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Recognizing Schubert Cells

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Abstract. We address the problem of distinguishing between different Schubert cells using vanishing patterns of generalized Plücker coordinates.

Keywords: Schubert cell, Schubert variety, flag manifold, Plücker coordinate, Bruhat cell, vanishing pattern

1. Introduction

This paper focuses on the properties of Schubert cells as quasi-projective subvarieties of a generalized flag variety. More specifically, we investigate the problem of distinguishing between different Schubert cells using vanishing patterns of generalized Plücker coordinates.

1.1. Formulations of the main problems

Let G be a simply connected complex semisimple Lie group of rank r with a fixed Borel subgroup B and a maximal torus $H \subset B$. Let $W = \mathrm{Norm}_G(H)/H$ be the Weyl group of G. The generalized flag manifold G/B can be decomposed into the disjoint union of *Schubert cells* $X_w^\circ = (BwB)/B$, for $w \in W$.

To any weight γ that is W-conjugate to some fundamental weight of G, one can associate a *generalized Plücker coordinate* p_γ on G/B (see [9] or Section 3 below). In the case of type A_{n-1} (i.e., $G = SL_n$), the p_γ are the usual Plücker coordinates on the flag manifold.

The closure of a Schubert cell X_w° is the *Schubert variety* X_w, an irreducible projective subvariety of G/B that can be described as the set of common zeroes of some collection of generalized Plücker coordinates p_γ. It is also known (see, e.g., Proposition 4.1 below) that every Schubert cell X_w° can be defined by specifying vanishing and/or non-vanishing of some collection of Plücker coordinates.

The main two problems studied in this paper are the following.

Problem 1.1 (Short descriptions of cells**)** Describe a given Schubert cell by as small as possible number of equations of the form $p_\gamma = 0$ and inequalities of the form $p_\gamma \neq 0$.

Problem 1.2 (Cell recognition**)** Suppose a point $x \in G/B$ is unknown to us, but we have access to an oracle that answers questions of the form: "$p_\gamma(x) = 0$, true or false?" How many such questions are needed to determine the Schubert cell x is in?

Problem 1.2 looks harder than Problem 1.1, since we do not fix a Schubert cell in advance. However, we will demonstrate that the complexity of the two problems is the same: informally speaking, it takes as much time to recognize a cell as it takes to describe it.

Our interest in these problems was originally motivated by their relevance to the theory of total positivity criteria. As shown in [5], these criteria take different form in different Bruhat cells BwB, so one has to first find out which cell an element $g \in G$ is in.

1.2. Overview of the paper

In Section 2, we illustrate our problems by working out the special case $G = SL_3$. Section 3 provides the necessary background on generalized Plücker coordinates, Bruhat orders, and Schubert varieties.

The number of equations of the form $p_\gamma = 0$ needed to define a Schubert variety is generally much larger than its codimension. In Proposition 6.3, we show that for certain Schubert variety X_w in the flag manifold of type A_{n-1}, one needs exponentially many (as a function of n) such equations to define it, even though $\operatorname{codim}(X_w) \leq \dim(G/B) = \binom{n}{2}$. Given this kind of "complexity" of Schubert varieties, it may appear surprising that every Schubert *cell* actually does have a short description in terms of vanishing or non-vanishing of certain Plücker coordinates. In Theorem 4.8, for the types A_r, B_r, C_r, and G_2, we provide a description of an arbitrary Schubert cell X_w° that only uses $\operatorname{codim}(X_w)$ equations of the form $p_\gamma = 0$ and at most r inequalities of the form $p_\gamma \neq 0$. Thus in these cases every Schubert cell is a "set-theoretic complete intersection." Our proof of this property relies on the new concept of an *economical* linear ordering of fundamental weights. For the type D, a description of Schubert cells is slightly more complicated; see Proposition 4.11. This completes our treatment of Problem 1.1.

In Section 5, we turn to Problem 1.2. Our main result is Algorithm 5.5 that recognizes a Schubert cell X_w° containing an element x. In the cases when an economical ordering exists (i.e., for the types A_r, B_r, C_r, and G_2), our algorithm ends up examining precisely the same Plücker coordinates of x that appear in Theorem 4.8. In the case of type A_{n-1}, recognizing a cell requires testing the vanishing of at most $\binom{n}{2}$ Plücker coordinates.

In Section 7, we discuss the problem of *cell recognition without feedback*, i.e., the problem of presenting a subset of Plücker coordinates whose vanishing pattern determines which cell a point is in. We show that such a subset must contain all but a negligible proportion of the Plücker coordinates. (Our proof of this result exhibits a surprising connection with coding theory.) In Section 8, we demonstrate that the situation changes radically if we only allow *generic* points in each cell. With this assumption, knowing the vanishing pattern of polynomially many Plücker coordinates (namely, the ones corresponding to the *base* of W, as defined by Lascoux and Schützenberger [13]), suffices to recognize a cell.

1.3. Comments

For the purposes of this paper, all the relevant information about any point on a flag variety can be extracted from a finite binary string—the vanishing pattern of its Plücker coordinates. No explicit description is known for the set of all possible vanishing patterns. For the

type A, a combinatorial abstraction of these patterns is provided by the notion of a matroid; for a general Coxeter group, such an abstraction was given by Gelfand and Serganova [9]. All results of the present paper can be directly extended to generalized matroids of [9] (irrespective of their realizability), and in fact to a more general combinatorial framework of "acceptable" binary vectors introduced in Definition 5.2.

Note that the "cell recognition" problem becomes much simpler if its input is an element gB represented by a matrix of g in some standard representation of G. For instance, if $G = SL_n$, then the Bruhat cell of a given matrix g can be easily determined via Gaussian elimination. The reader is referred to [11], where an even more general problem of classifying an arbitrary matrix (not necessarily invertible) is solved. (This was generalized to the classical series in [10].)

2. Example: $G = SL_3$

To illustrate our problems, let us look at a particular case of type A_2 where $G = SL_3$. In this case, a flag $x = (0 \subset F_1 \subset F_2 \subset F_3 = \mathbb{C}^3) \in G/B$ can be represented by a 3×2 matrix whose first column spans F_1 and whose first two columns span F_2. The homogeneous Plücker coordinates of x are:

(1) the matrix entries p_1, p_2, and p_3 in the first column of the matrix;
(2) the 2×2 minors on the first 2 columns: p_{12}, p_{13}, p_{23}.

The complete set of restrictions satisfied by the 6 Plücker coordinates consists of:

(a) the *Grassmann-Plücker relation* $p_1 p_{23} - p_2 p_{13} + p_3 p_{12} = 0$;
(b) non-degeneracy conditions: $(p_1, p_2, p_3) \neq (0, 0, 0)$, $(p_{12}, p_{13}, p_{23}) \neq (0, 0, 0)$.

The Weyl group here is the symmetric group $\mathcal{S}_3$, with generators $s_1 = (1, 2)$ and $s_2 = (2, 3)$ and relations $s_1^2 = s_2^2 = 1$ and $w_o = s_1 s_2 s_1 = s_2 s_1 s_2$. In Table 1, we show which Plücker coordinates must or must not vanish on each particular Schubert cell. In the table, 0 means "vanishes on X_w°," 1 means "does not vanish anywhere on X_w°," and the wildcard $*$ means that both zero and nonzero values do occur.

