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Association Schemes of Quadratic Forms and Symmetric Bilinear Forms*

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Abstract. Let X_n and Y_n be the sets of quadratic forms and symmetric bilinear forms on an n-dimensional vector space V over $\mathbb{F}_q$, respectively. The orbits of $GL_n(\mathbb{F}_q)$ on $X_n \times X_n$ define an association scheme Qua(n, q). The orbits of $GL_n(\mathbb{F}_q)$ on $Y_n \times Y_n$ also define an association scheme Sym(n, q). Our main results are: Qua(n, q) and Sym(n, q) are formally dual. When q is odd, Qua(n, q) and Sym(n, q) are isomorphic; Qua(n, q) and Sym(n, q) are primitive and self-dual. Next we assume that q is even. Qua(n, q) is imprimitive; when $(n, q) \neq (2, 2)$, all subschemes of Qua(n, q) are trivial, i.e., of class one, and the quotient scheme is isomorphic to Alt(n, q), the association scheme of alternating forms on V. The dual statements hold for Sym(n, q).

Keywords: association scheme, quadratic form, symmetric bilinear form

1. Introduction

The association schemes of sesquilinear (bilinear, alternating, and Hermitian) forms are all self-dual and primitive [1, 2]. They are important families of P-polynomial schemes, or equivalently, distance regular graphs. Now we consider two families of association schemes defined on quadratic forms and symmetric bilinear forms, respectively. Let $V = V_n(\mathbb{F}_q)$ be an n-dimensional vector space over $\mathbb{F}_q$. Let X_n be the set of quadratic forms on V. The general linear group $GL_n(\mathbb{F}_q)$ acts on X_n as follows: for $Q \in X_n$ and $g \in GL_n(\mathbb{F}_q)$, $Q^g(\boldsymbol{x}) = Q(\boldsymbol{x}^g)$, for all $\boldsymbol{x} \in V$. Let $C_0 = \{0\}, C_1, \ldots, C_d$ be the orbits. We define an association scheme on X_n using the orbits C_i: for $Q_1, Q_2 \in X_n$, $(Q_1, Q_2) \in R_i$ if $Q_1 - Q_2 \in C_i$. Then $(X_n, \{R_i\}_{0 \leq i \leq d})$ is indeed an association scheme, and we denote this scheme by Qua(n, q) (the notation Quad(n, q) is used for the Egawa scheme of quadratic forms in literature. Quad(n, q) is also defined on X_n but with $(Q_1, Q_2) \in R_i$ if rank$(Q_1 - Q_2) = 2i - 1$ or $2i$.)

Similarly we can define a family of association schemes on symmetric bilinear forms. Let Y_n be the set of symmetric bilinear forms V. $GL_n(\mathbb{F}_q)$ acts on Y_n as follows: for $B \in Y_n$ and $g \in GL_n(\mathbb{F}_q)$, $B^g(\boldsymbol{x}, \boldsymbol{y}) = B(\boldsymbol{x}^g, \boldsymbol{y}^g)$, where $\boldsymbol{x}, \boldsymbol{y} \in V$. We define an association scheme on Y_n using the $GL_n(\mathbb{F}_q)$-orbits in the same way. We use Sym(n, q) to represent this scheme.

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Each quadratic form Q has an associated symmetric bilinear form define by $B_Q(\boldsymbol{x}, \boldsymbol{y}) = Q(\boldsymbol{x}+\boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y})$. For q odd, Q can be defined by B_Q, and vice versa. For q even, B_Q is alternating. We define, for any given symmetric bilinear B,

$$\mathrm{Q}_B = \{Q \in X_n \mid B_Q = B\}. \tag{1.1}$$

In particular, we use Q_0 to denote Q_B defined by the zero bilinear form 0 [4].

The association scheme $\mathrm{Alt}(n, q)$ of alternating forms is defined on the set K_n of alternating forms on V, where, for $A_1, A_2 \in K_n$, $(A_1, A_2) \in R_i$ if $\mathrm{rank}(A_1 - A_2) = 2i$. $\mathrm{Qua}(n, q)$ was introduced in [4, 14] and $\mathrm{Sym}(n, q)$ in [8, 9, 13]. These two families are not P-polynomial schemes in general, but nevertheless they are closely related to two well known families of association schemes: $\mathrm{Alt}(n, q)$ and $\mathrm{Quad}(n, q)$. For example, $\mathrm{Alt}(n, q)$ appears as the quotient schemes of $\mathrm{Qua}(n, q)$ (see the main theorem), as an association subscheme [13] and a fusion scheme of $\mathrm{Sym}(n, q)$ [11]. $\mathrm{Quad}(n, q)$ is a fusion scheme of $\mathrm{Qua}(n, q)$ by definition. $\mathrm{Quad}(n, q)$ can be also constructed from $\mathrm{Alt}(n, q)$ for q even [11]. A fusion scheme is an association scheme which is obtained by fusing some classes of another association scheme.

