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On Near Hexagons and Spreads of Generalized Quadrangles

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Abstract. The glueing-construction described in this paper makes use of two generalized quadrangles with a spread in each of them and yields a partial linear space with special properties. We study the conditions under which glueing will give a near hexagon. These near hexagons satisfy the nice property that every two points at distance 2 are contained in a quad. We characterize the class of the "glued near hexagons" and give examples, some of which are new near hexagons.

Keywords: spread, generalized quadrangle, near polygon

1. Definitions

An *incidence structure* is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ (the point set) a nonempty set and $\mathcal{L}$ (the set of lines) a (possibly empty) set and I a symmetric incidence relation between those sets. Although the incidence relation is symmetric, we will write, in order not to overload the notation, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ or even use "$\in$" as incidence relation. The incidence structures which we will consider here are all finite. If x is a point, then $\Gamma_i(x)$ denotes the set of all points at distance i from x (in the point graph). We will denote $\Gamma(x) = \Gamma_1(x)$.

1. An incidence structure is called a *partial linear space* if the following conditions are satisfied.
   (a) Every line $L \in \mathcal{L}$ is incident with at least two points.
   (b) Two different points are incident with at most one line.

   A *linear space* is a partial linear space with the property that every two points are collinear.
2. An incidence structure of points and lines is *connected* if its point graph is connected.
3. A connected partial linear space is called *degenerate* if there is a point incident with exactly one line.
4. A *near polygon* $\mathcal{S}$ is a connected partial linear space satisfying the following conditions.
   (a) The diameter of the point graph Γ of $\mathcal{S}$ is finite.
   (b) For every point p and every line L, there is a unique point q on L, nearest to p (nearest with respect to the distance $d(.\,,\,.)$ in Γ).

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If d is the diameter of Γ then $\mathcal{S}$ is called a near $2d$-gon. A near 0-gon has only one point and no lines and a near 2-gon consists of one line with a number (≥ 2) of points on it. The near quadrangles are just the generalized quadrangles. A generalized quadrangle (GQ for short) is called degenerate if there is a point incident with exactly one line. The point-line dual of a nondegenerate GQ is again a nondegenerate GQ. If a nondegenerate GQ is neither a grid nor a dual grid, then it must have an order (s, t).

5. A GQ is called *bad* when it is degenerate or when it is a nonsymmetrical dual grid; otherwise it is called a *good* GQ. If $\mathcal{Q}$ is a good GQ, then every point of it is incident with the same number of lines, this number being denoted by $t_{\mathcal{Q}} + 1$.
6. An *ovoid* of a generalized quadrangle $\mathcal{Q}$ is a set O of points such that every line of $\mathcal{Q}$ is incident with exactly one element of O. If $\mathcal{Q}$ has order (s, t), then $|O| = 1 + st$. A set of $1 + st$ mutually noncollinear points of $\mathcal{Q}$ is always an ovoid of $\mathcal{Q}$. The dual notion is that of a *spread*. A spread is a set of lines of $\mathcal{Q}$ such that every point is incident with exactly one line of the set. For more details on generalized quadrangles, we refer to [6].
7. The incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called *affine* or *embedded in the finite affine space* $\mathcal{A}$ if $\mathcal{L}$ is a set of lines of $\mathcal{A}$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$ and the incidence relation is the one induced by that of $\mathcal{A}$. If $\mathcal{A}'$ is the subspace of $\mathcal{A}$ generated by all points of $\mathcal{P}$, then we say that $\mathcal{A}'$ is the *ambient space* of $\mathcal{S}$.

   A special type of affine embedding is the so-called *linear representation*. Let $\prod_\infty$ be a projective space of dimension $n \geq 0$ embedded as a hyperplane in the projective space $\prod$ and let $\mathcal{K}$ be a nonempty subset of the point set of $\prod_\infty$. The linear representation $T_n^*(\mathcal{K})$ is the geometry with points the affine points of $\prod$ (= the points not belonging to $\prod_\infty$). The lines of $T_n^*(\mathcal{K})$ are all the lines of $\prod$ which intersect $\prod_\infty$ in a (unique) point of $\mathcal{K}$. Incidence is the one derived from $\prod$.
8. If $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ are two partial linear spaces, then the direct product of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ and $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$. The point (x, y) is incident with the line $(a, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = a$ and $y\ \mathrm{I}_2\ L$ and it is incident with the line $(M, b) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $y = b$ and $x\ \mathrm{I}_1\ M$. We denote $\mathcal{S}$ also with $\mathcal{S}_1 \times \mathcal{S}_2$. Since $\mathcal{S}_1 \times \mathcal{S}_2 \simeq \mathcal{S}_2 \times \mathcal{S}_1$ and $(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3 \simeq \mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3)$, also the direct product of $k \geq 1$ partial linear spaces $\mathcal{S}_1, \ldots, \mathcal{S}_k$ is well-defined. If $\mathcal{S}_i$ $(i \in \{1, 2\})$ is a near $2d_i$-gon, then one can easily prove that $\mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(d_1 + d_2)$-gon.
9. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. A set $X \subseteq \mathcal{P}$ is called a *subspace* whenever all the points of a line are in X as soon as two of them are in X. Every such subspace induces a partial linear space $\mathcal{S}_X = (X, \mathcal{L}_X, \mathrm{I}')$ where $\mathcal{L}_X$ is the set of all lines of $\mathcal{L}$ which have all their points in X and I' is the restriction of I to $X \times \mathcal{L}_X$. A subspace X is called *geodetically closed* when all points of a shortest path between two points of X are also contained in X. A *quad* is a geodetically closed subset of $\mathcal{P}$ which induces a nondegenerate GQ. Since no confusion will be possible in the sequel, the GQ induced by a quad will also be called a quad. If a quad $\mathcal{Q}$ contains a unique point nearest a fixed point x, then this point is called the *projection* of x on $\mathcal{Q}$.

2. Some theorems

Theorem 2.1 ([7, 8]) *Let x and y be two points of a near polygon at mutual distance 2. If x and y have two common neighbours c and d such that the line xc contains at least three points, then x and y are in a unique (necessarily good) quad.*

Theorem 2.2 *Let $\mathcal{S}$ be a near polygon and let x be a point at distance at most 1 from a quad $\mathcal{Q}$, then there exists a unique point x' of $\mathcal{Q}$ nearest to x and $d(x, y) = d(x, x') + d(x', y)$ for all points y of $\mathcal{Q}$. Hence, if L is a line of $\mathcal{Q}$, then the unique point of L nearest to x is also the unique point of L nearest to x'.*

Proof: This follows from the fact that $\mathcal{Q}$ is geodetically closed. □

Corollary 2.3 *Let $\mathcal{Q}$ be a quad of a near polygon $\mathcal{S}$ and let x and y be two collinear points of $\mathcal{S}$ such that the line xy is disjoint with $\mathcal{Q}$. If x, respectively y, is collinear with $x' \in \mathcal{Q}$, respectively $y' \in \mathcal{Q}$, then $d(x', y') = 1$.*

Proof: By Theorem 2.2, we have that $2 = d(x', y) = d(x', y') + d(y', y) = 1 + d(x', y')$. □

Theorem 2.4 ([3]) *Let $\mathcal{S}$ be a near polygon with the property that every two points at distance 2 are contained in a good quad, then each point of $\mathcal{S}$ is incident with the same number of lines.*

Proof: Let x and y be two collinear points. The point x (respectively y) is incident with $t_x + 1$ (respectively $t_y + 1$) lines. Now

$$t_x + 1 = 1 + \sum t_{\mathcal{Q}} = t_y + 1,$$

where the summation ranges over all quads $\mathcal{Q}$ through the line xy. Hence x and y are incident with the same number of lines and the result follows by connectedness of $\mathcal{S}$. □

Theorem 2.5 ([3]) *Let $\mathcal{S}$ be a near polygon satisfying the following properties:*
(a) *every two points at distance 2 have at least two common neighbours,*
(b) *there are lines incident with a different number of points,*
then $\mathcal{S}$ is the direct product of a number of near polygons, each of which has a constant length for the lines.