Table 1. Schubert cells and Plücker coordinates in type A_2.

w		p_1	p_2	p_3	p_{12}	p_{13}	p_{23}	X_w°
e	123	1	0	0	1	0	0	$p_3 = p_2 = p_{13} = 0$
s_1	213	$*$	1	0	1	0	0	$p_{13} = p_{23} = 0$, $p_2 \neq 0$
s_2	132	1	0	0	$*$	1	0	$p_2 = p_3 = 0$, $p_{13} \neq 0$
$s_1 s_2$	231	$*$	1	0	$*$	$*$	1	$p_3 = 0$, $p_{23} \neq 0$
$s_2 s_1$	312	$*$	$*$	1	$*$	1	0	$p_{23} = 0$, $p_3 \neq 0$
w_o	321	$*$	$*$	1	$*$	$*$	1	$p_3 \neq 0$, $p_{23} \neq 0$

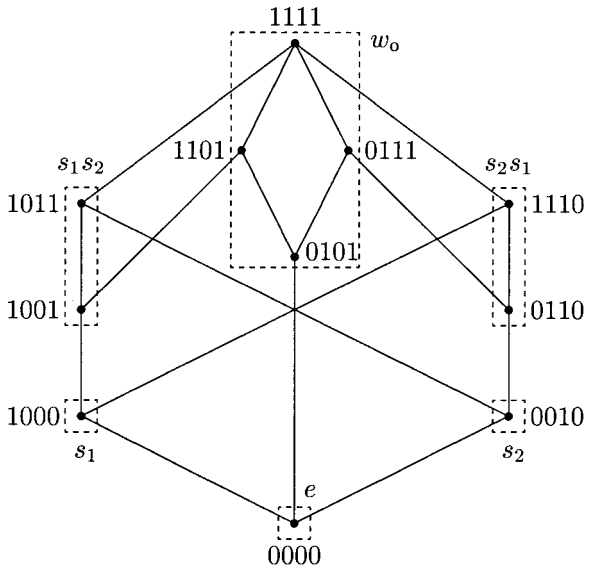


Figure 1. Vanishing patterns of Plücker coordinates p_2, p_3, p_{13}, p_{23}.

Concerning Problem 1.1, we see that each Schubert cell can be described in terms of the 4 Plücker coordinates p_2, p_3, p_{13}, p_{23} (these are exactly the "bigrassmannian" coordinates discussed in Section 8). Moreover, 3 equations/inequalities suffice to describe every single cell, as shown in the last column of Table 1.

Altogether, there are 11 possible vanishing patterns for the Plücker coordinates p_2, p_3, p_{13}, p_{23}. The classification of points on the flag variety according to the vanishing patterns of these coordinates provides a refinement of the Schubert cell decomposition. In figure 1, we represent this stratification by a graph (actually, the Hasse diagram of a poset) whose 11 vertices are labelled by the vanishing patterns and whose edges show how the subcells degenerate into each other when a condition of the form $p_I \neq 0$ is replaced by $p_I = 0$. The dashed boxes enclose the subsets making up individual Schubert cells. See Section 8 for further discussion of this poset.

The Schubert varieties X_w are defined by the *equalities* appearing in the last column of Table 1. Thus in this case the minimal number of equations of the form $p_\gamma(x) = 0$ that define a Schubert variety X_w as a subset of G/B is equal to its codimension. In general, however, such a statement is grossly false (see Section 6).

Turning to Problem 1.2, the best recognition algorithm is given in figure 2; it requires 3 questions. Notice that each branch of the tree provides a short description of the corresponding Schubert cell.

3. Preliminaries

In this section, we review basic facts about generalized Plücker coordinates, the Bruhat orders, and Schubert varieties. For general background on these topics, see, e.g., [9, Section 4], [1], and [6, §23.3, 23.4].

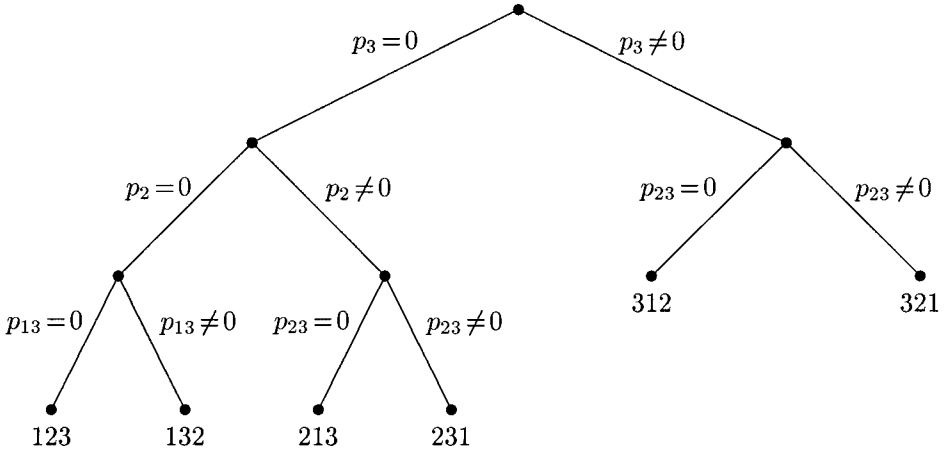


Figure 2. Cell recognition algorithm for the type A_2.

3.1. Generalized Plücker coordinates

Our approach to this classical subject is similar to the one of Gelfand and Serganova [9, Section 4.2]. Let us fix some linear ordering $\omega_1, \ldots, \omega_r$ of fundamental weights; the choice of this ordering will later become important. We will call the weights $\gamma \in W\omega_i$ *Plücker weights of level i*. Recall that the orbits of fundamental weights are pairwise disjoint, so the notion of level is well defined.

Let V_{ω_i} be the fundamental representation of G with highest weight ω_i. The Plücker weights γ of level i are precisely the extremal weights of V_{ω_i}. The corresponding weight subspaces $V_{\omega_i}(\gamma)$ are known to be one-dimensional. Let us fix an arbitrary nonzero vector $v_\gamma \in V_{\omega_i}(\gamma)$ for each such γ. In particular, v_{ω_i} is a highest weight vector in V_{ω_i}, and thus an eigenvector for the action of any $b \in B$; we will write $bv_{\omega_i} = b^{\omega_i} v_{\omega_i}$.

Definition 3.1 The *generalized Plücker coordinate* p_γ associated to a Plücker weight γ of level i is defined as follows. For $g \in G$, let $p_\gamma(g)$ be the coefficient of v_γ in the expansion of gv_{ω_i} into any basis of V_{ω_i} consisting of weight vectors. It follows that $p_\gamma(gb) = b^{\omega_i} p_\gamma(g)$ for any $g \in G$ and $b \in B$. Thus we can think of p_γ as a global section of the line bundle on G/B corresponding to the character $b \mapsto b^{\omega_i}$ of B. It then makes sense to talk about vanishing or non-vanishing of p_γ at any point $x = gB$ of the generalized flag manifold G/B.

Although the definition of p_γ depends on the choice of normalization for the vectors v_γ, this dependence is not very essential: a different choice of normalizations only changes each p_γ by a nonzero scalar multiple. In particular, the set of zeroes of each p_γ is a uniquely and unambiguously defined hypersurface in G/B.

We note that one natural choice of normalization is the following: define p_γ as the "generalized minor" Δ_{γ,ω_i}, in the notation of [5, Section 1.4].