Further study of $\mathrm{Qua}(n, q)$ and $\mathrm{Sym}(n, q)$ will contribute to the understanding of distance regular graphs on forms and dual polar graphs. In the present paper, we develop a systematic approach for further studying the association schemes of forms. We are also interested in $\mathrm{Qua}(n, q)$ and $\mathrm{Sym}(n, q)$ in their own rights. For instance, what are the fusion schemes in $\mathrm{Qua}(n, q)$ or $\mathrm{Sym}(n, q)$? New families of distance regular graphs might arise from the fusion schemes. In the present paper, we will prove the following theorem:

Main Theorem

(1) $\mathrm{Qua}(n, q)$ *and* $\mathrm{Sym}(n, q)$ *are formally dual.*
(2) *When q is odd*, $\mathrm{Qua}(n, q)$ *and* $\mathrm{Sym}(n, q)$ *are isomorphic*, *thus they are self-dual.*
(3) *When q is odd*, $\mathrm{Qua}(n, q)$ *and* $\mathrm{Sym}(n, q)$ *are primitive.*
(4) *Suppose q is even.* $\mathrm{Qua}(n, q)$ *is imprimitive. When* $(n, q) \neq (2, 2)$, *all subschemes of* $\mathrm{Qua}(n, q)$ *are given by* $\mathrm{Q}_B (B \in K_n)$ *and they are trivial. The quotient scheme is isomorphic to* $\mathrm{Alt}(n, q)$. *Dually*, $\mathrm{Sym}(n, q)$ *is imprimitive*; *all the subschemes of* $\mathrm{Sym}(n, q)$ *are isomorphic to* $\mathrm{Alt}(n, q)$, *and the quotient scheme is trivial.*
(5) *Suppose* $(n, q) = (2, 2)$. $\mathrm{Qua}(2, 2)$ *and* $\mathrm{Sym}(2, 2)$ *are isomorphic to the cube graph*, *which is bipartite and antipodal.*

The paper is organized as follows. Section 2 reviews some concepts of association schemes, and defines $\mathrm{Qua}(n, q)$ and $\mathrm{Sym}(n, q)$ in terms of matrices. In Section 3, we prove assertions (1) and (2) of the main theorem (see Propositions 3.4 and 3.5). In Section 4, the eigenmatrices of $\mathrm{Qua}(2, q)$ are computed when q is even. In Section 5, we discuss the primitivity of $\mathrm{Qua}(n, q)$ for odd q, and the imprimitivity of $\mathrm{Qua}(n, q)$ for even q. We prove assertions (4) and (5) of the main theorem (see Proposition 5.4).

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2. Definitions

A d-class *commutative association scheme* is a pair $\mathrm{X} = (X, \{R_i\}_{0\le i\le d})$, where X is a finite set, each R_i is a nonempty subset of $X \times X$ satisfying the following:

(a) $R_0 = \{(x, x) \mid x \in X\}$.
(b) $X \times X = R_0 \cup R_1 \cdots R_d$, $R_i \cap R_j = \emptyset$ if $i \neq j$.
(c) $R_i^{\mathrm{T}} = R_j$ for some j, $0 \le j \le d$, where $R_i^{\mathrm{T}} = \{(y, x) \mid (x, y) \in R_i\}$.
(d) There exist integers p_{ij}^k such that for all $x, y \in X$ with $(x, y) \in R_k$,

$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|,$$

and further, $p_{ij}^k = p_{ji}^k$.

X is referred as the vertex set of X, and the p_{ij}^k as the intersection numbers of X. In addition, if

(e) $R_i^{\mathrm{T}} = R_i$ for all i,

then we say that X is *symmetric*.

Let $\mathrm{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme. The i-th adjacency matrix A_i is defined to be the adjacency matrix of the digraph (X, R_i). By the Bose–Mesner algebra of X we mean the algebra A generated by the adjacency matrices $A_0, A_1, \ldots, A_d$ over the complex numbers $\mathbb{C}$. Since A consists of commutative normal matrices, there is a second basis consisting of the primitive idempotents $E_0, E_1, \ldots, E_d$. The Krein parameters q_{ij}^k's are the structure constants of E_i's with respect to entry-wise matrix multiplication: $E_i \circ E_j = \sum_{k=0}^d q_{ij}^k E_k$. Let

$$A_j = \sum_{i=0}^d p_j(i) E_i, \quad E_j = \frac{1}{|X|} \sum_{i=0}^d q_j(i) A_i,$$

and let P and Q be the $(d + 1) \times (d + 1)$ matrices the (i, j)-entries of which are $p_j(i)$ and $q_j(i)$, respectively. The matrices P and Q are called the *first* and *second eigenmatrix* of X, respectively. We use $k_j = p_j(0)$ and let m_j denote the rank of matrix E_j. The numbers k_i are called valencies and m_i multiplicities. We refer the readers [1, 2] for the theory of association schemes.