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near 2-gon or a good GQ, then $|\Gamma_i(p)|$ ($i \in \{0, 1, 2\}$) is independent of $p \in \mathcal{P}$. We derive a similar property for near hexagons.

Theorem 2.6 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near hexagon such that every two points at distance 2 are contained in a good quad, then $|\Gamma_i(p)|$ ($i \in \{0, 1, 2, 3\}$) is independent of $p \in \mathcal{P}$.*

Proof: If not all lines of $\mathcal{S}$ are incident with the same number of points, then Theorem 2.5 implies that $\mathcal{S}$ is the direct product of a line with a good GQ. It is straightforward to check that the result is true in this case. Hence we may suppose that all lines are incident with $s+1$ points. Theorem 2.4 implies then that $\mathcal{S}$ has an order (s, t). Now, let $p \in \mathcal{P}$ be a fixed point and put $n_i = |\Gamma_i(p)|$. Then $n_0 = 1, n_1 = s(t+1)$. Let V be the set of quads through p. Counting points in $\Gamma_2(p)$ we find

$$n_2 = s^2 \sum_{x \in V} t_x. \tag{1}$$

Counting edges between $\Gamma_2(p)$ and $\Gamma_3(p)$ we find that

$$n_3(t+1) = s^3 \sum_{x \in V} t_x(t - t_x). \tag{2}$$

Finally, counting triples $(L_1, L_2, \mathcal{Q})$ where L_1, L_2 are two different lines through p and $\mathcal{Q}$ is the quad through L_1 and L_2, yields

$$t(t+1) = \sum_{x \in V} t_x(t_x + 1). \tag{3}$$

Eliminating $\sum t_x$ and $\sum t_x^2$, we find that $n_3 = s(n_2 - s^2 t)$. Together with $v = n_0 + n_1 + n_2 + n_3$ this gives

$$n_2 = \frac{v}{s+1} - 1 + st(s-1), \tag{4}$$

$$n_3 = s\left(\frac{v}{s+1} - st - 1\right). \tag{5}$$

□

Corollary 2.7 *If $\mathcal{S}$ is a near hexagon satisfying the property that every two points at distance 2 are contained in a quad of order (s, t_1) or (s, t_2), $s \geq 1$ and $1 \leq t_1 < t_2$, then for each $i \in \{1, 2\}$, the number of quads of order (s, t_i) through a point is independent of that point.*

Proof: This follows from Eqs. (1), (3) and (4). □

Remark The previous corollary was proved in [2] in the case that $s = 2, t_1 = 1, t_2 = 2$ by using the same double countings as in the proof of Theorem 2.6.

Theorem 2.8 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space of order $(s, t) \neq (s, 0)$ satisfying*

1. *for every point p and every line L not through p, there exists at most one point on L collinear with p,*
2. *$a = |\Gamma_2(x)|$ is independent of the point $x \in \mathcal{P}$,*
3. *$d(x, L) \leq 2$ for all $x \in \mathcal{P}$ and $L \in \mathcal{L}$,*

then $b = |\Gamma_3(x)|$ *is also independent of* $x \in \mathcal{P}$ *and the following inequalities hold*:
- $a \geq s^2 t$,
- $b \leq s(a - s^2 t)$.

Moreover, $\mathcal{S}$ *is a generalized quadrangle if and only if* $a = s^2 t$ *and* $\mathcal{S}$ *is a near hexagon if and only if* $a > s^2 t$ *and* $b = s(a - s^2 t)$.

Proof: Clearly $|\Gamma_3(x)| = |\mathcal{P}| - 1 - s(t+1) - |\Gamma_2(x)|$ is independent of $x \in \mathcal{P}$. Take an arbitrary line L and let r be a point of L. There are a points in $\Gamma_2(r)$, $s^2 t$ of these are contained in $\Gamma_1(L)$. Hence $a \geq s^2 t$ and $\Gamma_2(L) \leq (s+1)(a - s^2 t)$. If $a = s^2 t$ then $\Gamma_2(L) = \emptyset$ implies that $\mathcal{S}$ is a generalized quadrangle. So, suppose that $a \neq s^2 t$, then $\mathcal{S}$ is a near hexagon if and only if $\Gamma_2(L) = (s+1)(a - s^2 t)$. From $|\Gamma_2(L)| = |\mathcal{P}| - (s+1) - st(s+1) = a + b - s^2 t$, it follows that $b \leq s(a - s^2 t)$ and equality appears if and only if $\mathcal{S}$ is a near hexagon. □

3. A possible construction for near hexagons

Let $\mathcal{Q}_i = (\mathcal{P}_i, \mathcal{L}_i, \mathrm{I}_i)$ (for each $i \in \{1, 2\}$) be a GQ of order (s, t_i), let $S_i = \{L_1^{(i)}, \ldots, L_{1+st_i}^{(i)}\} \subset \mathcal{L}_i$ be a spread of $\mathcal{Q}_i$ and let θ be a bijection from $L_1^{(1)}$ to $L_1^{(2)}$ (here we suppose that every line is a subset of the point set).

For every $i \in \{1, 2\}$ and every $j \in \{1, \ldots, 1 + st_i\}$, $\Phi_j^{(i)} : \mathcal{P}_i \mapsto L_j^{(i)}$ is defined such that $x \in \mathcal{P}_i$ is mapped to the unique point of $L_j^{(i)}$ nearest to x (in the generalized quadrangle $\mathcal{Q}_i$).

Let $\Gamma(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ (Γ for short if no confusion is possible) be the graph with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two different points $(x, L_i^{(1)}, L_j^{(2)})$ and $(y, L_k^{(1)}, L_l^{(2)})$ are adjacent whenever at least one of the following two conditions are satisfied:

(1) $j = l$ and $\Phi_i^{(1)}(x)$, $\Phi_k^{(1)}(y)$ are collinear points in $\mathcal{Q}_1$,
(2) $i = k$ and $\Phi_j^{(2)} \circ \theta(x)$, $\Phi_l^{(2)} \circ \theta(y)$ are collinear points in $\mathcal{Q}_2$.

If $i = k$ and $j = l$, then both (1) and (2) are satisfied. It is clear that $\Gamma(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta) \simeq \Gamma(\mathcal{Q}_2, \mathcal{Q}_1, S_2, S_1, L_1^{(2)}, L_1^{(1)}, \theta^{-1})$. For, $\Delta : (x, L_i^{(1)}, L_j^{(2)}) \mapsto (\theta(x), L_j^{(2)}, L_i^{(1)})$ defines an isomorphism. The definition of Γ is hence symmetric in $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Remark In the sequel, we will not write the symbol "$\circ$" between functions, i.e. with fg we mean the function $f \circ g$.