For the type A_{n-1}, the notion of a Plücker coordinate specializes to the ordinary one (see, e.g., [7]), as follows. Let us use the standard numeration of the fundamental weights, so

that $V_{\omega_1} = V = \mathbb{C}^n$ is the defining representation of $G = SL_n$, and $V_{\omega_i} = \Lambda^i V$. Plücker weights of level i are naturally identified with subsets $I \subset [1, n]$ of cardinality i: under this identification, the weight subspace $V_{\omega_i}(\gamma)$ is the one-dimensional subspace $\Lambda^i \mathbb{C}^I \subset \Lambda^i V$. The variety G/B is identified with the manifold of all complete flags $x = (0 \subset F_1 \subset \cdots \subset F_n = V)$ in V: for $x = gB \in G/B$, the subspace F_i is generated by the first i columns of the matrix g. The Plücker coordinate $p_I(x)$ is simply the minor of g with the row set I and the column set $[1, i] = \{1, \ldots, i\}$. It follows that p_I does *not* vanish at a flag x if and only if $F_i \cap \mathbb{C}^{[1,n]-I} = \{0\}$.

3.2. Bruhat orders

The Bruhat order can be defined for an arbitrary Coxeter group W. (Even though it seems to be well established that the Bruhat order is actually due to Chevalley, we stick with the traditional terminology to avoid misconceptions.) Let $S = \{s_1, \ldots, s_r\}$ be the set of simple reflections in W, and $\ell(w)$ be the length function. The Bruhat order on W is the transitive closure of the following relation: $w < wt$ for any reflection t (that is, a W-conjugate of a simple reflection) such that $\ell(w) < \ell(wt)$.

For every subset J of $[1, r]$, let W_J denote the parabolic subgroup of W generated by the simple reflections s_j with $j \in J$. Each coset in W/W_J has a unique representative which is minimal with respect to the Bruhat order. These representatives are partially ordered by the Bruhat order, inducing a partial order on W/W_J. This partial order is also called the Bruhat order on W/W_J.

We will be especially interested in the coset spaces modulo *maximal* parabolic subgroups $W_{\hat{i}} = W_{[1,r]-\{i\}}$. The following basic result is due to Deodhar [2, Lemma 3.6].

Lemma 3.2 *For $u, v \in W$, we have: $u \leq v$ if and only if $uW_{\hat{i}} \leq vW_{\hat{i}}$ for all i.*

From now on we assume that W is the Weyl group associated to a semisimple complex Lie group G. Then the stabilizer of a fundamental weight ω_i is the maximal parabolic subgroup $W_{\hat{i}}$. Thus the correspondence $w \mapsto w\omega_i$ establishes a bijection between the coset space $W/W_{\hat{i}}$ and the set $W\omega_i$ of Plücker weights of level i. This bijection transfers the Bruhat order from $W/W_{\hat{i}}$ to $W\omega_i$. Note that if γ and δ are two Plücker weights of the same level, with $\gamma \leq \delta$ with respect to the Bruhat order, then the weight $\gamma - \delta$ can be expressed as a sum of simple roots. The converse statement is true for type A but false in general. A counterexample for the type B_3 is given in [16, pp. 176, 177]; see also Deodhar [3] (we thank John Stembridge for providing this reference).

The Bruhat order on the Weyl group W also has the following well-known geometric interpretation in terms of Schubert cells and Schubert varieties:

$$u \leq v \Longleftrightarrow X_u^\circ \subset X_v \Longleftrightarrow X_u \subset X_v. \tag{3.1}$$

A similar interpretation exists for the Bruhat order on any coset space W/W_J: if P_J is the parabolic subgroup in G corresponding to W_J then the correspondence $w \mapsto wP_J$

establishes a bijection between W/W_J and G/P_J, and we have $uW_J \leq vW_J$ if and only if the "cell" $(BuP_J)/P_J$ is contained in the closure of $(BvP_J)/P_J$.

To illustrate the above concepts, consider the case of type A_{n-1} where $G = SL_n$, and W is the symmetric group $\mathcal{S}_n$. We have already seen that Plücker weights of level i are in natural bijection with the i-subsets of $[1, n]$. The Bruhat order on the i-subsets of $[1, n]$ can be explicitly described as follows: for two subsets $J = \{j_1 < \cdots < j_i\}$ and $K = \{k_1 < \cdots < k_i\}$, we have $J \leq K$ if and only if $j_1 \leq k_1, \ldots, j_i \leq k_i$. Let u and v be two permutations in $\mathcal{S}_n$. By Lemma 3.2, $u \leq v$ (in the Bruhat order) if and only if $u([1, i]) \leq v([1, i])$ for any i, in the sense just defined. (This is the original Ehresmann's criterion [4].)

3.3. Set-theoretic description of Schubert varieties

The following proposition is well known to experts although we were unable to find it explicitly stated in the literature.

Proposition 3.3 *A point $x \in G/B$ belongs to the Schubert variety X_w if and only if $p_\gamma(x) = 0$ for any Plücker weight γ (say, of level i) such that $\gamma \not\leq w\omega_i$ in the Bruhat order.*

We note that the much stronger results in [12, 15] provide a scheme-theoretic description of Schubert varieties.

We will show that Proposition 3.3 is an easy corollary of the following lemma.

Lemma 3.4 [12, 17]

1. *A point x on the Schubert variety X_w belongs to the Schubert cell X_w° if and only if $p_{w\omega_i}(x) \neq 0$ for all i.*
2. *A Plücker coordinate p_γ of level i identically vanishes on X_w if and only if $\gamma \not\leq w\omega_i$.*

Lemma 3.4 can be extracted from [17, Lemmas 3 and 4] and [12, Corollary 10.2]. (We thank the anonymous referee for providing these references.) To keep our presentation self-contained, and to spare the reader from the trouble of reconciling the notation and conventions of [17] with the ones in this paper, we provide short proofs of Lemma 3.4 and Proposition 3.3 below.

Proof: We begin by proving Lemma 3.4. Let us first show that $p_{w\omega_i}$ vanishes nowhere on X_w°. Recall the definition of $p_\gamma(gB)$: up to a nonzero scalar, this is the coefficient of v_γ in the expansion of gv_{ω_i} in any basis of V_{ω_i} consisting of weight vectors. Here V_{ω_i} is the fundamental representation of G with highest weight ω_i, and $v_\gamma \in V_{\omega_i}$ is a (unique up to a scalar) vector of weight γ. It follows that if $g \in BwB$ and $\gamma = w\omega_i$, then $p_\gamma(gB)$ is a nonzero scalar multiple of the coefficient of v_γ in the expansion of bv_γ for some $b \in B$. This coefficient is clearly nonzero, as desired.

Let us now assume that γ is a Plücker weight of level i such that $\gamma \leq w\omega_i$. Then $\gamma = u\omega_i$ for some $u \leq w$. We have just proved that p_γ vanishes nowhere on X_u°. But $X_u^\circ \subset X_w$ by (3.1); therefore p_γ does not identically vanish on X_w.

To complete the proof of Part 2, we need the converse statement: if p_γ does not identically vanish on X_w then $\gamma \leq w\omega_i$. (The following argument was shown to us by Peter Littelmann; it closely follows the proof of Proposition 1 in Gelfand and Serganova [9, Section 5].) Let $P(V_{\omega_i})$ denote the projectivization of the vector space V_{ω_i}, and let $[v] \in P(V_{\omega_i})$ denote the projectivization of a nonzero vector $v \in V_{\omega_i}$. Then the stabilizer of $[v_{\omega_i}]$ in G is the maximal parabolic subgroup $P_{\hat{i}}$, so the map $g \mapsto g[v_{\omega_i}]$ identifies the coset space $G/P_{\hat{i}}$ with the orbit $G[v_{\omega_i}] \subset P(V_{\omega_i})$.

We shall use the following well-known fact: the convex hull of all weights of the representation V_{ω_i} is a convex polytope whose vertices are precisely the Plücker weights of level i. It follows that, for every Plücker weight γ of level i, there exists a one-parameter subgroup $\chi : \mathbb{C}_{\neq 0} \to H$ such that

$$\lim_{t\to\infty} \chi(t)^\delta / \chi(t)^\gamma = 0 \tag{3.2}$$

for any weight $\delta \neq \gamma$ of V_{ω_i}.

