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On the Connection Between Macdonald Polynomials and Demazure Characters

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Abstract. We show that the specialization of nonsymmetric Macdonald polynomials at $t = 0$ are, up to multiplication by a simple factor, characters of Demazure modules for $\widehat{sl(n)}$. This connection furnishes Lie-theoretic proofs of the nonnegativity and monotonicity of Kostka polynomials.

Keywords: affine Lie algebras, Macdonald polynomials, Demazure character

1. Introduction

Macdonald defined a special class of polynomials $P_\lambda(z, q, t)$, called *symmetric Macdonald polynomials*, which form a basis of the symmetric polynomials in $\mathbb{C}(q, t)[z_1, \ldots, z_n]$. These polynomials are indexed by partitions $\lambda \in \mathbb{N}^n$, $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0$. They interpolate between several classes of classical polynomials: $P_\lambda(z, 0, t)$ are the Hall-Littlewood polynomials, which, in turn are the Schur functions when $t = 0$. By setting $q = t^\alpha$ and letting t go to 1, one obtains Jack polynomials. In [11], Macdonald mentions that there is no similar interpretation of $P_\lambda(z, q, 0)$. By using the theory of nonsymmetric Macdonald polynomials, we show that the $P_\lambda(z, q, 0)$ are the characters (up to factor) of certain Demazure modules of $\widehat{sl(n)}$. This interpretation allows us to obtain Lie-theoretic proofs of the nonnegativity and monotonicity of Kostka polynomials. In addition, it gives us a branching rule for the decomposition of certain integrable highest weight $\widehat{sl(n)}$-modules under the action of $sl(n)$.

The connection between Demazure characters and symmetric functions has already been explored in [8] using a path realization of the crystal basis. The results in this paper intersect somewhat with those in [8]. The main advantage of our approach is its simplicity and its explanation of the connection with Macdonald polynomials. Nonnegativity and positivity of Kostka polynomials have already been proven by Lascoux-Schützenberger [9], Butler [1], Lusztig [10]. The connection between the branching rule and Kostka polynomials was explored in [5]. A different representation-theoretic interpretation of $P_\lambda(z, q, 0)$ is given in [4].

2. Nonsymmetric Macdonald polynomials

These nonsymmetric analogues of the symmetric Macdonald polynomials were first introduced in [12, 14]. Nonsymmetric Macdonald polynomials $E_\lambda(z, q, t)$ are indexed by compositions $\lambda \in \mathbb{N}^n$ and form a basis of $\mathbb{C}(q, t)[z_1, \ldots, z_n]$. (See [2, 5] for their precise definition). In [6], Knop gives a recursive description of the $E_\lambda(z,q,t)$. We describe this recursion for when $t=0$. In this case, we have $E_\lambda(z,q,0)\in\mathbb{Z}[q,q^{-1}][z_1,\ldots,z_n]$. For ease of notation, we will denote $E_\lambda(z,q,0)$ simply by E_λ from now on. For $i\in[1,\ldots,n-1]$ let s_i be the simple reflection that interchanges z_i and z_{i+1}. Consider the following operators on $\mathbb{Z}[q,q^{-1}][z_1,\ldots,z_n]$:

$$\bar{H}_i := s_i - z_{i+1}\frac{(1-s_i)}{(z_i-z_{i+1})} \quad \text{for } i\in[1,\ldots,n-1]$$
$$\Phi f(z_1,\ldots,z_n) := z_n f(q^{-1}z_n, z_1,\ldots,z_{n-1})$$
$$\bar{H}_0 := \Phi\bar{H}_1\Phi^{-1} = \Phi^{-1}\bar{H}_{n-1}\Phi$$

Then the recursion relations are given by [6]

