |H|$. Now η belongs to H, since for every i in Kerφ, $\eta(i) = t_i t_{-1}^{-1} = t_i t_i^{-1} = 1$, and hence η belongs to G. □

6. Conclusion

We hope to have convinced the reader that it is worthwhile to consider two-weight spin models as a special case of four-weight spin models, because we then have at our disposal the powerful tool of gauge transformations. However we believe that general four-weight spin models are also interesting for their own sake. To support this belief it would be nice to exhibit a link invariant which can be obtained from a four-weight spin model and not from a two-weight spin model. The main difficulty here is to find a criterion to show that a link invariant cannot be associated with some two-weight spin model, which does not show at the same time that the link invariant cannot be associated with some four-weight spin model. But another difficulty is the lack of known examples of four-weight spin models (excluding of course those which can be obtained from two-weight spin models by gauge transformations). See however [2, 15, 28].

Thus more research on explicit constructions of four-weight spin models is needed. However, some theoretical aspects are worth investigating as well. In particular the present work paves the way to new axiomatizations of the notion of four-weight spin model, which would bear on one of the pairs (W_1, W_3) or (W_2, W_4) alone, or even better on the 3-tensors appearing in (8). Maybe such axiomatizations could lead to a better topological understanding of the associated link invariants.

References

1. E. Bannai and E. Bannai, "Spin models on finite cyclic groups," *J. Alg. Combin.* **3** (1994), 243–259.
2. E. Bannai and E. Bannai, "Generalized spin models (four-weight spin models)," *Pacific J. of Math.* **170** (1995), 1–16.
3. E. Bannai, E. Bannai, and F. Jaeger, "On spin models, modular invariance, and duality," *J. Algebraic Combin.* **6** (1997), 203–228.
4. Ei. Bannai and T. Ito, *Algebraic Combinatorics, Vol. I: Association Schemes*, Benjamin/Cummings, Menlo Park, 1984.
5. Et. Bannai and A. Munemasa, "Duality maps of finite abelian groups and their applications to spin models," preprint, 1995.
6. A.E. Brouwer, A.M. Cohen, and A. Neumaier, "*Distance-Regular Graphs*, Springer-Verlag, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 18, 1989.
7. G. Burde and H. Zieschang, *Knots*, de Gruyter, Berlin, New York, 1985.
8. R.H. Crowell and R.H. Fox, *Introduction to Knot Theory*, Springer, Berlin, 1963.
9. E. Date, M. Jimbo, K. Miki, and T. Miwa, "Braid group representations arising from the generalized chiral Potts models," *Pacific J. of Math.* **154**(1) (1992), 37–66.
10. T. Deguchi, "Generalized spin models associated with exactly solvable models," Progress in Algebraic Combinatorics, Mathematical Society of Japan, 1996, pp. 81–100. Advanced Studies in Pure Mathematics, Vol. 24.
11. D.M. Goldschmidt and V.F.R. Jones, "Metaplectic link invariants," *Geom. Dedicata* **31** (1989), 165–191.
12. F. Jaeger, "Spin models for link invariants," in *Surveys in Combinatorics 1995*, Peter Rowlinson (Ed.), Cambridge University Press 1995, pp. 71–101. London Mathematical Society Lecture Notes Series, Vol. 218.
13. F. Jaeger, "On spin models, triply regular association schemes, and duality," *J. Alg. Combin.* **4** (1995), 103–144.
14. E. Jaeger, "Towards a classification of spin models in terms of association schemes," Progress in Algebraic Combinatorics, Mathematical Society of Japan, 1996, pp. 197–225. Advanced Studies in Pure Mathematics, Vol. 24.
15. F. Jaeger, "New constructions of models for link invariants," *Pacific J. Math.* **176** (1996), 71–116.
16. F. Jaeger, M. Matsumoto, and K. Nomura, "Bose-Mesner algebras related to type II matrices and spin models," *J. Algebraic Combin.* **8** (1998), 39–72.

17. V.F.R. Jones, "A polynomial invariant for knots via Von Neumann algebras," *Bull. Am. Math. Soc.* **12** (1985), 103–111.
18. V.F.R. Jones, "On knot invariants related to some statistical mechanical models," *Pacific J. of Math.* **137** (1989) 2, 311–334.
19. V.G. Kac and M. Wakimoto, "A construction of generalized spin models," *Perspectives in Mathematical Physics*, International Press, 1994, pp. 125–150. Conf. Proc. Lecture Notes Math. Phys. III.
20. L.H. Kauffman, *On Knots*, Princeton University Press, Princeton, New Jersey, 1987. Annals of Mathematical Studies, Vol. 115.
21. K. Kawagoe, A. Munemasa, and Y. Watatani, "Generalized spin models," *J. of Knot Theory and its Ramifications* **3** (1994), 465–476.
22. W.B.R. Lickorish and K. Millett, "A polynomial invariant of oriented links," *Topology* **26** (1987), 107–141.
23. H.R. Morton and P.R. CROMWELL, "Distinguishing mutants by knot polynomials," *J. of Knot Theory and its Ramifications* **5** (1996), 225–238.
24. K. Nomura, "An algebra associated with a spin model," preprint, 1994.
25. N. Yu. Reshetikhin and V.G. Turaev, "Ribbon graphs and their invariants derived from quantum groups," *Commun. Math. Phys.* **127** (1990), 1–26.
26. V.G. Turaev, "The Yang-Baxter equation and invariants of links," *Invent. Math.* **92** (1988), 527–553.
27. F.Y. Wu, P. Pant, and C. King, "New link invariant from the chiral Potts model," *Phys. Rev. Lett.* **72** (1994), 3937–3940.
28. M. Yamada, "The construction of four-weight spin models by using Hadamard matrices and M-structure," *Australasian J. of Comb.* **10** (1994), 237–244.