Now suppose p_γ does not identically vanish on X_w. By definition, this means that v_γ appears with nonzero coefficient in the expansion of gv_{ω_i} for some $g \in BwB$. Using (3.2), we see that

$$[v_\gamma] = \lim_{t\to\infty} \chi(t) g[v_{\omega_i}].$$

It follows that $[v_\gamma]$ lies in the closure of $(BwB)[v_{\omega_i}]$. We have $\gamma = u\omega_i$ for some $u \in W$. Identifying as above the orbit $G[v_{\omega_i}]$ with the coset space $G/P_{\hat{i}}$ we conclude that the coset $uP_{\hat{i}}$ is contained in the closure of $(BwP_{\hat{i}})/P_{\hat{i}}$. As explained in Section 3.2, this implies that $\gamma = u\omega_i \leq w\omega_i$, and Part 2 is proved.

To complete the proof of Part 1, it remains to show the following: if $x \in X_w \setminus X_w^\circ$, then $p_{w\omega_i}(x) = 0$ for some i. Any $x \in X_w \setminus X_w^\circ$ belongs to X_u for some $u < w$. By Lemma 3.2, we have $u\omega_i < w\omega_i$ for some i. We have just proved that this implies that $p_{w\omega_i}$ vanishes on X_u. In particular, $p_{w\omega_i}(x) = 0$, completing the proof of Lemma 3.4.

To deduce Proposition 3.3 from Lemma 3.4, we let $X \subset G/B$ denote the variety defined by the equations $p_\gamma(x) = 0$ for all Plücker weights γ of any level i such that $\gamma \not\leq w\omega_i$. The inclusion $X_w \subset X$ follows from Lemma 3.4.2. Now assume that $x \notin X_w$; say, $x \in X_v^\circ$ with $v \not\leq w$. By Lemma 3.2, there exists i such that $v\omega_i \not\leq w\omega_i$. By Lemma 3.4.1, $p_{v\omega_i}(x) \neq 0$. Therefore $x \notin X$, as desired. □

4. Short descriptions of Schubert cells

This section is devoted to set-theoretic descriptions of Schubert cells. Such a description of X_w° can be obtained by combining Lemma 3.4 (1) with the set-theoretic description of X_w in Proposition 3.3. However, the following proposition shows that we can do better.

Proposition 4.1 *An element $x \in G/B$ belongs to a Schubert cell X_w° if and only if, for every $i \in [1, r]$, the following conditions hold*:

(1) $p_{w\omega_i}(x) \neq 0$;
(2) $p_\gamma(x) = 0$ *for all* $\gamma \in wW_{[i,r]}\omega_i$ *such that* $\gamma > w\omega_i$.

Proof: In view of Lemma 3.4, these conditions are certainly neccesary. Let us prove that they are also sufficient. Suppose that (1)–(2) hold, and let $x \in X_u^\circ$. Our goal is to show that $u = w$. First, by (1) and Lemma 3.4, we have $w\omega_i \leq u\omega_i$ for all i, hence $w \leq u$ by Lemma 3.2. Now suppose that $w < u$. Then at least one of the inequalities $w\omega_i \leq u\omega_i$ is strict; take the minimal index i such that $w\omega_i < u\omega_i$. The equalities $w\omega_j = u\omega_j$ for $j < i$ imply that $w^{-1}u \in \bigcap_{j=1}^{i-1} W_{\hat{j}}$. Using the equality $W_{J_1} \cap \cdots \cap W_{J_k} = W_{J_1 \cap \cdots \cap J_k}$ valid in any Coxeter group (see [1]), we conclude that

$$\bigcap_{j=1}^{i-1} W_{\hat{j}} = W_{[i,r]}. \tag{4.1}$$

It follows that the weight $\gamma = u\omega_i$ satisfies both conditions in (2), so we must have $p_{u\omega_i}(x) = 0$. But this contradicts the last statement in Lemma 3.4, and we are done. □

Notice that condition (2) in Proposition 4.1 depends on the choice of ordering of fundamental weights. We will introduce a special class of *economical* orderings that lead to the minimal possible number of equations in (2).

For any i, let $R(i)$ denote the set of positive roots whose expansion into the sum of simple roots contains the simple root α_i.

Proposition 4.2 *The correspondence* $\alpha \mapsto s_\alpha\omega_i$ *is an embedding of* $R(i)$ *into* $W\omega_i - \{\omega_i\}$.

Proof: Let α be a positive root. We have

$$\omega_i - s_\alpha\omega_i = (\omega_i, \alpha^\vee)\alpha, \tag{4.2}$$

where $(,)$ is a W-invariant scalar product of weights, and $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ is the dual root. By definition of fundamental weights, $(\omega_i, \alpha^\vee)$ is the coefficient of $\alpha_i^\vee$ in the expansion of $\alpha^\vee$ into the sum of dual simple roots. Clearly this coefficient is nonzero precisely when $\alpha \in R(i)$. Since no two positive roots are proportional to each other, the vectors $(\omega_i, \alpha^\vee)\alpha$ for $\alpha \in R(i)$ are distinct nonzero vectors, proving the proposition. □

Definition 4.3 We say that an index $i \in [1, r]$ (or the corresponding fundamental weight ω_i) is *economical* for W if the correspondence in Proposition 4.2 is a *bijection* between $R(i)$ and $W\omega_i - \{\omega_i\}$. This is equivalent to

$$1 + |R(i)| = |W\omega_i| = |W|/|W_i|. \tag{4.3}$$

Here is a classification of all economical fundamental weights in irreducible Weyl groups.

Proposition 4.4 *Let* W *be an irreducible Weyl group of rank* r *with the set of simple reflections ordered as in* [1]. *An index* i *is economical for* W *precisely in the following three cases*:

(1) $r \leq 2$, *and* i *is arbitrary.*
(2) W *is of type* A_r *for* $r > 2$, *and* $i = 1$ *or* $i = r$.
(3) W *is of type* B_r *or* C_r *for* $r > 2$, *and* $i = 1$.

Proof: First let us show that an index i is indeed economical in each of the cases (1)–(3). The statement is trivial for type A_1. If $r = 2$ then W is of type A_2, B_2 or G_2, i.e., is a dihedral group of cardinality $2d$ where $d = 3$, 4 or 6, respectively. We have $|W|/|W_{\hat{i}}| = d$ for any i since $W_{\hat{i}}$ is the two-element group. On the other hand, d is the number of positive roots in each case which implies that $|R(i)| = d - 1$ (the only positive root not in $R(i)$ is the simple root different from α_i). Thus our statement follows from (4.3).

In case (2), we have $W = \mathcal{S}_{r+1}$ and $W_{\hat{i}} = \mathcal{S}_r$ for $i = 1$ or $i = r$. Therefore, $|W|/|W_{\hat{i}}| = (r+1)!/r! = r + 1$. On the other hand, $R(1)$ (resp. $R(r)$) consists of r roots $\varepsilon_1 - \varepsilon_{j+1}$ (resp. $\varepsilon_j - \varepsilon_{r+1}$) for $j = 1, \ldots, r$, in the standard notation of [1]. Thus both $i = 1$ and $i = r$ are economical.

Similarly, in case (3), the index $i = 1$ (in the usual numeration) is economical because $|W|/|W_{[2,r]}| = (2^r r!)/(2^{r-1}(r-1)!) = 2r$, while $R(1)$ (say, for type B_r) consists of $2r - 1$ roots: $\varepsilon_1 \pm \varepsilon_j$ $(j = 2, \ldots, r)$ and ε_1.