Two association schemes are said to be *formally dual* if the P matrix of one is the Q matrix of the other possibly with a reordering of the rows and columns of Q, or equivalently, the Krein parameters of one are the intersection numbers of the other. If an association scheme has the property that its P matrix is equal to its Q matrix possibly with a reordering of its primitive idempotents, then it is said to be *self-dual*. The Hamming and the Johnson schemes are two such well known examples.

An association scheme $\mathrm{X} = (X, \{R_i\}_{0\le i\le d})$ is *primitive* if all the digraphs $(X, R_i)(1 \le i \le d)$ are connected, and otherwise it is *imprimitive*. For an imprimitive association scheme, its association subschemes and quotient schemes are defined [1].

We introduce the association scheme of quadratic forms in terms of matrices. Let $\mathbb{F}_q$ be a finite field of q elements and $n \geq 2$ be an integer. We use $M_{n,n}(\mathbb{F}_q)$ to denote the set of all $n \times n$ matrices over $\mathbb{F}_q$. $M_{n,n}(\mathbb{F}_q)$ is an algebra and we are mainly interested in its additive group structure. Let K_n be the set of alternating matrices in $M_{n,n}(\mathbb{F}_q)$ (recall the matrix (a_{ij}) is alternating if $a_{ij} = -a_{ji} (i \neq j)$ and $a_{ii} = 0$). K_n is an additive subgroup of $M_{n,n}(\mathbb{F}_q)$. Let X_n be the collection of the K_n-cosets in $M_{n,n}(\mathbb{F}_q)$, for A in $M_{n,n}(\mathbb{F}_q)$, $[A]$ is the coset which contains A. The quadratic form $f = \sum_{i \leq j} a_{ij} x_i x_j$ in $x_1, \ldots, x_n$ over $\mathbb{F}_q$ corresponds to $[A]$, where $A = (a_{ij})$ is upper triangular. This correspondence is one-to-one. So X_n can be identified with the set of quadratic forms over $\mathbb{F}_q$.

The general linear group $GL_n(\mathbb{F}_q)$ acts on X_n as follows: for $T \in GL_n(\mathbb{F}_q)$ and $[X] \in X_n$,

$$\begin{aligned} GL_n(\mathbb{F}_q) \times X_n &\to X_n \\ (T, [X]) &\to T[X]T^{\mathrm{T}} := [TXT^{\mathrm{T}}]. \end{aligned} \tag{2.1}$$

It is easy to see that this action is well-defined. Two $n \times n$ matrices A and B are said to be *cogredient* if there is a $T \in GL_n(\mathbb{F}_q)$ such that $TAT^{\mathrm{T}} \equiv B (\mathrm{mod}\ K_n)$. It is not hard to see that this is an equivalence relation which partitions $M_{n,n}(\mathbb{F}_q)$ into equivalence classes. X_n is the collection of classes of cogredient matrices. Let $G_1 = GL_n(\mathbb{F}_q) \cdot X_n$, the semidirect product of $GL_n(\mathbb{F}_q)$ with X_n. G_1 acts on X_n transitively: for $(T, [A]) \in G_1$ and $[X] \in X_n$,

$$\begin{aligned} G_1 \times X_n &\to X_n \\ ((T, [A]), [X]) &\to [TXT^{\mathrm{T}}] + [A]. \end{aligned} \tag{2.2}$$

Thus this action determines the *association scheme of quadratic forms*, denoted by $\mathrm{Qua}(n, q)$. Two pairs of quadratic forms $([A], [B])$ and $([C], [D])$ are in the same class of $\mathrm{Qua}(n, q)$ if and only if, there exists a $T \in GL_n(\mathbb{F}_q)$ such that $T(A-B)T^{\mathrm{T}} \equiv C-D (\mathrm{mod}\ K_n)$.

We now define the association scheme of symmetric matrices (or symmetric bilinear forms). Let Y_n be the set of all $n \times n$ symmetric matrices over $\mathbb{F}_q$ and $G_2 = GL_n(\mathbb{F}_q) \cdot Y_n$ the semidirect product of $GL_n(\mathbb{F}_q)$ with Y_n. G_2 acts transitively on Y_n as follows: for $(T, A) \in G_2$ and $X \in Y_n$,

$$\begin{aligned} G_2 \times Y_n &\to Y_n \\ ((T, A), X) &\to TXT^{\mathrm{T}} + A. \end{aligned} \tag{2.3}$$

This action also determines an association scheme, denoted by $\mathrm{Sym}(n, q)$. For $A, B \in Y_n$, if there is a $T \in GL_n(\mathbb{F}_q)$ such that $TAT^{\mathrm{T}} = B$, we also say that A and B are cogredient. By counting the incogredient norm forms (see [12]) of symmetric matrices (quadratic forms), we know that when q is odd, $\mathrm{Sym}(n, q)$ and $\mathrm{Qua}(n, q)$ have $2n + 1$ classes, and when q is even, $\mathrm{Sym}(n, q)$ and $\mathrm{Qua}(n, q)$ have $n + \lfloor n/2 \rfloor + 1$ classes. Moreover, when q is even or $q \equiv 1 (\mathrm{mod}\ 4)$, $\mathrm{Sym}(n, q)$ is symmetric; when $q \equiv 3 (\mathrm{mod}\ 4)$, $\mathrm{Sym}(n, q)$ is not symmetric yet commutative [8].