Theorem 1 *The E_λ are generated by application of the $\bar{H}_i$ $(0\le i<n)$ and Φ to* 1. *More precisely, set $E_{(0^n)}:=1$. The action of Φ and the $\bar{H}_i$ on the set of E_λ for $\lambda\in\mathbb{N}^n$ is as follows*:

$$q^{\lambda_1}\Phi E_{(\lambda_1,\ldots,\lambda_n)} = E_{(\lambda_2,\ldots,\lambda_n,\lambda_1+1)}$$
$$\bar{H}_i E_\lambda = \begin{cases} E_{s_i\lambda} & \text{if } \lambda_i<\lambda_{i+1} \\ E_\lambda & \text{if not}\end{cases} \quad \text{for } 1\le i\le n-1$$

where $s_i\lambda$ is the composition λ with λ_i and λ_{i+1} interchanged.

$$q^{\lambda_1-\lambda_n+1}\bar{H}_0 E_\lambda = \begin{cases} E_{(\lambda_n-1,\lambda_2,\ldots,\lambda_{n-1},\lambda_1+1)} & \text{if } \lambda_1>\lambda_n-1 \\ E_\lambda & \text{if not}\end{cases}$$

To ease notation, we define the operators $\tilde{H}_0$ and $\tilde{\Phi}$ on the set of nonsymmetric Macdonald polynomials:

$$\tilde{H}_0 E_{(\lambda_1,\ldots,\lambda_n)} := q^{\lambda_1-\lambda_n+1}\bar{H}_0 E_{(\lambda_1,\ldots,\lambda_n)} = E_{(\lambda_n-1,\lambda_2,\ldots,\lambda_{n-1},\lambda_1+1)}$$
$$\tilde{\Phi} E_{(\lambda_1,\ldots,\lambda_n)} := q^{\lambda_1}\Phi E_{(\lambda_1,\ldots,\lambda_n)} = E_{(\lambda_2,\ldots,\lambda_n,\lambda_1+1)}$$

Although this definition of nonsymmetric Macdonald polynomials is given for only $\lambda\in\mathbb{N}^n$, we can easily extend it to compositions $\lambda\in\mathbb{Z}^n$ by defining

$$E_\lambda := \tilde{\Phi}^{-mn}E_{\lambda+(m^n)} = q^{-(m|\lambda|+nm(m+1)/2)}\Phi^{-mn}E_{\lambda+(m^n)}$$

where m is chosen large enough so that $\lambda+(m^n)=(\lambda_1+m,\ldots,\lambda_n+m)$ is in $\mathbb{N}^n$. The E_λ are well-defined (don't depend on the choice of m). In fact, let m_1 and m_2, with $m_1\le m_2$, be two such choices. Then,

$$\tilde{\Phi}^{-m_2n}E_{\lambda+(m_2^n)} = \tilde{\Phi}^{-m_1n}\tilde{\Phi}^{-(m_2-m_1)n}E_{\lambda+(m_2^n)} = \tilde{\Phi}^{-m_1n}E_{\lambda+(m_1^n)},$$

the last equality following from the well-definedness of the E_μ for $\mu\in\mathbb{N}^n$. Note that the E_λ are elements of $\mathbb{Z}[q,q^{-1}][z_1,z_1^{-1},\ldots,z_n,z_n^{-1}]$.

We now check that, for $\lambda \in \mathbb{Z}^n \backslash \mathbb{N}^n$, the E_λ satisfy the recursion relations. For $i \neq 0$,

$$E_{s_i \cdot \lambda} = \tilde{\Phi}^{-mn} E_{s_i \cdot \lambda + (m^n)} = \tilde{\Phi}^{-mn} \bar{H}_i \tilde{\Phi}^{mn} E_\lambda = \bar{H}_i E_\lambda$$

Now, let $\lambda^* := (\lambda_n - 1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 + 1)$ and choose m such that $\lambda^* + (m^n) \in \mathbb{N}^n$. Then

$$E_{\lambda^*} = \tilde{\Phi}^{-mn} E_{\lambda^* + (m^n)} = \tilde{\Phi}^{-mn} \tilde{H}_0 \tilde{\Phi}^{mn} E_\lambda = q^{\lambda_1 - \lambda_n + 1} \bar{H}_0 E_\lambda,$$

the last equality following from the commutativity of the $\bar{H}_i$ with $\tilde{\Phi}^n$. This proves that, for all $\lambda \in \mathbb{Z}^n$, the E_λ satisfy the relations of Theorem 1.