To show that cases (1)–(3) exhaust all economical indices, we use the following observation: if i is economical for W then, in particular, we have

$$\omega_i - w_0\omega_i = (\omega_i, \alpha^\vee)\alpha$$

for some positive root α (cf. (4.2)), where w_0 is the maximal element of W. Since w_0 sends positive roots to negative ones, it follows that $-w_0\omega_i$ is also a fundamental weight (possibly equal to ω_i), and so α must be a dominant weight. If W is simply-laced, i.e., all roots are of the same length, then it is known that W acts on the set R of roots transitively. Therefore, there is a unique root which is a dominant weight: the maximal root $\alpha_{\max}$. The tables in [1] show that if W is simply-laced but not of type A_r then $\alpha_{\max}$ is proportional to some fundamental weight ω_i, so only this fundamental weight has a chance to be economical. But then we have

$$|W\omega_i| = |W\alpha_{\max}| = |R| = 2|R_+| > |R(i)| + 1,$$

so, for a simply laced W not of type A_r, there are no economical indices.

If W is not simply-laced then there are precisely two roots which are dominant weights: the maximal long root and the maximal short root. Leaving aside cases (1) and (3) that we already considered, this leaves only three more possibilites for an economical index: $i = 2$ for W of type B_r with $r > 2$; and $i = 1$ or $i = 4$ for W of type F_4. Since the root system of type F_4 is self-dual, we have $|W\omega_1| = |W\omega_4| = |R_+|$, while $|R(i)| \leq |R_+| - 3$ for any i (since $R(i)$ does not contain three simple roots different from α_i). As for W of type B_r and $i = 2$, the set $W\omega_2$ consists of $2r(r-1)$ weights of the form $\pm(\varepsilon_i \pm \varepsilon_j)$, $1 \leq i < j \leq r$, and we have

$$|R(2)| + 1 \leq |R_+| - r + 2 = r(r-1) + 2 < 2r(r-1) = |W\omega_2|.$$

We see that, in each of the three cases, $|R(i)| + 1 < |W\omega_i|$, i.e., i is not economical, and we are done. □

Proposition 4.5 *If a fundamental weight ω_i is economical for W then the Bruhat order on $W\omega_i$ is linear.*

Proof: Let $\gamma = w\omega_i$ and δ be two distinct Plücker weights of level i. Then $w^{-1}\delta \neq \omega_i$, which by Definition 4.3 implies that $w^{-1}\delta = t\omega_i$ for some reflection t. Since wt and w are comparable in the Bruhat order, the same is true for $\delta = wt\omega_i$ and $\gamma = w\omega_i$. □

According to V. Serganova (private communication), the converse of Proposition 4.5 is also true: the Bruhat order on $W\omega_i$ is linear precisely in one of the cases (1)–(3) in Proposition 4.4.

Definition 4.6 A linear ordering of fundamental weights is called *economical* if, for each i, the index i is economical for the group $W_{[i,r]}$.

This definition can be restated as follows. For a positive root α, let $\mu(\alpha)$ denote the smallest index i such that $\alpha \in R(i)$. (In other words, the expansion of α does not contain the simple roots $\alpha_1, \ldots, \alpha_{i-1}$ but does contain α_i.) The ordering of fundamental weights is economical if and only if, for every $i \in [1, r]$, the map $\alpha \mapsto s_\alpha\omega_i$ is a bijection between

(i) the set of positive roots α with $\mu(\alpha) = i$ and
(ii) the set $W_{[i,r]}\omega_i - \{\omega_i\}$.

Repeatedly using Proposition 4.4, we obtain the following corollary.

Corollary 4.7 *An irreducible Weyl group possesses an economical ordering of fundamental weights if and only if it is of one of the types A_r, B_r, C_r, or G_2. In each of these cases, the standard ordering of fundamental weights given in* [1] *is economical.*

For an economical ordering, Proposition 4.1 can be refined as follows.

Theorem 4.8 *Suppose the fundamental weights are ordered in an economical way. Then an element $x \in G/B$ belongs to a Schubert cell X_w° if and only if*:

$$p_{w\omega_i}(x) \neq 0 \text{ for all } i \text{ such that there exists a positive root } \alpha \text{ with } \mu(\alpha) = i \text{ and } w\alpha \text{ negative}; \tag{4.4}$$

$$p_{ws_\alpha\omega_{\mu(\alpha)}}(x) = 0 \text{ for all positive roots } \alpha \text{ such that } w\alpha \text{ is also positive.} \tag{4.5}$$

Proof: Recall that, for $\alpha > 0$, the root $w\alpha$ is positive if and only if $ws_\alpha > w$. In view of this, Proposition 4.1 shows that conditions (4.4)–(4.5) are indeed necessary.

Assume that (4.4)–(4.5) hold. To prove that $x \in X_w^\circ$, it suffices to show that $p_{w\omega_i}(x) \neq 0$ for *all* $i \in [1, r]$. Suppose otherwise, and let i be the minimal index such that $p_{w\omega_i}(x) = 0$.

By (4.4), we have $w\alpha > 0$ (thus $ws_\alpha > w$) for all positive roots α with $\mu(\alpha) = i$. In view of the definition of economical ordering, the weight $w\omega_i$ is the minimal element of $wW_{[i,r]}\omega_i$. Now (4.5) implies that $p_\gamma(x) = 0$ for all $\gamma \in wW_{[i,r]}\omega_i - \{w\omega_i\}$.

Suppose $x \in X_u^\circ$. The same argument as in the proof of Proposition 4.1 shows that $u \in wW_{[i,r]}$. Since $p_{u\omega_i}(x) \neq 0$, the weight $u\omega_i$ must coincide with $w\omega_i$, which contradicts the assumption $p_{w\omega_i}(x) = 0$. □

The number of equations in (4.5) is equal to the number of positive roots α such that $w\alpha$ is also positive; this is precisely the codimension $\dim(G/B) - \ell(w)$ of X_w° in the flag variety. Furthermore, the number of inequalities in (4.4) is at most $\min(r, \ell(w))$. Applying Corollary 4.7, we obtain the following solution of Problem 1.1 for types A, B, C, and G_2.

Corollary 4.9 *For each of the types A_r, B_r, C_r, and G_2, conditions* (4.4)–(4.5) (*with the standard ordering of fundamental weights*) *describe an arbitrary Schubert cell X_w° using* $\dim(G/B) - \ell(w)$ *equations and at most* $\min(r, \ell(w))$ *inequalities.*

As a special case, we obtain the following enhancement of [5, Proposition 4.1].

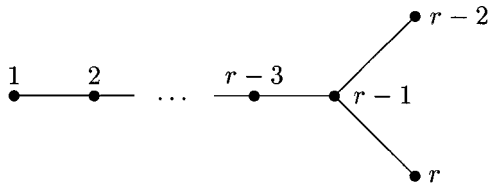
Corollary 4.10 *For the type A_{n-1}, an element $x \in G/B$ belongs to the Schubert cell X_w° if and only if it satisfies the following conditions*:

$$p_{w([1,i])}(x) \neq 0 \text{ for all } i \text{ such that there exists } j > i \text{ with } w(j) < w(i); \tag{4.6}$$

$$p_{w([1,i-1]\cup\{j\})}(x) = 0 \text{ whenever } 1 \leq i < j \leq n \text{ and } w(i) < w(j). \tag{4.7}$$

Thus X_w° can be described by at most $\binom{n}{2}$ equations and inequalities of the form $p_I = 0$ or $p_I \neq 0$.