3. The duality between Qua(n, q) and Sym(n, q)

We will prove assertions (1) and (2) of the main theorem in this section. As in Section 2, X_n and Y_n are the additive groups of the quadratic forms and the $n \times n$ symmetric matrices over $\mathbb{F}_q$, respectively. Now we give a map between Y_n and the character group X_n^* of X_n. Let χ be a fixed non-trivial complex character of $\mathbb{F}_q$ as an additive group. For a symmetric matrix $A = (a_{ij}) \in Y_n$, we define a map ϕ_A from X_n to $\mathbb{C}$ by

$$\phi_A([X]) = \chi\left(\sum_{i,j=1}^{n} a_{ij}x_{ij}\right), \quad \text{for all } [X] \in X_n,$$

where $X = (x_{ij})$ is a representative of $[X]$. Note this map is well defined. It is also easy to see that ϕ_A is a character of X_n and $\phi_{A+B} = \phi_A\phi_B$.

Proposition 3.1 *$\phi_A = \phi_B$ if and only if $A = B$; the mapping $A \mapsto \phi_A$ is an isomorphism between Y_n and X_n^*.*

Proof: We prove the necessity of the first assertion, since the sufficiency is trivial. Suppose $\phi_A = \phi_B$ with $A = (a_{ij})$ and $B = (b_{ij})$. So

$$\phi_A([X]) = \phi_B([X]), \quad \text{for any } [X] \in X_n,$$

i.e.,

$$\chi\left(\sum_{i,j=n}^{n} a_{ij}x_{ij}\right) = \chi\left(\sum_{i,j=n}^{n} b_{ij}x_{ij}\right), \quad \text{for any } x_{ij} \in \mathbb{F}_q.$$

For i, j take $x_{kl} = 0, k \neq i, j \neq l$, and then $\chi(a_{ij}x_{ij}) = \chi(b_{ij}x_{ij})$, for any $x_{ij} \in \mathbb{F}_q$. So

$$\chi((a_{ij} - b_{ij})x_{ij}) = 1.$$

Since χ is a non-trivial character, we have $a_{ij} = b_{ij}$ $(i, j = 1, \ldots, n)$ and thus $A = B$.

The second assertion follows from $\phi_{A+B} = \phi_A\phi_B$, and that X_n^* and Y_n have the same cardinality. □

The following theorem says that the actions of $GL_n(\mathbb{F}_q)$ on Y_n and X_n^* are compatible under the map $A \mapsto \phi_A$.

Proposition 3.2 *For $A \in Y_n$, $[X] \in X_n$, $T \in GL_n(\mathbb{F}_q)$, $\phi_{TAT^{\mathrm{T}}}([X]) = \phi_A(T^{\mathrm{T}}[X]T)$.*

Proof: Let $TAT^{\mathrm{T}} = (a_{ij}^*)$, and $a_{ij}^* = \sum_{k,l=1}^{n} t_{ik}a_{kl}t_{jl}$, $a_{ij}^* = a_{ji}^*$. Pick a representative X of $[X]$, $X = (x_{ij})$.

$$\begin{aligned}\phi_{TAT^{\mathrm{T}}}([X]) &= \chi\left(\sum_{i,j=1}^{n} a_{ij}^* x_{ij}\right)\\ &= \chi\left(\sum_{i,j=1}^{n}\sum_{k,l=1}^{n} t_{ik}a_{kl}t_{jl}x_{ij}\right)\end{aligned}$$

$$= \chi\left(\sum_{k,l=1}^{n} a_{kl} \sum_{i,j=1}^{n} t_{ik}x_{ij}t_{jl}\right)$$
$$= \phi_A([T^{\mathrm{T}}XT])$$
$$= \phi_A(T^{\mathrm{T}}[X]T). \qquad \square$$

For a quadratic form $[X] \in X_n$, we define a map from Y_n to $\mathbb{C}$ by

$$\psi_{[X]}(A) = \phi_A([X]), \quad \text{for all } A \in Y_n.$$

Then since the character group $(X_n^*)^*$ of X_n^* is canonically identified with X_n, it follows from Proposition 3.1 that $\psi_{[X]}$ is an irreducible character of Y_n, and $[X] \mapsto \psi_{[X]}$ is an isomorphism between X_n and the character group Y_n^* of Y_n. Thus we can regard X_n as the character group of Y_n and further by Proposition 3.2 we have

$$\psi_{T[X]T^{\mathrm{T}}}(A) = \phi_A(T[X]T^{\mathrm{T}}), \quad \text{for all } A \in Y_n.$$