Lemma 3.1 *Through every two adjacent vertices of* Γ, *there is a unique maximal clique. This clique has size* $s + 1$.

Proof: Let $a = (x, L_i^{(1)}, L_j^{(2)})$ and $b = (y, L_k^{(1)}, L_l^{(2)})$ be two fixed adjacent vertices; we determine what the common neighbours $(z, L_m^{(1)}, L_n^{(2)})$ look like. If $i = k \neq m$, then $j = n = l$ and $\Phi_i^{(1)}(x) \sim \Phi_m^{(1)}(z) \sim \Phi_i^{(1)}(y)$ implies that $x = y$ and hence $a = b$, a contradiction. Similarly, $j = l \neq n$ is impossible. If $i = k = m$, then $\Phi_j^{(2)}\theta(x) \sim \Phi_n^{(2)}\theta(z) \sim \Phi_l^{(2)}\theta(y)$ implies that $\Phi_n^{(2)}\theta(z)$ is an element of the line of $\mathcal{Q}_2$ through $\Phi_j^{(2)}\theta(x)$ and $\Phi_l^{(2)}\theta(y)$. This yields $s-1$ common neighbours of a and b and they are all mutually adjacent. Together with the vertices a and b, they yield a clique of size $s+1$. A similar reasoning holds in the case $j = l = n$. □

Let $\mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ be the partial linear space with points the vertices of Γ and with lines the maximal cliques of Γ. The incidence is the natural one. Again, we will write $\mathcal{S}$ when no confusion is possible.

Definition 3.2 A line L is said to be of *type I*, if there exists a fixed j, such that every point of L is of the form $(x, L_i^{(1)}, L_j^{(2)})$. A line M is said to be of *type II*, if there exists a fixed i, such that every point of M is of the form $(x, L_i^{(1)}, L_j^{(2)})$. Remark that there are lines which are of both types, namely the lines $\{(x, L_i^{(1)}, L_j^{(2)}) \mid x \in L_1^{(1)}\}$, where i and j are fixed. These lines partition the point set of $\mathcal{S}$ (hence they form a spread of $\mathcal{S}$).

Lemma 3.3

(a) *For a fixed* $j \in \{1, \ldots, 1+st_2\}$, *the set* $\{(x, L_i^{(1)}, L_j^{(2)}) \mid x \in L_1^{(1)}, 1 \leq i \leq 1+st_1\}$ *is a quad isomorphic to* $\mathcal{Q}_1$.

(b) *For a fixed* $i \in \{1, \ldots, 1+st_1\}$, *the set* $\{(x, L_i^{(1)}, L_j^{(2)}) \mid x \in L_1^{(1)}, 1 \leq j \leq 1+st_2\}$ *is a quad isomorphic to* $\mathcal{Q}_2$.

Proof: The isomorphisms are given by $\Delta_1 : (x, L_i^{(1)}, L_j^{(2)}) \mapsto \Phi_i^{(1)}(x)$ for (a) and $\Delta_2 : (x, L_i^{(1)}, L_j^{(2)}) \mapsto \Phi_j^{(2)}\theta(x)$ for (b). □

Definition 3.4

(1) The previous lemma shows that several GQ's (isomorphic to $\mathcal{Q}_1$ or $\mathcal{Q}_2$) are glued together to form the geometry $\mathcal{S}$. For this reason the above construction is called *glueing* and $\mathcal{S}$ will be called a *glued geometry*.

(2) A *quad of type I, respectively II* is a quad that arises like in (a), respectively (b) of the previous lemma. The following properties hold then.

- Every line contained in a quad of type A ($\in$ {I, II}) is also of type A.
- Two quads of the same type are equal or disjoint.
- Two quads of different type meet each other in a line which is of both types.
- Through every point of $\mathcal{S}$, there is a unique quad of each type.
- Every line of type A ($\in$ {I, II}) is contained in a unique quad of type A.

Lemma 3.5 *$\mathcal{S}$ has order (s, t_1+t_2) and satisfies properties* 1 *and* 3 *of Theorem* 2.8.

Proof: Let p be an arbitrary point of $\mathcal{S}$. The quad of type I (respectively type II) through p contains t_1+1 (respectively t_2+1) lines through p and both quads have exactly one line in common. Hence $\mathcal{S}$ has order (s, t_1+t_2).

Property 1 clearly holds by Lemma 3.1, so let x and M be a point and a line of $\mathcal{S}$, both arbitrarily chosen. Through M, there is a quad $\mathcal{R}_1$ of type $A \in$ {I, II}. Take the unique quad

$\mathcal{R}_2$ through p of type B such that $\{A, B\} = \{\text{I}, \text{II}\}$. On the intersection line of $\mathcal{R}_1$ and $\mathcal{R}_2$ there is a unique point nearest to x. This point has distance at most 1 to x and M. This proves the lemma. □

Definition 3.6

- For all $i, j \in \{1, \ldots, 1 + st_1\}$, $\phi_{i,j}^{(1)}$ is the permutation of $L_1^{(1)}$ equal to the restriction of $\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}$ to $L_1^{(1)}$. The group of permutations of $L_1^{(1)}$ generated by the elements $\phi_{i,j}^{(1)}$ is denoted by G_1.
- For all $i, j \in \{1, \ldots, 1 + st_2\}$, $\phi_{i,j}^{(2)}$ is the permutation of $L_1^{(2)}$ equal to the restriction of $\Phi_1^{(2)}\Phi_j^{(2)}\Phi_i^{(2)}$ to $L_1^{(2)}$. The group of permutations of $L_1^{(2)}$ generated by the elements $\phi_{i,j}^{(2)}$ is denoted by G_2.

Remark

- $\phi_{i,i}^{(1)}, \phi_{i,i}^{(2)}$ are identity permutations,
- $\phi_{i,j}^{(k)}$ and $\phi_{j,i}^{(k)}$ ($k \in \{1, 2\}$) are inverse permutations.

Theorem 3.7 *$\mathcal{S}$ is a near hexagon if and only if $[G_1, \theta^{-1}G_2\theta] = 0$. (Here 0 stands for the trivial group and $[G_1, \theta^{-1}G_2\theta]$ is the group generated by all commutators $[g_1, \theta^{-1}g_2\theta]$ with $g_1 \in G_1$ and $g_2 \in G_2$.)*

Proof: Suppose that $\mathcal{S}$ is a near hexagon. It suffices to prove that $\phi_{i,j}^{(1)}$ commutes with $\theta^{-1}\phi_{k,l}^{(2)}\theta$ for all possible i, j, k, l with $i \neq j$ and $k \neq l$. If $x \in L_1^{(1)}$, then we have the following adjacencies:

$$\begin{aligned}&\left(\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}\theta^{-1}\Phi_1^{(2)}\Phi_l^{(2)}\Phi_k^{(2)}\theta(x), L_j^{(1)}, L_l^{(2)}\right)\\ &\quad\sim\left(\theta^{-1}\Phi_1^{(2)}\Phi_l^{(2)}\Phi_k^{(2)}\theta(x), L_i^{(1)}, L_l^{(2)}\right)\\ &\quad\sim\left(x, L_i^{(1)}, L_k^{(2)}\right)\\ &\quad\sim\left(\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}(x), L_j^{(1)}, L_k^{(2)}\right)\\ &\quad\sim\left(\theta^{-1}\Phi_1^{(2)}\Phi_l^{(2)}\Phi_k^{(2)}\theta\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}(x), L_j^{(1)}, L_l^{(2)}\right).\end{aligned}$$