Let B_m denote the $\mathbb{Z}[q, q^{-1}]$-vector space generated by all E_λ with $|\lambda| = m$. Then the $\bar{H}_i B_m \subset B_m$ for all i and $\Phi B_m \subset B_{m+1}$. The action of the $\bar{H}_i$ $(i \neq 0)$ and $\tilde{H}_0$ on the E_λ is related to the action of the affine Weyl group on compositions:

$$s_i \cdot (\lambda_1, \ldots, \lambda_n) := (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_n)$$
$$s_o \cdot (\lambda_1, \ldots, \lambda_n) := (\lambda_n - 1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 + 1)$$

The connection between compositions of a given degree, say m, and elements of the affine Weyl group is as follows. Let $m = kn + i$ where $k \geq 0$ and $0 \leq i < n$. Then the smallest composition is $\eta_m := (k, \ldots k, k+1, \ldots, k+1)$ (i factors of $k+1$ and $n-i$ factors of k). Every composition λ of degree m equals $w \cdot \eta_m$ where w is an affine Weyl group element.

For $w = s_{i_1}, \ldots, s_{i_j}$ a reduced decomposition, we define $\bar{H}_w := \bar{H}_{i_1}, \ldots, \bar{H}_{i_j}$ and $\tilde{H}_w$ the same expression but with $\bar{H}_0$ replaced by $\tilde{H}_0$.

For a composition $\lambda \in \mathbb{Z}^n$, let $\mathrm{u}(\lambda) := \sum_i \frac{\lambda_i(\lambda_i - 1)}{2}$.

Theorem 2 *We can write* $E_\lambda = \tilde{H}_w \tilde{\Phi}^{|\lambda|} \cdot 1 = q^{\mathrm{u}(\lambda)} \bar{H}_w \Phi^{|\lambda|} \cdot 1$ *where* w *is determined by* $\lambda = w\eta_{|\lambda|}$.

Proof: By the commuting relations of Φ and the $\bar{H}_i$ [6],

$$\begin{cases} \Phi \bar{H}_{i+1} = \bar{H}_i \Phi & i = 1, \ldots, n-2 \\ \Phi^2 \bar{H}_1 = \bar{H}_{n-1} \Phi^2 \end{cases}$$

we need only prove that the power of q is $\mathrm{u}(\lambda)$ by induction. For $i \geq 1$, the actions of the $\bar{H}_i$ do not involve any powers of q. The operator $\bar{H}_0$ equals $\Phi \bar{H}_1 \Phi^{-1}$ by definition. Therefore, we need only check that this holds for Φ. Let $\mu = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + 1)$. We have

$$E_\mu = q^{\lambda_n} \Phi E_\lambda = q^{\lambda_n} q^{\mathrm{u}(\lambda)} \Phi H_w \Phi^{|\lambda|} \cdot 1 = q^{\mathrm{u}(\mu)} H_{w'} \Phi^{|\mu|} \cdot 1$$

where $H_{w'}$ is determined by the above commutation relations. □

Remark We note that $\tilde{\Phi}^{nk+i} \cdot 1 = q^{\mathrm{u}(\eta_{nk+i})}(z_1, \ldots, z_n)^k z_{n-i+1} \cdots z_n$. The $\bar{H}_i$ (all i) commute with multiplication by q and the symmetric function $z_1 \cdots z_n$. Therefore, $E_\lambda = q^{\mathrm{u}(\eta_{nk+i})}(z_1, \ldots, z_n)^k \tilde{H}_w z_{n-i+1} \cdots z_n$. We will use this information in Section 4.