Before we prove assertion (1) of the main theorem, let us introduce S-rings ([1, Section II.6]). Let G be a finite abelian group, and let $G_0 = \{0\}, G_1, \ldots, G_d$ be a partition of G with the following properties:

(a) Let $G_i^{-1} = \{a \in G \mid -a \in G_i\}$. Then $G_i^{-1} = G_{i'}$ for some i'.
(b) $\mathbb{G}_i\mathbb{G}_j = \sum_{k=0}^{d} c_{ij}^k \mathbb{G}_k$, where $\mathbb{G}_i$ is the element $\sum_{x\in G_i} x$ in the group ring $\mathbb{C}G$.

The subalgebra S of $\mathbb{C}G$ spanned by $\mathbb{G}_0, \mathbb{G}_1, \ldots, \mathbb{G}_d$ is called an S-ring. Now we define an association scheme on G by defining the relations on G as follows:

$$(x, y) \in R_i \quad \text{if } y - x \in G_i.$$

Then $\mathrm{X}(G) = (G, \{G_i\}_{0\le i\le d})$ is a commutative association scheme whose Bose–Mesner algebra is isomorphic to the S-ring by the correspondence of A_i to $\mathbb{G}_i$, where A_i is the adjacency matrix of the digraph (X, R_i).

Theorem 3.3 ([1, II.6.3]) *Let* S *be an S-ring over a finite abelian group* X *and let* Y *be the character group of* G. *Let* $\sim$ *be the equivalence relation on* Y *defined by* $\delta_\alpha \sim \delta_\beta$ *if and only if the restriction of* δ_α *and* δ_β *to* X *coincide. Let* $Y_0, Y_1, \ldots, Y_d$ *be the equivalence classes, and let* $\mathbb{Y}_i = \sum_{\delta_\alpha\in Y_i} \delta_\beta$. *Then the subalgebra* S^* (*of* $\mathbb{C}[Y]$) *spanned by* $\mathbb{Y}_0, \mathbb{Y}_1, \ldots, \mathbb{Y}_d$ *becomes an S-ring with the property that* $\dim \mathrm{S} = \dim \mathrm{S}^*$ *and the intersection number of* $\mathrm{X}(S^*)$ *are the Krein parameters of* $\mathrm{X}(S)$.

Proposition 3.4 *Assertion* (1) *of the main theorem holds.*

Proof: Let $\mathrm{R} = \{R_i \mid 0 \le i \le d\}$ be the classes of $\mathrm{Qua}(n, q)$, where $d = 2n$ or $n + \lfloor n/2 \rfloor$ depends on q being odd or even. Fix the quadratic form 0, and let

$$R_i(0) = \{[X] \in X_n \mid (0, [X]) \in R_i\}, \quad 0 \le i \le d.$$

Then $C = \{R_i(0) \mid 0 \le i \le d\}$ is a partition of X_n, and in fact they are the cogredience classes of X_n. So

$$R_i(0) = \{T[X]T^{\mathrm{T}} \mid T \in GL_n(\mathbb{F}_q)\} \quad \text{for some } [X] \in R_i(0).$$

The partition C induces a partition C^* on Y_n. For $A, B \in Y_n$, A and B are in the same cell of C^* if

$$\sum_{[X]\in R_i(0)} \phi_A([X]) = \sum_{[X]\in R_i(0)} \phi_B([X]) \quad \text{for all } i, 0 \le i \le d.$$

If A and B are cogredient, then A and B are in the same cell of C^* by Proposition 3.2. So each cell of C^* is the union of cogredience classes of X_n^*. On the other hand, $|C^*| = |C|$ by Theorem 3.3, and $|C|$ is the number of cogredience classes of X_n, which is equal to the number of cogredience classes of Y_n. Consequently, C^* coincides with the family of cogredience classes of Y_n. Therefore Qua(n, q) and Sym(n, q) are formally dual by Theorem 3.3. □

Proposition 3.5 *Assertion* (2) *of the main theorem holds.*

When $\mathbb{F}_q$ is of odd characteristic, quadratic forms have a representation in terms of symmetric matrices. It is well known that Qua(n, q) and Sym(n, q) are isomorphic when q is odd. Thus assertion (2) follows. But when q is even, Qua(n, q) and Sym(n, q) are not isomorphic in general (see next section.)

Remark 1 In characteristic 2, X_n can be identified with the dual space of Y_n in a way compatible with the action of $GL_n(\mathbb{F}_q)$. When represented with respect to appropriate $\mathbb{F}_2$-bases, the actions of $GL_n(\mathbb{F}_q)$ on X_n and Y_n are contragredient, that is, their matrices are transpose of each other. Thus Sym(n, q) and Qua(n, q) fit Example II.6.5 of [1].