Let p be the point $(x, L_i^{(1)}, L_k^{(2)})$ and L be the line $\{(x, L_j^{(1)}, L_l^{(2)}) \mid x \in L_1^{(1)}\}$ (this is a line of type I and of type II). Since there is only one point of L at distance 2 from p, it follows that

$$\theta^{-1}\phi_{k,l}^{(2)}\theta\phi_{i,j}^{(1)} = \phi_{i,j}^{(1)}\theta^{-1}\phi_{k,l}^{(2)}\theta.$$

Conversely, suppose that $[G_1, \theta^{-1}G_2\theta]$ is the trivial group. Let x be an arbitrary point of $\mathcal{S}$. Through x, there is a unique quad $\mathcal{R}_1$ of type I and a unique quad $\mathcal{R}_2$ of type II. In $\mathcal{R}_1 \cup \mathcal{R}_2$, there are $s^2(t_1 + t_2)$ points of $\Gamma_2(x)$. The points of $\mathcal{S}$ not in $\mathcal{R}_1 \cup \mathcal{R}_2$ are partitioned by $s^2t_1t_2$ lines which have both types. The previous reasoning shows that each of these lines contains a unique point at distance 2 from x. Hence $a = |\Gamma_2(x)| = s^2(t_1t_2 + t_1 + t_2)$ is independent of the point x. From this it follows that $b = |\Gamma_3(x)| = (s+1)(st_1+1)(st_2+1) - 1 - |\Gamma_1(x)| - |\Gamma_2(x)| = s^3t_1t_2$. Since $a > s^2(t_1 + t_2)$ and $b = s(a - s^2(t_1 + t_2))$, it follows from Theorem 2.8 that $\mathcal{S}$ is a near hexagon. □

Above, we defined $\mathcal{S} = \mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$. Take now an arbitrary line $L_i^{(1)}$ in S_1 and an arbitrary line $L_j^{(2)}$ in S_2. If we define $\theta_{i,j}$ as the restriction of $\Phi_j^{(2)}\theta\Phi_1^{(1)}$ to $L_i^{(1)}$, then we can define

$$\mathcal{S}_{i,j} = \mathcal{S}\big(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_i^{(1)}, L_j^{(2)}, \theta_{i,j}\big).$$

Theorem 3.8 *If $\mathcal{S}$ is a near hexagon, then $\mathcal{S}_{i,j}$ is isomorphic to $\mathcal{S}$ for all $i \in \{1, \ldots, 1+st_1\}$ and all $j \in \{1, \ldots, 1+st_2\}$.*

Proof: We prove that $\Delta : L_1^{(1)} \times S_1 \times S_2 \mapsto L_i^{(1)} \times S_1 \times S_2, (x, L_k^{(1)}, L_l^{(2)}) \mapsto (\Phi_i^{(1)}\phi_{k,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x), L_k^{(1)}, L_l^{(2)})$ is an isomorphism between $\mathcal{S}$ and $\mathcal{S}_{i,j}$. This map is clearly a bijection and it suffices to prove that adjacency is preserved in the point graph of the geometries. Consider the two adjacent vertices $a = (x, L_k^{(1)}, L_l^{(2)})$ and $b = (y, L_k^{(1)}, L_m^{(2)})$ in $\mathcal{S}$, then $y = \theta^{-1}\phi_{l,m}^{(2)}\theta(x)$ and

$$\Delta(a) = \big(\Phi_i^{(1)}\phi_{k,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x), L_k^{(1)}, L_l^{(2)}\big),$$
$$\Delta(b) = \big(\Phi_i^{(1)}\phi_{k,i}^{(1)}\theta^{-1}\phi_{m,j}^{(2)}\theta(y), L_k^{(1)}, L_m^{(2)}\big).$$

Now, $\Delta(a) \sim \Delta(b)$ (in $\mathcal{S}_{i,j}$) if and only if

$$\Phi_l^{(2)}\Phi_j^{(2)}\theta\Phi_1^{(1)}\Phi_i^{(1)}\phi_{k,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x) \sim \Phi_m^{(2)}\Phi_j^{(2)}\theta\Phi_1^{(1)}\Phi_i^{(1)}\phi_{k,i}^{(1)}\theta^{-1}\phi_{m,j}^{(2)}\theta(y)$$
$$\Phi_l^{(2)}\Phi_j^{(2)}\theta\phi_{k,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x) \sim \Phi_m^{(2)}\Phi_j^{(2)}\theta\phi_{k,i}^{(1)}\theta^{-1}\phi_{m,j}^{(2)}\theta(y)$$
$$\Phi_l^{(2)}\Phi_j^{(2)}\phi_{l,j}^{(2)}\theta\phi_{k,i}^{(1)}(x) \sim \Phi_m^{(2)}\Phi_j^{(2)}\phi_{m,j}^{(2)}\theta\phi_{k,i}^{(1)}(y)$$
$$\Phi_l^{(2)}\theta\phi_{k,i}^{(1)}(x) \sim \Phi_m^{(2)}\theta\phi_{k,i}^{(1)}(y)$$
$$\theta^{-1}\phi_{l,m}^{(2)}\theta\phi_{k,i}^{(1)}(x) = \phi_{k,i}^{(1)}(y)$$
$$\theta^{-1}\phi_{l,m}^{(2)}\theta\phi_{k,i}^{(1)}(x) = \phi_{k,i}^{(1)}\theta^{-1}\phi_{l,m}^{(2)}\theta(x).$$

Consider the two adjacent vertices $a = (x, L_k^{(1)}, L_l^{(2)})$ and $b = (y, L_m^{(1)}, L_l^{(2)})$ in $\mathcal{S}$, then $y = \phi_{k,m}^{(1)}(x)$ and

$$\Delta(a) = \big(\Phi_i^{(1)}\phi_{k,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x), L_k^{(1)}, L_l^{(2)}\big),$$
$$\Delta(b) = \big(\Phi_i^{(1)}\phi_{m,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(y), L_m^{(1)}, L_l^{(2)}\big).$$

Now, $\Delta(a) \sim \Delta(b)$ (in $\mathcal{S}_{i,j}$) if and only if

$$\begin{aligned}\Phi_k^{(1)}\Phi_i^{(1)}\phi_{k,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x) &\sim \Phi_m^{(1)}\Phi_i^{(1)}\phi_{m,i}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(y)\\ \Phi_k^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x) &\sim \Phi_m^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(y)\\ \phi_{k,m}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x) &= \theta^{-1}\phi_{l,j}^{(2)}\theta(y)\\ \phi_{k,m}^{(1)}\theta^{-1}\phi_{l,j}^{(2)}\theta(x) &= \theta^{-1}\phi_{l,j}^{(2)}\theta\phi_{k,m}^{(1)}(x).\end{aligned}$$

□

Theorem 3.9 *If $\mathcal{S}$ is a near hexagon, then any two points at distance* 2 *are contained in a quad.*

Proof: Let $p_1 = (x, L_i^{(1)}, L_j^{(2)})$ and $p_2 = (y, L_k^{(1)}, L_l^{(2)})$ denote the two points at distance 2. If $i = k$ (respectively $j = l$), then p_1 and p_2 are contained in a quad of type II (respectively I). If $i \neq k$ and $j \neq l$, then the adjacencies of Theorem 3.7 show that p_1 and p_2 have two common neighbours p_3 and p_4. If $s \geq 2$, then Theorem 2.1 implies that p_1 and p_2 are contained in a quad (which is a $(s+1) \times (s+1)$-grid in this case). If $s = 1$, then p_1 and p_2 are contained in a quad, since $\{p_1, p_2, p_3, p_4\}$ is geodetically closed and induces a (2×2)-grid. □

Definition 3.10 Suppose $\mathcal{S}$ is a near hexagon. The quads in $\mathcal{S}$, different from the above defined quads of type I and II are called the quads of type III. These quads are $(s+1) \times (s+1)$-grids.