We conclude this section by addressing Problem 1.1 for type D_r. We note that for $r \geq 4$, there are no economical indices. The index $i = 1$ (in the standard numeration) is "one root short" of being economical: $|W|/|W_{[2,r]}| = 2r$ while $R(1)$ consists of $2r - 2$ roots $\varepsilon_1 \pm \varepsilon_j$ ($j = 2, \ldots, r$). As a consequence, we have to add extra equations to those in (4.5) in order to describe X_w°. To minimize the number of these equations, we use the following ordering of fundamental weights, which is somewhat different from the one in [1]:



Theorem 4.8 and Corollary 4.9 then have the following analogues (with similar proofs).

Proposition 4.11 *Let G be of type D_r, $r \geq 4$, and let the fundamental weights be ordered as above. Then an element $x \in G/B$ belongs to a Schubert cell X_w° if and only it satisfies*

conditions (4.4)–(4.5), *along with the condition*

$$p_\gamma(x) = 0 \text{ whenever } \gamma = w(\varepsilon_1 + \cdots + \varepsilon_{i-1} - \varepsilon_i) > w(\varepsilon_1 + \cdots + \varepsilon_i),\ i \le r-3. \tag{4.8}$$

Thus X_w° *can be described using at most* $\dim(G/B) - \ell(w) + r - 3$ *equations and at most* $\min(r, \ell(w))$ *inequalities.*

We note that $\gamma = w(\varepsilon_1 + \cdots + \varepsilon_{i-1} - \varepsilon_i)$ in (4.8) is indeed a Plücker weight of level i, since $\gamma = wss'\omega_i$, where s and s' are the reflections corresponding to the roots $\varepsilon_i - \varepsilon_{i+1}$ and $\varepsilon_i + \varepsilon_{i+1}$, respectively.

5. Cell recognition algorithms

Our approach to the cell recognition problem (Problem 1.2) will be based on Proposition 4.1 and Theorem 4.8.

Suppose that the binary string (b_γ) is the vanishing pattern of all Plücker coordinates at some point $x \in G/B$:

$$b_\gamma = b_\gamma(x) = \begin{cases} 0 & \text{if } p_\gamma(x) = 0; \\ 1 & \text{if } p_\gamma(x) \neq 0. \end{cases} \tag{5.1}$$

The following lemma is a reformulation of Lemma 3.4.

Lemma 5.1 *For any* $x \in G/B$ *and any* $i \in [1, r]$, *the set of all Plücker weights* γ *of level* i *such that* $b_\gamma(x) = 1$ *has a unique maximal element with respect to the Bruhat order on* $W\omega_i$. *Furthermore, if* x *belongs to the Schubert cell* $X_w^\circ = (BwB)/B$, *then this maximal element is equal to* $w\omega_i$.

In view of Lemma 5.1, any vector $b_\gamma(x)$ is "acceptable" according to the following definition.

Definition 5.2 A binary vector (b_γ), where γ runs over all Plücker weights, is called *acceptable* if

$$\text{for any } i \in [1, r], \text{ the set } \{\gamma \in W\omega_i : b_\gamma = 1\} \text{ is nonempty, and has a unique maximal element } \gamma_i \text{ with respect to the Bruhat order;} \tag{5.2}$$

$$\text{there exists } w \in W \text{ such that } \gamma_i = w\omega_i \text{ for any } i. \tag{5.3}$$

It is immediate from Lemma 3.2 that the element w in (5.3) is unique.

We will now study the following purely combinatorial problem that includes Problem 1.2 as a special case.

Problem 5.3 For a given acceptable vector (b_γ), compute the element w in (5.3) by testing the minimal number of bits b_γ.

For $\gamma \in W\omega_i$, let us denote $W(\gamma) = \{u \in W : u\omega_i = \gamma\}$. Thus $W(\gamma)$ is a left coset in W with respect to the stabilizer of ω_i (i.e., with respect to $W_{\hat{i}}$). Our approach to Problem 5.3 will be based on the following lemma, which follows from (4.1).

Lemma 5.4 *Let (b_γ) be an acceptable binary vector. In the notation of Definition* 5.2, *for every i, we have*:

$$W(\gamma_1) \cap \cdots \cap W(\gamma_{i-1}) = wW_{[i,r]};$$

also, γ_i is the maximal element of $wW_{[i,r]}\omega_i$ such that $b_{\gamma_i} = 1$.

The following algorithm for Problem 5.3 is based on Lemma 5.4; it successively computes the weights $\gamma_1, \gamma_2, \ldots$, and in the end obtains w as the sole element in the intersection $W(\gamma_1) \cap \cdots \cap W(\gamma_r)$.

Algorithm 5.5
Input: acceptable binary vector (b_γ).
Output: the element $w \in W$ given by (5.3).
$U := W$;
for i **from** 1 **to** r **do**
fix a linear order $U\omega_i = \{\eta_1 < \cdots < \eta_m\}$ compatible with the Bruhat order;
$j := m$;
while $b_{\eta_j} = 0$ **do** $j := j - 1$; **od**;
comment: $\eta_j = \gamma_i = \max\{\gamma \in U\omega_i : b_\gamma = 1\}$
$U := U \cap W(\eta_j)$;
od;
return(U);

In particular, this algorithm can be used to solve Problem 1.2: if the input vector (b_γ) is the vanishing pattern (5.1) for a point $x \in G/B$, then the algorithm returns the element $w \in W$ such that $x \in X_w^\circ$.

The algorithm depends on the choice of the ordering of fundamental weights. As in Section 4, the best results are achieved for economical orderings. In this case, Proposition 4.5 implies that the set of weights $U\omega_i = wW_{[i,r]}\omega_i$ appearing in Algorithm 5.5 is linearly ordered by the Bruhat order, making the third line of the algorithm redundant.

In particular, in the case of type A_{n-1}, the standard ordering of the fundamental weights, and an acceptable vector defined by (5.1), Algorithm 5.5 takes the following form. (As before, we identify the Plücker weights with subsets in $[1, n]$.)

Algorithm 5.6
Input: vanishing pattern of Plücker coordinates of a complete flag x in $\mathbb{C}^n$.
Output: permutation $w \in \mathcal{S}_n$ such that $x \in X_w^\circ$.

$I := \emptyset$;
for i **from** 1 **to** n **do**
  $k := n$;
  while $k > \min([1,n]-I)$ **and** ($k \in I$ **or** $p_{I\cup\{k\}}(x) = 0$) **do** $k := k-1$**; od**;
  $w(i) := k$;
  $I := I \cup \{k\}$;
  comment: $I = w([1,i])$
od;

To convince oneself that Algorithm 5.6 is a specialization of Algorithm 5.5, it suffices to observe the following: the weights in $wW_{[i,r]}\omega_i$ correspond to the i-subsets of the form $w([1,i-1]) \cup \{k\}$, and the Bruhat order on $wW_{[i,r]}\omega_i$ corresponds to the usual ordering of the values k.

In the special case of type A_2, we recover the algorithm presented in figure 2.

Algorithm 5.6 agrees completely with the description of Schubert cells given in Corollary 4.10: to arrive at any w, we need to check exactly the same Plücker coordinates that appear in (4.6)–(4.7). We thus obtain the following result.

Proposition 5.7 *For a complete flag x in $\mathbb{C}^n$, Algorithm* 5.6 *recognizes the Schubert cell x is in by testing at most $\binom{n}{2}$ bits of the vanishing pattern of its Plücker coordinates.*

We omit the type B (or C) analogues of Algorithm 5.6 and Proposition 5.7, which can be obtained in a straightforward way.