3. Demazure modules of $\widehat{sl(n)}$

Let Λ be a dominant integral weight. Let $V = V(\Lambda)$ be the unique (up to isomorphism) irreducible highest weight $\widehat{sl(n)}$-module with highest weight Λ. Let W be the Weyl group of $\widehat{sl(n)}$. For each $w \in W$, the weight space $V_{w(\Lambda)}$ of weight $w(\Lambda)$ is one-dimensional. We consider $E_w(\Lambda)$, the $\underline{\mathbf{b}}$-module generated by $V_{w(\Lambda)}$, where $\underline{\mathbf{b}}$ is the Borel subalgebra. The $E_w(\Lambda)$, called *Demazure modules*, are finite-dimensional vector spaces which form a filtration of V which is compatible with the Bruhat order on W: $w \leq w' \Leftrightarrow E_w(\Lambda) \subseteq E_{w'}(\Lambda)$.

To each Demazure module $E_w(\Lambda)$, we can associate its character $\chi(E_w(\Lambda)$:

$$\chi(E_w(\Lambda)) := \sum_{\mu \text{ weight}} (\dim E_w(\Lambda)_\mu) e^\mu$$

Since the $E_w(\Lambda)$ are finite dimensional, the $\chi(E_w(\Lambda))$ are polynomials in the n simple roots α_i and lie in the group ring for the weight lattice P.

We now define *Demazure operators*. For each α_i, we define an operator Δ_i on P:

$$\Delta_i := \frac{1 - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}$$

where s_i is the simple reflection with respect to α_i. Let $w = s_{i_1} s_{i_2} \cdots s_{i_j}$ be a reduced decomposition. Then, we can define $\Delta_w := \Delta_{i_1} \Delta_{i_2} \cdots \Delta_{i_j}$ and Δ_w does not depend on the choice of reduced decomposition. The connection between characters and Demazure operators is given by [3, 7, 13]:

Theorem 3 $\chi(E_w(\Lambda)) = \Delta_w(e^\Lambda)$.

4. Macdonald polynomials and Demazure module characters

Let $\Lambda_0, \ldots, \Lambda_{n-1}$ denote the n fundamental weights of $\widehat{sl(n)}$ defined by $(\Lambda_i, \alpha_j) = \delta_{ij}$. Let $\delta = \sum_{i=0}^{n-1} \alpha_i$. Let π be the ring homomorphism $\pi : \mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] \to P$ defined by: $\pi(z_i) = e^{\Lambda_i - \Lambda_{i-1}}$ for $i < n$, $\pi(z_n) = e^{\Lambda_0 - \Lambda_{n-1}}$ and $\pi(q) = e^{-\delta}$.

Theorem 4 *The operator $\bar{H}_i$ is equivalent to the Demazure operator Δ_i in the sense that the following diagram commutes*:

$$\begin{array}{ccc}
\mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] & \xrightarrow{\pi} & P \\
\bar{H}_i \downarrow & & \downarrow \Delta_i(e^{\Lambda_0} \cdot \quad) \\
\mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] & \xrightarrow{\pi} & P
\end{array}$$

Proof: We have that $\bar{H}_i$, $i \neq 0$ (resp. $\bar{H}_0$) commutes with multiplication by z_j for $j \neq i$ or $i+1$ (resp. z_1 or z_n). Therefore, one only needs to verify this equivalence on the monomials $z_i^a z_{i+1}^b$ (resp. $z_1^a z_n^b$). This is done by direct computation. □

Let C be the following "change of basis" operator on P: $C(e^{\Lambda_0}) = e^{\Lambda_{n-1}}$ and $C(e^{\Lambda_i}) = e^{\Lambda_{i-1}-\delta}$ for $1 \leq i \leq n-1$.