Remarks

(a) For $i \in \{1, 2\}$ fixed, let $\mathcal{Q}_i$ be an $(s+1) \times (s+1)$-grid and S_i be one of the two spreads of $\mathcal{Q}_i$. Since $\phi_{j,k}^{(i)}$ is the identity permutation for all $j, k \in \{1, \ldots, 1+s\}$, one has that $[G_1, \theta^{-1}G_2\theta] = 0$, hence $\mathcal{S}$ is a near hexagon. It is straightforward to check that $\mathcal{S}$ is the direct product of $\mathcal{Q}_{3-i}$ with a line of size $s+1$.

(b) For every $t \in \mathbb{N}\backslash\{0\}$, there is a unique GQ of order $(1, t)$. This GQ contains several spreads which are all equivalent. Since G_2 is a commutative group, the above construction with $s = 1, t_1, t_2 \geq 1$ will yield a thin near hexagon.

(c) In the next sections we will construct near hexagons using two generalized quadrangles ($\mathcal{Q}_1$ and $\mathcal{Q}_2$) and certain spreads in them (S_1 and S_2 respectively). In the definition of $\mathcal{S}$, we took in each spread two special lines (namely $L_1^{(1)}$ and $L_1^{(2)}$). Theorem 3.8 says (in the case that $\mathcal{S}$ is a near hexagon) that those special lines are in fact not so special. One can obtain the same near hexagon starting with two arbitrary lines (one in each spread) by taking a suitable θ.

(d) We will not study the problem of determining suitable spreads and suitable maps θ. Also, the above construction can be generalized to obtain other near polygons (e.g. near octagons). These two problems will be considered in forthcoming papers.

4. A new construction for $(T_2^*(O_1), T_2^*(O_2))$

4.1. *The generalized quadrangle $T_2^*(O)$*

Consider a hyperoval O in $PG(2, q)$ with q even. Embed $PG(2, q)$ as a hyperplane in $PG(3, q)$, then $T_2^*(O)$ is a generalized quadrangle of order $(q-1, q+1)$, see [1, 5, 6]. Let p be a fixed point of O, then the set of lines of $PG(3, q)$ intersecting O in p defines a spread $\mathcal{S}$ of $T_2^*(O)$. Consider now the model of $PG(3, q)$ where the points are the 1-dimensional subspaces of $V(4, q)$ and let L_1, L_2, L_3 denote three arbitrary (but different) lines of $\mathcal{S}$. The plane $\langle L_i, L_j\rangle$ ($i \neq j$ and $i, j \in \{1, 2, 3\}$) intersects $PG(2, q)$ in a line through p. Let $\langle \bar{c}_{ij}\rangle$ denote the second point of O on that line. Take $\bar{a}, \bar{b} \in V(4, q)$ such that $p = \langle \bar{a}\rangle$ and $L_1 = \langle \bar{a}, \bar{b}\rangle$ and let $x = \langle \alpha\bar{a} + \bar{b}\rangle$ with $\alpha \in \mathbb{F}_q$ be an arbitrary point of L_1. The projection (in $T_2^*(O)$) of x on L_2 is equal to $\Phi_2(x) = \langle \alpha\bar{a} + \bar{b} + \beta\bar{c}_{12}\rangle$ where $\beta \in \mathbb{F}_q$ is independent of α. In the same way, we will find that $\Phi_1\Phi_3\Phi_2(x) = \langle \alpha\bar{a} + \bar{b} + \beta\bar{c}_{12} + \gamma\bar{c}_{23} + \delta\bar{c}_{31}\rangle$ where γ, δ are independent of α. Now $\beta\bar{c}_{12} + \gamma\bar{c}_{23} + \delta\bar{c}_{31} = \mu\bar{a}$ where μ is independent of α. Hence the map $\phi_{2,3}$ (which is equal to the restriction of $\Phi_1\Phi_3\Phi_2$ to L_1) maps the point $\langle \alpha\bar{a} + \bar{b}\rangle$ to $\langle (\alpha + \mu)\bar{a} + \bar{b}\rangle$ where μ is independent of $\alpha \in \mathbb{F}_q$.

4.2. *The near hexagon $(T_2^*(O_1), T_2^*(O_2))$*

In [4] the following near hexagon was described. Let $\prod_\infty$ be a $PG(4, q)$, with q even, embedded as a hyperplane in the 5-dimensional space $\prod$. Consider in $\prod_\infty$ two planes α_1 and α_2 meeting each other in a point p and consider in α_i ($i = 1, 2$) a hyperoval O_i containing p. It was proved in [4] that $T_4^*(O_1 \cup O_2)$ is a near hexagon and it was denoted there by $(T_2^*(O_1), T_2^*(O_2))$.

Theorem 4.1 *The near hexagon $(T_2^*(O_1), T_2^*(O_2))$ is glued.*

Proof: Let a be a fixed affine point of $\prod$ and put $A_i = \langle a, \alpha_i\rangle$ ($i \in \{1, 2\}$). For every affine point $x \in \prod$, we define $\mathcal{Q}_i(x)$ ($i \in \{1, 2\}$) as the GQ with the affine points of $\langle x, \alpha_i\rangle$ as points, two points are collinear in the GQ whenever they are collinear in $T_4^*(O_1 \cup O_2)$. These GQ's are quads of $T_4^*(O_1 \cup O_2)$ and each point of $T_4^*(O_1 \cup O_2)$ has distance at most one to each such quad. For $i = \{1, 2\}$, let $\mathcal{Q}_i = \mathcal{Q}_i(a)$, let S_i be the set of lines of A_i intersecting $\prod_\infty$ in p, let $L_1^{(1)} = L_1^{(2)} = pa$ and finally let θ be the identity map. In the previous paragraph we determined what $\phi_{i,j}^{(1)}$ and $\phi_{i,j}^{(2)}$ look like. We can conclude that $[G_1, G_2] = 0$, hence we can define a near hexagon $\mathcal{S} = \mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, pa, pa, \theta)$. We will construct now an isomorphism Δ between $T_4^*(O_1 \cup O_2)$ and $\mathcal{S}$. Let x be an arbitrary affine point of $\prod$. The quad $\mathcal{Q}_1(x)$ (respectively $\mathcal{Q}_2(x)$) intersects $\mathcal{Q}_2$ (respectively $\mathcal{Q}_1$) in a line $\delta_2(x)$ (respectively $\delta_1(x)$) of S_2 (respectively S_1). We put $\gamma(x)$ equal to the unique point of pa nearest to x (in $T_4^*(O_1 \cup O_2)$). The point of $\mathcal{Q}_i$ nearest to x is then equal to the projection (in $\mathcal{Q}_i$) of $\gamma(x)$ on the line $\delta_i(x) \in S_i$, see Theorem 2.2. If we put $\Delta(x) = (\gamma(x), \delta_1(x), \delta_2(x))$, then we will prove that Δ is an isomorphism. Let $(a, L_1, L_2) = (\gamma(x), \delta_1(x), \delta_2(x))$ and put a_i ($i \in \{1, 2\}$) equal to the projection of a on the line L_i of $\mathcal{Q}_i$. If $L_1 = pa$, then $x = a_2$; if $L_2 = pa$, then $x = a_1$; if $L_1 \neq pa \neq L_2$, then x is the common neighbour of a_1 and a_2 (in $T_4^*(O_1 \cup O_2)$) different from a. This proves that Δ is a bijection. Since both geometries have the same order, it suffices to prove that Δ preserves adjacency in the point graph of the geometries. Let r and r' be two adjacent points of $T_4^*(O_1 \cup O_2)$. If the line rr' intersects $\prod_\infty$ in a point of O_i, then $\delta_{3-i}(r) = \delta_{3-i}(r')$ and the result follows from Corollary 2.3 by considering the projection on the quad $\mathcal{Q}_i$. □