6. On the number of equations defining a Schubert variety

Problem 1.1 is closely related to the classical problem of describing Schubert varieties X_w as algebraic subsets of G/B.

Problem 6.1 (Short descriptions of Schubert varieties) Define an arbitrary Schubert variety X_w (as a subset of G/B) by as small as possible number of equations of the form $p_\gamma = 0$.

The aim of this section is to demonstrate that, for a certain Schubert variety X_w of type A_{n-1}, one needs exponentially many (as a function of n) such equations to define X_w (set-theoretically).

Throughout this section, $G = SL_n$ and $W = \mathcal{S}_n$. Any Schubert cell X_w° has the special representative π_w: it is a complete flag in $\mathbb{C}^n$ formed by the coordinate subspaces $\mathbb{C}^{w([1,i])}$ for $i = 1,\ldots,n$. The following obvious observation will be useful in obtaining lower bounds.

Lemma 6.2 *For $w \in \mathcal{S}_n$, a Plücker coordinate p_I does not vanish at π_w if and only if $I = w([1,|I|])$.*

Proposition 6.3 *Suppose that $n = 4k$ is divisible by 4. Let $w \in \mathcal{S}_n$ be the maximal element of the parabolic subgroup $W_{\widehat{2k}} = \mathcal{S}_{2k} \times \mathcal{S}_{2k} \subset \mathcal{S}_n$ (thus w puts the elements in each of the blocks $[1, 2k]$ and $[2k+1, 4k]$ in the reverse order). Suppose the set $\mathcal{I}$ is such that*

$$X_w = \{x \in G/B \,:\, p_I(x) = 0 \text{ for } I \in \mathcal{I}\}.$$

Then

$$|\mathcal{I}| \geq \binom{2k}{k}. \tag{6.1}$$

Note that the right-hand side of (6.1) grows as $2^{n/2}/\sqrt{n}$, while the codimension of this particular Schubert variety X_w equals $(n/2)^2$.

Proof: Our lower bound for $|\mathcal{I}|$ is based on the following idea. Suppose a permutation $u \in \mathcal{S}_n$ is such that $u \not\leq w$. Then the flag π_u does not belong to the Schubert variety X_w, so there must exist $I \in \mathcal{I}$ such that $p_I(\pi_u) \neq 0$. By Lemma 6.2, this means that $I = u([1, |I|])$. In view of Lemma 3.4, the membership $I \in \mathcal{I}$ also implies that $I \not\leq w([1, |I|])$. We conclude that, in order to prove (6.1), it suffices to construct a subset $U \subset \mathcal{S}_n$ satisfying the following three properties:

(1) $u \not\leq w$ for any $u \in U$;
(2) $|U| = \binom{2k}{k}^2$;
(3) for every subset $I \subset [1, n]$ such that $I \not\leq w([1, |I|])$, there are at most $\binom{2k}{k}$ permutations $u \in U$ such that $I = u([1, |I|])$.

Define U to be the set of all permutations u that send $[1, k] \cup [2k+1, 3k]$ onto $[1, 2k]$, and increase on each of the blocks $[1, k]$, $[k+1, 2k]$, $[2k+1, 3k]$, and $[3k+1, 4k]$. Each $u \in U$ is uniquely determined by two k-subsets $A = u([1, k]) \subset [1, 2k]$ and $B = u([k+1, 2k]) \subset [2k+1, 4k]$; we write $u = u_{A,B}$. Now (2) is obvious. Since $u_{A,B}([1, 2k]) = A \cup B > [1, 2k] = w([1, 2k])$, we have $u \not\leq w$ for any $u \in U$, so U satisfies (1).

It remains to prove (3). Let $I \subset [1, n]$ be such that $I \not\leq w([1, |I|])$. We need to show that there are at most $\binom{2k}{k}$ permutations $u_{A,B} \in U$ such that $I = u_{A,B}([1, |I|])$. First of all, we have $u_{A,B}([1, i]) \leq w([1, i])$ for $i \leq k$ or $i \geq 3k$. Therefore, we may assume that $k < |I| < 3k$. Let us consider two cases.

Case 1. $|I| = k + l$ for some $l \in [1, k]$. The equality $I = u_{A,B}([1, |I|])$ means that I is the union of A and the set of l smallest elements of B. Thus $A = [1, 2k] \cap I$ is uniquely determined by I, while the number of choices for B is $\binom{4k - \max I}{k-l}$, which is less than $\binom{2k}{k}$.

Case 2. $|I| = 2k + l$ for some $l \in [1, k-1]$. Now the equality $I = u_{A,B}([1, |I|])$ means that I is the union of A, B, and the set of l smallest elements of $[1, 2k] - A$. Thus $B = [2k+1, 4k] \cap I$ is uniquely determined by I, while the number of choices for A is $\binom{k+l}{k} < \binom{2k}{k}$.

This concludes the proof of (6.1). □

Corollary 6.4 *There exist elements $u < v$ in $W = \mathcal{S}_{4k}$ such that X_u has codimension 1 in X_v, while defining X_u inside X_v requires at least $\frac{1}{4k^2}\binom{2k}{k}$ equations of the form $p_I = 0$.*

Proof: Consider a saturated chain $w = v_0 < v_1 < \cdots < v_N = w_o$ in the Bruhat order, where w is the same as in Proposition 6.3. (thus $N = 4k^2$). If $M(u, v)$ denotes the minimal number of equations of the form $p_I = 0$ defining X_u inside X_v, then obviously $M(w, w_o) \leq \sum M(v_i, v_{i+1}) \leq N \cdot \max_i(M(v_i, v_{i+1}))$. Combining this with the lower bound on $M(w, w_o)$ obtained in Proposition 6.3 completes the proof. □

7. On cell recognition without feedback

In this section, we examine the following problem.

Problem 7.1 (Cell recognition without feedback) Find a subset of Plücker coordinates of smallest possible cardinality whose vanishing pattern at any point $x \in G/B$ uniquely determines the Schubert cell of x.

Notice that, unlike in Problem 1.1, the Schubert cell is not fixed in advance; and in contrast to Problem 1.2, we have to present the entire list of Plücker coordinates right away (i.e., there is no feedback).

Example 7.2 Consider the special case of $G = SL_3$. Analyzing Table 1 in Section 2, we discover that the list in question must contain the Plücker coordinates p_3 (to distinguish between vanishing patterns of generic elements of Schubert cells labelled by s_1s_2 and w_o), p_2 (same reason, for e and s_1), p_{13} (for e and s_2), and p_{23} (for s_2s_1 and w_o). The vanishing pattern of these 4 Plücker coordinates does indeed determine the cell a point is in (see last column of Table 1). Hence this 4-element collection of Plücker coordinates provides the unique solution to Problem 7.1 for the type A_2.

The following result shows that for the type A, the subset asked for in Problem 7.1 must contain an overwhelming proportion of all Plücker coordinates.

Proposition 7.3 *For the type A_{n-1}, any subset satisfying the requirements in Problem 7.1 contains at least the $\frac{n-1}{n+1}$ proportion of all Plücker coordinates.*

Note that there are $2^n - 2$ Plücker coordinates altogether in this case.

Proof: We will actually show more: that this many Plücker coordinates are needed to distinguish between the vanishing patterns of any two different elements of the form π_w, for $w \in W = \mathcal{S}_n$ (we use the notation introduced at the beginning of Section 6). Let $\mathcal{I}$ be a collection of subsets $I \subset [1, n]$ such that the vanishing patterns of the Plücker coordinates $p_I(\pi_w)$, for $I \in \mathcal{I}$, are distinct for all elements $w \in W$. In view of Lemma 6.2, this means that for any distinct $u, v \in W$, there exists an index $i \in [1, n]$ such that the subsets $u([1, i])$ and $v([1, i])$ are distinct, and at least one of them belongs to $\mathcal{I}$.