4. The eigenmatrices of Qua$(2, q)$ (q even)

Throughout this section, we assume that q is even. Qua$(2, q)$ is distance regular and thus we could compute the eigenmatrices of Qua$(2, q)$ using its intersection numbers. The purpose of this section is to show how duality can help the calculation. We remark this can done in general, which has been in [10].

We take the upper triangular matrices as the representatives of the quadratic forms. X_2 has four cogredience classes, and we may take their representatives as follows (see Lemma 5.2):

$$A_0 = O, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{2^+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{2^-} = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix},$$

where $\alpha \in \mathbb{F}_q$ is a fixed element such that $\alpha \notin N = \{x^2 + x \mid x \in \mathbb{F}_q\}$. Let C_i be the cogredience class with representative $A_i (i = 0, 1, 2^+, 2^-)$. Then we have

$$C_0 = \{O\}, \quad C_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \middle| x \text{ and } z \text{ are not both zero} \right\},$$
$$C_{2^+} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| y \neq 0, y^{-2}xz \in N \right\},$$
$$C_{2^-} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| y \neq 0, y^{-2}xz \notin N \right\}.$$

We denote C_{2+} and C_{2-} by C_2 and C_3. It is easy to compute the valencies of Qua$(2, q)$.

$$k_0 = |C_0| = 1, \quad k_1 = |C_1| = q^2 - 1, \quad k_2 = |C_{2^+}| = \frac{1}{2}q(q^2 - 1),$$
$$k_3 = |C_{2^-}| = \frac{1}{2}q(q - 1)^2.$$

For the cogredience classes of Y_2, we may take their representatives as follows (see [12, 13]):

$$S_0 = O, \quad S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\phi_i := \phi_{S_i}, i = 0, 1, 2, 3$. Note $\phi_0 = 1$, the trivial character. Then $\phi_i (i = 0, 1, 2, 3)$ is a set of representatives of cogredient classes of the character group X_2^* of X_2. Then the P matrix of Qua$(2, q)$ is given by $P = (\phi_j(C_i))$ (see [7, Lemma 12.9.2]), where

$$\phi_j(C_i) = \sum_{X \in C_i} \phi_j(X)$$

is the (i, j)-entry of P. Now we compute $\phi_j(C_i)$'s, which will use the fact $|N| = q/2$ and the following identity:

$$\sum_{x \in \mathbb{F}_q} \chi(x) = 0.$$

It is easy to see that

$$\phi_j(C_0) = 1, \quad j = 0, 1, 2, 3.$$
$$\phi_0(C_i) = k_i, \quad j = 0, 1, 2, 3.$$
$$\phi_1(C_1) = \sum_{X \in C_1} \phi_1(X) = \sum_{(x,z) \neq (0,0,)} \chi(x) = (q - 1) \sum_{x \in \mathbb{F}_q} \chi(x) + \sum_{x \in \mathbb{F}_q^*} \chi(x) = -1,$$

$$\phi_1(C_3) = \sum_{X\in C_3} \phi_1(X) = \sum_{\substack{y\neq 0\\ xz\notin y^2N}} \chi(x) = \sum_{x\neq 0} \chi(x)[q(q-1)/2] = -\frac{1}{2}q(q-1),$$

$$\phi_1(C_2) = \sum_{X\in C_2} \phi_1(X) = \sum_{\substack{y\neq 0\\ xz\in y^2N}} \chi(x) = \sum_{\substack{y\neq 0\\ x=0\\ z\in\mathbb{F}_q}} 1 + \sum_{\substack{y\neq 0\\ x\neq 0\\ z\in x^{-1}y^2N}} \chi(x)$$

$$= q(q-1) + \frac{1}{2}q(q-1)\sum_{x\neq 0}\chi(x) = q(q-1) - \frac{1}{2}q(q-1) = \frac{1}{2}q(q-1).$$

Similarly, we can get

$$\phi_2(C_1) = -1, \quad \phi_2(C_2) = -\frac{1}{2}q, \quad \phi_2(C_3) = \frac{1}{2}q,$$

$$\phi_3(C_1) = q^2-1, \quad \phi_3(C_2) = -\frac{1}{2}q(q+1), \quad \phi_3(C_3) = -\frac{1}{2}q(q-1).$$

We get

$$P = \begin{pmatrix} 1 & q^2-1 & \frac{1}{2}q(q^2-1) & \frac{1}{2}q(q-1)^2 \\ 1 & -1 & \frac{1}{2}q(q-1) & -\frac{1}{2}q(q-1) \\ 1 & -1 & -\frac{1}{2}q & \frac{1}{2}q \\ 1 & q^2-1 & -\frac{1}{2}q(q+1) & -\frac{1}{2}q(q-1) \end{pmatrix}$$

The second eigenmatrix of Qua$(2, q)$ is

$$Q = q^3P^{-1} = \begin{pmatrix} 1 & q^2-1 & (q-1)(q^2-1) & q-1 \\ 1 & -1 & -(q-1) & q-1 \\ 1 & q-1 & -(q-1) & -1 \\ 1 & -(q+1) & q+1 & -1 \end{pmatrix}$$

Note that P can not be obtained from Q by switching the rows and columns of Q when $q \neq 2$. So when $q \neq 2$, Qua$(2, q)$ is not self-dual. Since Sym$(2, q)$ has Q as its first eigenmatrix ([11]), Qua$(2, q)$ and Sym$(2, q)$ are not isomorphic. Qua$(2, q)$ and Sym$(2, q)$ are isomorphic to the cube graph if $q = 2$.