5. New example related to $Q(5, q)$

The generalized quadrangle $Q(5, q)$ is the GQ of the points and the lines of a nonsingular elliptic quadric in PG$(5, q)$. Its order is (q, q^2). The corresponding dual generalized quadrangle is the GQ of the points and the lines of a nonsingular Hermitian variety $H(3, q^2)$ in PG$(3, q^2)$, see [6]. If we intersect this variety with a nontangent plane, then we get a set O of $q^3 + 1$ mutually noncollinear points in $H(3, q^2)$, hence O is an ovoid of $H(3, q^2)$. This ovoid O dualizes to a spread S of $Q(5, q)$.

Take now $Q = Q(5, q)$ and let L be an arbitrary line of S. The following theorem holds then (1_L denotes the identity permutation of the set of points of L).

Theorem 5.1 $\mathcal{S} = \mathcal{S}(Q, Q, S, S, L, L, 1_L)$ *is a near hexagon.*

Proof: We determine the permutations $\phi_{i,j}^{(1)} = \phi_{i,j}^{(2)}$ while reasoning in the dual GQ. The points of $H(3, q^2)$ are 1-dimensional subspaces of $V(4, q^2)$. Consider a nonsingular Hermitian form $(\cdot\,,\,\cdot)$ in $V(4, q^2)$, i.e. $(\sum_i \lambda_i v_i, \sum_j \mu_j\, w_j) = \sum_i \sum_j \lambda_i \mu_j^q (v_i, w_j)$, and let ζ be the corresponding polarity of PG$(3, q^2)$. Take now a nontangent plane π and let $\pi^\zeta = \langle \bar{u} \rangle$. Take three arbitrary (but different) points $\langle \bar{a} \rangle, \langle \bar{b} \rangle, \langle \bar{c} \rangle$ of $O = \pi \cap H(3, q^2)$. The tangent plane at $\langle \bar{a} \rangle$ intersects π in a line $\langle \bar{a}, \bar{v} \rangle$. Let $L = \langle \bar{a}, \bar{u} + \lambda \bar{v} \rangle$ be an arbitrary line of $H(3, q^2)$ through $\langle \bar{a} \rangle$. Since $(\bar{u} + \lambda \bar{v}, \bar{u} + \lambda \bar{v}) = 0$, one finds that $\lambda^{q+1} = -\frac{(\bar{u},\bar{u})}{(\bar{v},\bar{v})}$. We determine the line L' of $H(3, q^2)$ through $\langle \bar{b} \rangle$ intersecting L. This line looks like $\langle \bar{b}, \bar{u} + \lambda \bar{v} + \beta \bar{a} \rangle$. An easy calculation yields $\beta = -\lambda \frac{(\bar{v},\bar{b})}{(\bar{a},\bar{b})}$. Hence $L' = \langle \bar{b}, \bar{u} + \lambda \bar{v}' \rangle$ with $\langle \bar{v}' \rangle \in \pi \cap \langle \bar{b} \rangle^\zeta$ independent of λ. Similarly, if we project L' to a line L'' through $\langle \bar{c} \rangle$ and finally L'' to a line L''' through $\langle \bar{a} \rangle$, we will find that $L''' = \langle \bar{a}, \bar{u} + \lambda \bar{v}''' \rangle$ with $\langle \bar{v}''' \rangle \in \pi \cap \langle \bar{a} \rangle^\zeta$ independent of λ. Now $\bar{v}''' = \gamma_1 \bar{a} + \gamma_2 \bar{v}$, hence $L''' = \langle \bar{a}, \bar{u} + \lambda \gamma_2 \bar{v} \rangle$ where γ_2 is independent of λ. Just like before, one has that $(\lambda \gamma_2)^{q+1} = -\frac{(\bar{u},\bar{u})}{(\bar{v},\bar{v})}$ or $\gamma_2^{q+1} = 1$. It is now clear that $[G_1, G_2] = 0$, hence $\mathcal{S}$ is a near hexagon. □

6. New example related to $AS(q)$

For every odd prime power q, there exists a generalized quadrangle of order $(q - 1, q + 1)$ denoted by $AS(q)$, see [1, 6]. The points of $AS(q)$ are the points of the affine space AG$(3, q)$. The lines of $AS(q)$ are the following curves of AG$(3, q)$:

(1) $x = \sigma, \quad y = a, \quad z = b;$
(2) $x = a, \quad y = \sigma, \quad z = b;$
(3) $x = c\sigma^2 - b\sigma + a, \quad y = -2c\sigma + b, \quad z = \sigma.$

Here, the parameter σ ranges over GF(q) and a, b, c are arbitrary elements of GF(q). The incidence is the natural one. The set S which consists of all lines of type (1) is a spread of $AS(q)$. If L is an arbitrary line of S, then we have the following theorem.

Theorem 6.1 $\mathcal{S} = \mathcal{S}(AS(q), AS(q), S, S, L, L, 1_L)$ *is a near hexagon.*

Proof: Let $a, b, c, d \in$ GF(q) be fixed. Consider then the lines $M = \{(\sigma, a, b) \mid \sigma \in \mathrm{GF}(q)\}$ and $N = \{(\sigma, c, d) \mid \sigma \in \mathrm{GF}(q)\}$. Let $p = (\alpha, a, b)$ be an arbitrary point of M and let $p' = (\beta, c, d)$ be its projection on N. If $b = d$, then $\beta = \alpha$ and there is a line of type (2) through p and p'. If $b \neq d$, then the line through p and p' must necessarily be of type (3). Let $x = m\sigma^2 - l\sigma + k$, $y = -2m\sigma + l$, $z = \sigma$ be that line. Then we get the following equations:

$$\begin{aligned} \alpha &= mb^2 - lb + k, \\ \beta &= md^2 - ld + k, \\ a &= -2mb + l, \\ c &= -2md + l. \end{aligned}$$

Since $b \neq d$, m and l are completely determined by a, b, c and d. We have that $\beta = \alpha + m(d^2 - b^2) + l(b - d)$.