Let I be a nonempty proper subset of $[1, n]$ of cardinality i. Choose $u \in W$ so that $u([1, i]) = I$, and let $v = us_i$. Then $u([1, j]) = v([1, j])$ unless $j = i$, implying that $\mathcal{I}$ must contain either $u([1, i]) = I$ or $v([1, i]) = I \setminus \{u(i)\} \cup \{u(i{+}1)\}$ (or both). We conclude that for any two subsets $I, J \subset [1, n]$ of the same cardinality which are Hamming distance 2 from each other (i.e., one is obtained from another by exchanging a single element), the collection $\mathcal{I}$ has to contain either I or J.

Let $\bar{\mathcal{I}}_i$ denote the collection of all i-subsets of $[1, n]$ not in $\mathcal{I}$. Then $\bar{\mathcal{I}}_i$ does not contain two subsets at Hamming distance 2 from each other. Such collections of subsets are called *binary codes of constant weight detecting single errors*, and they were an object of extensive study in coding theory. In particular, various upper bounds on the cardinality of such a code have been obtained; see, for example, [14, Chapter 17]. (We thank Richard Stanley for providing this reference.) For our purposes, it will suffice to have a very simple upper bound

$$|\bar{\mathcal{I}}_i| \leq \frac{1}{i}\binom{n}{i-1} = \frac{1}{n+1}\binom{n+1}{i}. \tag{7.1}$$

Although this bound is immediate from a sharper [14, Ch. 17, Corollary 5], we will give a proof for the sake of completeness.

To prove (7.1), note that all $(i-1)$-subsets contained in various i-subsets in $\bar{\mathcal{I}}_i$ must be distinct. Each $I \in \bar{\mathcal{I}}_i$ contains i such subsets, implying that $i \cdot |\bar{\mathcal{I}}_i| \leq \binom{n}{i-1}$, as desired.

The proof of Proposition 7.3 can now be completed as follows:

$$\begin{aligned}
|\mathcal{I}| &= 2^n - 2 - \sum_{i=1}^{n-1} |\bar{\mathcal{I}}_i| \\
&\geq 2^n - 2 - \frac{1}{n+1} \sum_{i=1}^{n-1} \binom{n+1}{i} \\
&= 2^n - 2 - \frac{1}{n+1}(2^{n+1} - n - 3) \\
&= \frac{n-1}{n+1}(2^n - 1).
\end{aligned}$$

□

8. Generic vanishing patterns

In the course of the above proof of Proposition 7.3, we have actually shown the following: assuming there is no feedback, "almost all" Plücker coordinates are needed to distinguish between special representatives π_w of Schubert cells. We will now demonstrate that the situation changes dramatically if we replace these "most special" representatives by the "most generic" ones.

In what follows, W is an arbitrary Weyl group. We associate to any $w \in W$ the *generic* vanishing pattern $(b_\gamma^{\mathrm{gen}}(w))$ defined by

$$b_\gamma^{\mathrm{gen}}(w) = \begin{cases} 1 & \text{if } \gamma \leq w\omega_i; \\ 0 & \text{if } \gamma \not\leq w\omega_i, \end{cases} \tag{8.1}$$

where γ runs over all Plücker weights of any level i. By Lemma 3.4, this is the vanishing pattern $(b_\gamma(x))$ (cf. (5.1)) of Plücker coordinates for a generic element $x \in X_w^\circ$.

Problem 8.1 (Recognizing generic points without feedback**)** Find a minimal subset of Plücker coordinates whose vanishing pattern distinguishes between the generic patterns $(b_\gamma^{\text{gen}}(w))$.

Our solution of this problem will be based on the techniques developed by Lascoux and Schützenberger [13], and further enhanced by Geck and Kim [8]. Let us first recall the main definitions and results of these papers.

Let P be a finite poset with unique minimal and maximal elements. We say that $a \in P$ is the *supremum* of a subset $Q \subset P$ if $a \geq q$ for any $q \in Q$, and moreover $a < b$ for any other element $b \in P$ with this property.

Definition 8.2 The *base* $B = B(P)$ of P is the subset of P consisting of all elements $a \in P$ which cannot be obtained as the supremum of a subset of P not containing a.

Proposition 8.3 [13] *The map $a \mapsto \{b \in B : b \leq a\}$ is an embedding of P (as an induced subposet) into the boolean algebra of all subsets of $B = B(P)$. Moreover, any other subset $B' \subset P$ with this property contains B.*

The following result appeared in [13, Théorème 3.6]; another proof was given in [8, Theorem 2.5].

Theorem 8.4 [13] *For every element u in the base of a finite Coxeter group W, there are unique simple reflections s_i and s_j such that $us_i < u$ and $s_j u < u$.*

Let $\mathfrak{B}(W)$ denote the subset of Plücker weights which correspond to the elements of the base $B(W)$, as follows:

$$\mathfrak{B}(W) = \{u\omega_i : u \in B(W),\ us_i < u\}.$$

Proposition 8.5 *The correspondence $w \mapsto (b_\gamma^{\text{gen}}(w))$, where γ runs over $\mathfrak{B}(W)$, is an embedding of W (as an induced subposet) into the Boolean lattice of all binary vectors of the corresponding length. Moreover, $\mathfrak{B}(W)$ is a minimal subset of Plücker weights that has this property.*

Thus the set of the Plücker coordinates p_γ, with $\gamma \in \mathfrak{B}(W)$, provides a solution of Problem 8.1.

Proof: Let $u \in B(W)$, and let $\gamma = u\omega_i \in \mathfrak{B}(W)$ be the corresponding weight. Since u is the minimal representative of the coset $uW_{\hat{i}}$, it follows that for any $w \in W$, the condition "$\gamma \leq w\omega_i$" is equivalent to "$u \leq w$." Therefore, (8.1) becomes

$$b_\gamma^{\text{gen}}(w) = \begin{cases} 1 & \text{if } u \leq w; \\ 0 & \text{if } u \not\leq w. \end{cases} \tag{8.2}$$

Thus the set of non-vanishing Plücker coordinates p_γ, $\gamma \in \mathcal{B}(W)$, at a generic point in X_w° corresponds exactly to the set of elements in the base $B(W)$ that are less than or equal than w in the Bruhat order. The proposition then follows from Proposition 8.3. □

The bases $B(W)$ were explicitly described and enumerated in [13] (for the types A and B) and [8] (for all other types). As shown in [8, 13], if W is of one of the classical types A_r, B_r, and D_r, then the cardinality of $B(W)$ is a cubic polynomial in r. In particular, for the type A_{n-1} when $W = \mathcal{S}_n$, the base consists of the $\binom{n+1}{3}$ "bigrassmannian" permutations: every triple of integers $0 \le a < b < c \le n$ gives rise to a such a permutation that acts identically on each of the blocks $[1, a]$ and $[c + 1, n]$ while interchanging the blocks $[a + 1, b]$ and $[b + 1, c]$. The corresponding *bigrassmannian Plücker coordinate* is $p_{[1,a]\cup[b+1,c]}$. Proposition 8.5 tells that the vanishing pattern of these $\binom{n+1}{3}$ Plücker coordinates uniquely determines the Schubert cell of a given complete flag x in $\mathbb{C}^n$, provided we know that x is generic within its cell. In the special case $n = 3$, the bigrassmannian Plücker coordinates are exactly the four coordinates p_2, p_3, p_{13}, p_{23} involved in Example 7.2 and in the descriptions of Section 2.

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