5. The primitivity and imprtivity

The scheme $\mathrm{X} = (X, \{R_i\}_{0\leq i\leq d})$ is said to primitive if if all the digraphs $(X, R_i)(1 \leq i \leq d)$ are connected, and otherwise it is imprimitive. We will consider the connectivity of (X_n, R_i) for q even. For q odd, one can argue that each $(X_n, R_i)_{i\neq 0}$ is connected. We state the result for Qua(n, q) for q odd without proof.

Proposition 5.1 *If q is odd, all the digraphs $(X_n, R_i)_{i\neq 0}$ are connected. Thus assertion* (3) *of the main theorem holds.*

Throughout the rest of this section, we assume that q is even. Since we will use the norm forms for the quadratic forms, we give the following lemma.

Lemma 5.2 ([12]) *Suppose q is even. Any $n \times n$ matrix over $\mathbb{F}_q$ is cogredient to a matrix of one and only one of the following norm forms*

$$\begin{pmatrix} 0 & I^{(\nu)} & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I^{(\nu)} & & & \\ & 0 & & & \\ & & \alpha & 1 & \\ & & & \alpha & \\ & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I^{(\nu)} & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix},$$

where α is a fixed element of $\mathbb{F}_q$ not in $N = \{x^2 + x \mid x \in \mathbb{F}_q\}$.

The three matrices in the lemma above have 'rank' 2ν, $2\nu + 2$, and $2\nu + 1$, respectively. We further distinguish the norm form of even rank by their types. We say that the first matrix has '$+$' type and the second one '$-$' type. The rank and type of a quadratic form determine its norm form. Both rank and type are invariants under cogredience. To be brief, we say the first two matrices have types $(2\nu)^+$, $(2\nu + 2)^-$, respectively, and the third one $2\nu + 1$.

For any quadratic form Q, the associated symmetric bilinear $B_Q(\boldsymbol{x}, \boldsymbol{y}) = Q(\boldsymbol{x} + \boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y})$. Since q is even, B_Q is alternating. For any alternating matrix $B \in K_n$, we define

$$\mathrm{Q}_B = \{Q \in K_n \mid B_Q = B\}.$$

For the alternating $n \times n$ matrix $B = (b_{ij})$, one can obtain Q_B by taking the upper triangular part of B and then adding the main diagonal.

$$\mathrm{Q}_B = \left\{ \left[\begin{pmatrix} a_1 & b_{12} & b_{13} & \cdots & b_{1n} \\ & a_2 & b_{23} & \cdots & b_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & a_{n-1} & b_{n-1\, n} \\ & & & & a_n \end{pmatrix} \right] \,\middle|\, a_1, \ldots, a_n \in \mathbb{F}_q \right\}$$

In particular, Q_0 consists of all quadratic forms of rank ≤ 1 and is an additive subgroup of X_n.

For the digraphs of Qua(n, q), we have the following theorem

Theorem 5.3 *Suppose q is even and $(n, q) \neq (2, 2)$. The digraphs (X_n, R_i) are connected for $i \neq 0, 1$. (X_n, R_1) is disconnected with connected components $\mathrm{Q}_B (B \in K_n)$.*

Proof: Let $\Gamma_i = (X_n, R_i)$ be the graph on X_n with edge set R_i.

(a) Consider $\Gamma_1 = (X_n, R_1)$. The connected component containing the zero quadratic form 0 is the set of all quadratic forms of rank ≤ 1, i.e., Q_0, which is a maximal clique in Γ_1. Thus Γ_1 is a union of maximal cliques, and there are $q^{n(n-1)/2}$ such cliques. All clique are $\mathrm{Q}_B(B \in K_n)$.

(b) Consider $\Gamma_{2^+} = (X_n, R_{2^+})$. We want to show that Γ_{2^+} is connected. It suffices to show that there exists a path between any quadratic form and the zero quadratic form 0, which holds if and only if any quadratic form can be written as a sum of quadratic forms of type 2^+.

Let $f = \sum_{i \leq j} a_{ij} x_i x_j$. Let $f_{ij} = a_{ij} x_i x_j$ when $a_{ij} \neq 0$. Then f_{ij} has type 2^+ for $i \neq j$. We can write the quadratic form f_{ii} as sum of two quadratic forms of type 2^+ (for instance, $f_{11} = (a_{11}x^2 + x_1x_2) + (x_1x_2)$, where $a_{11}x^2 + x_1x_2$ and x_1x_2 are of type 2^+.) Therefore, we can write f as a sum of quadratic forms of type 2^+. So Γ_{2^+} is connected.