It is now clear that the maps $\phi_{i,j}^{(1)} = \phi_{i,j}^{(2)}$ are translations of the line L. This proves that $[G_1, G_2] = 0$, hence $\mathcal{S}$ is a near hexagon. □

Remark All the near hexagons with lines of size 3 and quads through every two points at distance 2 were classified in [2]. The near hexagons derived here from $AS(3)$ and $Q(5, 2)$ are both isomorphic to example (vi) of [2]. (Notice that $AS(3) \simeq Q(5, 2)$.)

7. Characterizations

7.1. The local space

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near hexagon satisfying the property that every two points at distance 2 are contained in a unique quad. For $x \in P$, we define the following incidence structure $\mathcal{S}_x$.

- The points of $\mathcal{S}_x$ are the lines of $\mathcal{S}$ through x.
- A line of $\mathcal{S}_x$ is the set of lines of $\mathcal{S}$ through x in a quad on x.
- Incidence is the symmetrized containment.

The space $\mathcal{S}_x$ is linear and is called *the local space at* x.

For $u, v \in \mathbb{N}\setminus\{0\}$, let $\mathcal{S}_{u,v} = (\mathcal{P}_{u,v}, \mathcal{L}_{u,v}, \mathrm{I}_{u,v})$ be the following linear space:

- $\mathcal{P}_{u,v} = \{\alpha, \beta_1, \ldots, \beta_u, \gamma_1, \ldots, \gamma_v\}$,
- $\mathcal{L}_{u,v} = \{\{\alpha, \beta_1, \ldots, \beta_u\}, \{\alpha, \gamma_1, \ldots, \gamma_v\}\} \cup \{\{\beta_i, \gamma_j\} \mid 1 \leq i \leq u \text{ and } 1 \leq j \leq v\}$,
- $\mathrm{I}_{u,v}$ is the symmetrized containment.

$\mathcal{S}_{u,v}$ is a linear space with a thin point (namely α). Conversely, every linear space with a thin point is obtained in this way. If $\mathcal{S}$ is a glued near hexagon, then $\mathcal{S}_x \simeq \mathcal{S}_{t_1,t_2}$ for all points x of $\mathcal{S}$.

Theorem 7.1 *Let $\mathcal{S}$ be a near hexagon satisfying the following properties*:
- *every two points at distance* 2 *are contained in a quad*,
- *if all lines of $\mathcal{S}$ are thin, then all quads are good*,
- *there exists a point x of $\mathcal{S}$ such that $\mathcal{S}_x \simeq \mathcal{S}_{1,r}$ for some $r \in \mathbb{N}\setminus\{0\}$*,

then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ.

Proof: If not all lines of $\mathcal{S}$ have the same number of points, then $\mathcal{S}$ is the direct product of a line with a GQ, see Theorem 2.5. Hence, by Theorem 2.4, we may assume that $\mathcal{S}$ has order (s, t) with $t = r + 1$. Consider through x a quad R_x containing t lines through x and let L_x be the remaining line through x. Every point z of R_x is incident with exactly one line L_z which is not in R_x. Let $y \in L_x\setminus\{x\}$ be fixed. Let M_1 and M_2 be two lines through y different from L_x and let R_y be the quad through M_1 and M_2. The quad through M_i $(i \in \{1, 2\})$ and L_x intersects R_x in a line M_i'. Now, let u be one of the $s^2(t-1)$ points of R_x at distance 2 from x. Let u_i $(i \in \{1, 2\})$ be the unique point on M_i' collinear with u. The quad through uu_i and L_{u_i} is a grid. Let u_i' be the intersection of L_{u_i} with M_i and let v_i be the unique neighbour of u_i' and u different from u_i. The point v_i is then the unique point of L_u at distance 2 from y. This implies that $v = v_1 = v_2$. Since v is collinear with the points u_1' and u_2' of R_y, v is itself contained in R_y. Hence $|\Gamma_2(y) \cap R_y| \geq s^2(t-1)$. This implies that R_y is a GQ of order $(s, t-1)$ containing all lines through y, except the line L_x and that $R_y \simeq R_x$. The result follows now immediately. □

7.2. Characterizations of the new class of near polygons

Theorem 7.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near hexagon satisfying the following properties*:
- *every two points at distance* 2 *are contained in a quad*,
- *if all lines of $\mathcal{S}$ are thin*, *then all quads are good*,
- *there exists a point x such that $\mathcal{S}_y$ has a thin point for all $y \in \Gamma(x)$*,

then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ *or $\mathcal{S}$ is a glued near hexagon.*

Proof: If not all lines of $\mathcal{S}$ have the same number of points, then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ. Hence, by Theorem 2.4 we may assume that $\mathcal{S}$ has an order (s, t). If $\mathcal{S}_y$ (with $y \in \mathcal{P}$) is a linear space with a thin point, then we may suppose that $\mathcal{S}_y$ contains a unique thin point which we denote by L_y, otherwise the result would follow from Theorem 7.1. The line L_y is then contained in exactly two quads. The following properties hold now.

(a) If y is a point for which $\mathcal{S}_y$ is a linear space with a thin point, then $\mathcal{S}_{y'} \simeq \mathcal{S}_y$ and $L_{y'} = L_y$ for all points $y' \in L_y$.

Proof: Suppose $\mathcal{S}_y \simeq \mathcal{S}_{t_1,t_2}$ with $t_1, t_2 > 1$ and $t = t_1 + t_2$. The point L_y of $\mathcal{S}_{y'}$ is contained in exactly two lines of $\mathcal{S}_{y'}$, one line has $t_1 + 1$ points, the other $t_2 + 1$ points. Since there are exactly $t_1 + t_2 + 1$ points in $\mathcal{S}_{y'}$, it follows that $\mathcal{S}_{y'} \simeq \mathcal{S}_{t_1,t_2}$. □

(b) If y_1, y_2 are points such that $\mathcal{S}_{y_1}, \mathcal{S}_{y_2}$ are linear spaces with a thin point, then L_{y_1} and L_{y_2} are equal or disjoint.

Proof: This follows immediately from (a). □

(c) There exists a point $y \in \Gamma(x)$ such that $x \in L_y$.

Proof: Suppose that this is not true. Let $y \in \Gamma(x)$ be fixed. Let $\mathcal{Q}$ be the quad of order (s, t') through xy and L_y. There are $s(t'+1)$ points $z_i \in \mathcal{Q}$ collinear with x. These give rise to $s(t'+1)$ lines L_{z_i} and all these lines are different and hence disjoint by (b). Suppose that L_z is not contained in $\mathcal{Q}$ for a certain $z \in \Gamma(x) \cap \mathcal{Q}$, then $\mathcal{S}_z$ contains at least three thick lines (namely the line defined by $\mathcal{Q}$ and the two lines of $\mathcal{S}_z$ through L_z), a contradiction since $\mathcal{S}_z$ is a linear space with a unique thin point. Hence, all lines L_z are contained in $\mathcal{Q}$ and there are at least $(s+1)(st'+s)$ points in $\mathcal{Q}$, but this is again impossible. □

Let $y \in \Gamma(x)$ such that $x \in L_y$. Hence $\mathcal{S}_x$ is also a linear space with a unique thin point L_x. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the two quads through L_x with respective orders (s, t_1) and (s, t_2). In $\mathcal{Q}_i$, there are st_i points z collinear with x and not on L_x. These give rise to st_i disjoint lines L_z which together with L_x form a spread S_i of $\mathcal{Q}_i$. Put $S_i = \{L_1^{(i)}, \ldots, L_{1+st_i}^{(i)}\}$ with $L_1^{(i)} = L_x$. Finally, let θ be the identity permutation of L_x. We prove now that $\mathcal{S} \simeq \mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$.