Theorem 5 *The operator Φ is equivalent to the operator C in the sense that the following diagram commutes*:

$$\begin{array}{ccc} \mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] & \xrightarrow{\pi} & P \\ \Phi\downarrow & & \downarrow C \\ \mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] & \xrightarrow{\pi} & P \end{array}$$

Proof: By direct computation. □

Theorems 1, 4, 5 along with the preceding Remark give us our main result:

Theorem 6 *Through the π homomorphism, we can identify $q^{-\mathrm{u}(\lambda)+\mathrm{u}(\eta_{|\lambda|})}E_\lambda$ with $\chi(E_w(\Lambda_i))$ where $i = |\lambda| \bmod n$ and where w is an affine Weyl group element defined by $\lambda = w\eta_{|\lambda|}$.*

Proof: We have that

$$E_\lambda = q^{\mathrm{u}(\lambda)}\bar{H}_w\Phi^{|\lambda|} \cdot 1 = q^{\mathrm{u}(\lambda)-\mathrm{u}(\eta_{|\lambda|})}(z_1 \cdots z_n)^k \bar{H}_w z_{n-i+1} \cdots z_n.$$

We have $\pi(z_1 z_2 \cdots z_n) = 1$ and $\pi(z_{n-i+1} \cdots z_n) = e^{\Lambda_i}$. Therefore,

$$\pi(E_\lambda) = q^{\mathrm{u}(\lambda)-\mathrm{u}(\eta_{|\lambda|})}\Delta_w e^{\Lambda_i}.$$

□

Remark

1. $\pi(E_\lambda)$ having nonnegative coefficients implies that E_λ has nonnegative coefficients.
2. By setting $q = 1$, one obtains the *real character* of a Demazure module (see [15]). For λ a partition, we have the factorization ([11], p. 324)

$$P_\lambda(z, 1, 0) = e_{\lambda'}(z) = \prod_{i=1}^{n} e_i^{\lambda_i - \lambda_{i+1}}(z)$$

where $e_i(z)$ is the ith elementary symmetric function. This gives us a similar factorization of

$$\chi(E_w(\Lambda)) = q^{-\mathrm{u}(\lambda)+\mathrm{u}(\eta_{|\lambda|})} \prod_{i=1}^{n-1} e_i(\pi(z))^{\lambda_i - \lambda_{i+1}}.$$

Previous examples of this factorization are found in [8, 15].

5. Positivity and monotonicity of Kostka polynomials

Recall that $P_\lambda(z, q, t)$ denotes the symmetric Macdonald polynomial associated to the partition λ.

Theorem 7 *For λ a partition, we have $E_\lambda(z, q, 0) = P_\lambda(z, q, 0)$.*

Proof: Consider $\sum_{w\in W} \bar{H}_w E_\lambda(z, q, t)$. It is symmetric and satisfies the same defining conditions as $P_\lambda(z, q, t)$ (see [6]), therefore is a scalar multiple of it. When $t = 0$, we have $\bar{H}_w E_\lambda(z, q, 0) = E_\lambda(z, q, 0)$. By comparing coefficients of the leading coefficient z^λ in both $E_\lambda(z, q, 0)$ and $P_\lambda(z, q, 0)$, we see that we have equality. □

Recall that one has the following order relation on partitions: two partitions γ and μ such that $|\gamma| = |\mu|$ satisfy $\gamma < \mu$ if $\gamma_1 + \cdots + \gamma_i \leq \mu_1 + \cdots + \mu_i$ for all i with strict inequality for some i.

It is known [[11], VI (8.11)] that $P_\lambda(z, q, 0) = \sum_{\mu\leq\lambda} K_{\mu\lambda}(q, 0)s_\mu(z)$ where K is the Kostka function and the s_μ are the Schur functions. In addition, it is known [[11], p. 355] that $K_{\mu\lambda}(q, 0) = K_{\mu'\lambda'}(q)$ where μ' (resp. λ') is the dual partition of μ (resp. λ). It follows that $P_\lambda(z, q, 0) = \sum_{\mu\leq\lambda} K_{\mu'\lambda'}(q)s_\mu(z)$.