First we prove that every point u of $\mathcal{S}$ has distance at most 1 to each $\mathcal{Q}_i$ ($i \in \{1,2\}$). Let u' be the unique point of L_x nearest to u; we may suppose that $d(u, u') = 2$. Since $\mathcal{S}_{u'} \simeq \mathcal{S}_x$, it follows that the quad through u and u' intersects each $\mathcal{Q}_i$ in a line. This proves that each $\mathcal{Q}_i$ contains a point collinear with u. For $i \in \{1,2\}$ and $u \in \mathcal{P}$, let $p_i(u)$ denote the unique point of $\mathcal{Q}_i$ nearest to u.

Next we prove that all the local spaces $\mathcal{S}_u$ are isomorphic to $\mathcal{S}_{t_1,t_2}$. Since for all $u \in \mathcal{Q}_i$, L_u is contained in exactly two quads ($\mathcal{Q}_i$ and another quad), we have that $G_u \simeq \mathcal{S}_{t_1,t_2}$. Let u be a point of $\mathcal{S}$ not contained in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. Let $u' = p_1(u)$ and $u'' = p_2(u)$. The local space $\mathcal{S}_u$ contains $t_1 + t_2 + 1$ points, a line with $t_1 + 1$ points (determined by the quad through uu'' and $L_{u''}$) and a line with $t_2 + 1$ points (determined by the quad through uu' and $L_{u'}$). From this it follows that $\mathcal{S}_u \simeq \mathcal{S}_{t_1,t_2}$. Hence L_u is defined for all $u \in \mathcal{P}$ and all these lines determine a spread of $\mathcal{S}$. Each L_u is contained in exactly two quads. One quad intersects $\mathcal{Q}_2$ in a line and is isomorphic to $\mathcal{Q}_1$. The other quad intersects $\mathcal{Q}_1$ and is isomorphic to $\mathcal{Q}_2$. Note that the isomorphisms are defined by the projections p_i, $i \in \{1,2\}$.

We consider now the following map $\Delta : \mathcal{P} \mapsto L_x \times S_1 \times S_2$, $\Delta(u) = (\gamma(u), \delta_1(u), \delta_3(u))$, where $\gamma(u)$ is the unique point of L_x nearest to u and $\delta_i(u)$ ($i \in \{1,2\}$) is the unique line of S_i incident with $p_i(u)$. By Theorem 2.2, it follows that $p_i(u)$ is the projection (in $\mathcal{Q}_i$) of $\gamma(u)$ on the line $\delta_i(u)$. Let $(a, L_1, L_2) = (\gamma(u), \delta_1(u), \delta_2(u))$ and put a_i ($i \in \{1,2\}$) equal to the projection of a on the line L_i of $\mathcal{Q}_i$. If $L_1 = L_x$, then $u = a_2$; if $L_2 = L_x$, then $u = a_1$; if $L_1 \neq L_x \neq L_2$, then u is the common neighbour of a_1 and a_2 different from a. This proves that Δ is a bijection. Since both geometries have the same order, it suffices to prove that Δ preserves adjacency in the point graph of the geometries. Let x and x' be two adjacent points. If x and x' are contained in a quad intersecting $\mathcal{Q}_2$, then

$\delta_2(x) = \delta_2(x')$ and the result follows from Corollary 2.3 by projection on the quad $\mathcal{Q}_1$. If x and x' are contained in a quad intersecting $\mathcal{Q}_1$, then $\delta_1(x) = \delta_1(x')$ and the result follows from Corollary 2.3 by projection on the quad $\mathcal{Q}_2$. □

Theorem 7.3 *Let $\mathcal{S}$ be a near hexagon satisfying the following properties*:

- *every two points at distance* 2 *are contained in a quad*,
- *if all lines of $\mathcal{S}$ are thin*, *then all quads are good*,
- *there exists a point x such that $\mathcal{S}_x$ has a thin point and such that $\mathcal{S}_y$ contains the same number of lines for all $y \in \Gamma(x)$*,

then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ *or $\mathcal{S}$ is a glued near hexagon.*

Proof: Just like before, we may suppose that $\mathcal{S}$ has an order (s, t). Theorem 2.6 implies that the number of points in $\Gamma_2(y)$ is independent of the point y of $\mathcal{S}$. For $y \in \Gamma(x)$, let V_y denote the set of quads through y. Now,

$$\sum_{\mathcal{Q}\in V_y} 1, \quad \sum_{\mathcal{Q}\in V_y} s^2 t_{\mathcal{Q}}, \quad \sum_{\mathcal{Q}\in V_y} t_{\mathcal{Q}}(t_{\mathcal{Q}}+1),$$

are respectively equal to the number of quads through y, the number of points in $\Gamma_2(y)$ and $t(t+1)$, hence these quantities are independent of $y \in \Gamma(x)$. Let L_x be a thin point of $\mathcal{S}_x$ and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the two quads through L_x with respective orders (s, t_1) and (s, t_2). One has that $t = t_1 + t_2$. Let $z \neq x$ be a second point of L_x. If $y \in \mathcal{Q}_1 \cap \Gamma(x)$, then

$$\sum_{\mathcal{Q}\in V_y} (t_{\mathcal{Q}}-1)(t_2-t_{\mathcal{Q}}) = \sum_{\mathcal{Q}\in V_z} (t_{\mathcal{Q}}-1)(t_2-t_{\mathcal{Q}}) = (t_1-1)(t_2-t_1).$$

Let $V'_y = V_y \setminus \{\mathcal{Q}_1\}$, then

$$\sum_{\mathcal{Q}\in V'_y} (t_{\mathcal{Q}}-1)(t_2-t_{\mathcal{Q}}) = 0.$$

Since there are only $t+1$ lines through y and $\mathcal{Q}_1$ has t_1+1 lines through y, one has that $1 \leq t_{\mathcal{Q}} \leq t_2$ for all $\mathcal{Q} \in V'_y$. This implies that $t_{\mathcal{Q}} = 1$ or $t_{\mathcal{Q}} = t_2$ for all $\mathcal{Q} \in V'_y$. By Theorem 7.1, we may suppose that $t_1, t_2 \neq 1$. From

$$\sum_{\mathcal{Q}\in V_y} 1 = \sum_{\mathcal{Q}\in V_z} 1,$$

and

$$\sum_{\mathcal{Q}\in V_y} t_{\mathcal{Q}} = \sum_{\mathcal{Q}\in V_z} t_{\mathcal{Q}},$$

it follows now that the number of quads $\mathcal{Q}$ of V'_y with $t_{\mathcal{Q}} = t_2$ is equal to 1. This implies that $\mathcal{S}_y \simeq \mathcal{S}_{t_1,t_2}$ for all $y \in \Gamma(x) \cap \mathcal{Q}_1$. A similar reasoning shows that this is also true for $y \in \Gamma(x) \cap \mathcal{Q}_2$. The result follows now from the previous theorem. □

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