Theorem 8 *The $K_{\mu\lambda}(q)$ have positive coefficients.*

Proof: We have that $P_\lambda(z, q, 0)$ is invariant under the $\bar{H}_i$ (for $i \neq 0$). This is equivalent to saying that the Demazure module $E_w(\Lambda_0)$ decomposes as a direct sum of simple $sl(n)$-modules. In fact, we have the following decomposition:

$$E_w(\Lambda_0) = \oplus_{j\in\mathbb{Z}}(E_w(\Lambda_0))_{j\delta}$$

where $(E_w(\Lambda_0))_{j\delta}$ is just the direct sum of weight spaces whose weights are of the form $\upsilon = \kappa + j\delta$ where κ is some weight for $sl(n)$. (In other words, these are all weights that satisfy $\langle\upsilon, d\rangle = j$ where d is the scaling element.) Since δ is orthogonal to the Cartan subalgebra of $sl(n)$, each $(E_w(\Lambda_0))_{j\delta}$ is a direct sum of irreducible $sl(n)$-modules. Let $\lambda = w\nu_{|\lambda|}$. The $P_\lambda(z, q, 0)$ merely represents the character $\chi(E_w(\Lambda_0))$ as seen in this light; since the $s_\mu(z)$ is a character of an irreducible $sl(n)$-module, the coefficient of q^j in $K_{\mu'\lambda'}(q)$ is the multiplicity of the $sl(n)$-module of highest weight $\mu - j\delta$ in $E_w(\Lambda_0)$. Therefore, the $K_{\mu'\lambda'}(q)$ have positive coefficients. □

Remark A consequence of this theorem is that the Kostka numbers $K_{\mu\lambda}(1)$ are the multiplicities of the (finite-dimensional) $sl(n)$-modules in the Demazure modules $E_w(\Lambda)$.

Recall that $V = V(\Lambda_i)$ is the irreducible highest weight $\widehat{sl(n)}$-module of highest weight Λ_i. We have that $\chi(V) = \lim_{\ell(w)\to\infty} \chi(E_w(\Lambda_i))$. We can now describe the branching rule for V in terms of Kostka polynomials (see [5]). Let $\{\lambda^j\}$ be an "increasing" sequence of partitions in the sense that $\lambda^j := w_j\nu_{|\lambda^j|}$ where $\lim_j \ell(w_j) = \infty$ and where $|\lambda^j| = i (\bmod n)$. We must choose $\nu_{|\lambda^j|}$ such that the resulting λ^j are still partitions.

Corollary 1 *The multiplicity of the $sl(n)$-module of weight μ in V is given by*

$$\lim_{j\to\infty} q^{-\mathrm{u}(\lambda^j)+\mathrm{u}(\nu_{|\lambda^j|})} K_{\mu'\lambda^{j\prime}}(q)$$

We also have a monotonicity result. Let $\tilde{K}_{\lambda\mu}(q) := q^{-\mathrm{u}(\mu)} K_{\lambda\mu}(q)$. Recall that if $\lambda = w\nu_m$ and $\gamma = w'\nu_m$, $\lambda \neq \gamma$ are partitions, then $\lambda < \gamma$ if and only if $w < w'$ in the Bruhat order, where w and w' are chosen to have smallest length.

Theorem 9 $\tilde{K}_{\lambda\mu}(q) - \tilde{K}_{\lambda\nu}(q)$ *has nonnegative coefficients when* $\nu \geq \mu$.

Proof: Let $\upsilon < \gamma$ be two partitions such that $\upsilon = w'\eta_{|\lambda|}$ and $\gamma = w\eta_{|\lambda|}$. Then $w' < w$ and $E_{w'}(\Lambda) \subset E_w(\Lambda)$. The coefficient of q^j in $\tilde{K}_{\nu'\gamma'}(q) - \tilde{K}_{\nu'\upsilon'}(q)$ is the multiplicity of the $sl(n)$-module of weight $\nu - j\delta$ ($j \in \mathbb{Z}$) in $E_w(\Lambda)/E_{w'}(\Lambda)$. Therefore, it has positive coefficients. □

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