(c) Suppose $n \geq 3$. Consider the graph $\Gamma_i (i \neq 1, 2^+, 2^-)$. We want to prove that Γ_i is connected. Again, it suffices to show that there exists a path between any quadratic form f and the zero quadratic form 0. By the connectedness of Γ_{2^+}, there exists a path from 0 to f in Γ_{2^+}. Let (f_j, f_{j+1}) be any edge on this path. Then $f_j - f_{j+1}$ has type 2^+. If we can show that the intersection number $p^{2^+}_{i\ i} \neq 0$, then a path exists between f_j and f_{j+1} in Γ_i. It follows that there is a path in Γ_i from 0 to f.

Let $f = x_1 x_n$, which has type 2^+. We choose g with following matrix representation

$$\begin{pmatrix} 0^{(\nu)} & I^{(\nu)} & & \\ & 0^{(\nu)} & & \\ & & \Delta & \\ & & & 0^{(n-2\nu-d)} \end{pmatrix}$$

where Δ is chosen according to $i = (2\nu)^+, 2\nu + 1$, or $(2\nu + 2)^-$. Then $\nu \geq 1$, and both g and $g + f$ have type i. So $p^{2^+}_{ii} \neq 0$. Hence Γ_i is connected.

(d) The only case left is $i = 2^-$. Now we consider Γ_{2^-}.

Let's consider the case when $n = 2$ first. Using the second eigenmatrix Q in Section 4 and the formula

$$p^k_{ij} = \frac{k_i k_j}{|X_2|} \sum_{\nu=0}^{d} q_\nu(i) q_\nu(j) q_\nu(k) / m^2_\nu,$$

we obtain that $p^{2^+}_{2^-\ 2^-} = q(q-1)(q-2)/4$. When $q \neq 2$, $p^{2^+}_{2^-\ 2^-} \neq 0$. Similarly as in case (c) above, we can show that Γ_{2^-} is connected.

Now we consider the case when $n \geq 3$. When $q > 2$, we can embed any 2×2 matrix into a $n \times n$ matrix by putting it at the upper-left corner and zero else where. As in the case $n = 2$, we can show that $p^{2^+}_{2^-\ 2^-} \neq 0$. When $q = 2$, we may take $f = x_1 x_3 + x_3^2$ and $g = x_1^2 + x_1 x_2 + x_2^2$. Then f has type 2^+, and both g and $g + f$ have type 2^-. So we also have $p^{2^+}_{2^-\ 2^-} \neq 0$. Therefore Γ_{2^-} is connected. We complete the proof of this theorem. $\square$

From the above theorem, we deduce the Assertion (4) of the main theorem.

Proposition 5.4 *Assertion* (4) *of the main theorem holds.*

Proof: Since Γ_1 is not connected, Qua(n, q) is not primitive. All connected component $Q_B(B \in K_n)$ with $\{R_0, R_1\}$ are trivial subschemes. And they are all isomorphic. All subschemes of Qua(n, q) arise in this way, since Γ_1 is its only disconnected digraph.

Next we show that the quotient scheme of Qua(n, q) is isomorphic to Alt(n, q). We construct a map from X_n to K_n by $\gamma([X]) = X - X^{\mathrm{T}}$. It is not hard to see that γ is well defined and γ is a surjective homomorphism.

What about the kernel(γ)? It turns out that kernel$(\gamma) = Q_0$. For $Q = \sum_{i \leq j} a_{ij} x_i x_j$, $\gamma(Q) = (a_{ij})$ is alternating. If $\gamma(g) = 0$, then $a_{ij} = 0 (i \neq j)$ and thus $Q \in Q_0$. Therefore, kernel$(\gamma) = Q_0$.

$\overline{X_n}$ is the system of imprimitivity. The homomorphism γ induces an isomorphism $\bar{\gamma}$ on the quotient group $\overline{X_n}$. The action of $G_1 = GL_n(\mathbb{F}_q) \cdot X_n$ on X_n induces an action on $\overline{X_n}$. It is not hard to see the kernel of this action is the subgroup $\{(I_n, X) \mid X \in Q_0\}$. Let $\overline{G_1}$ be the quotient group of G_1 modulo $\{(I_n, X) \mid X \in Q_0\}$. Then $\overline{G_1}$ acts faithfully on $\overline{X_n}$. $\overline{G_1}$ can actually be identified with the semidirect product $GL_n(\mathbb{F}_q) \cdot \overline{X_n}$. The quotient scheme Qua$(n, q)/Q_0$ is determined by the action of $\overline{G_1}$ on $\overline{X_n}$ ([1, Example II.9.5].

Now we want to show that Qua$(n, q)/Q_0$ is isomorphic to Alt(n, q). Let $G_3 = GL_n(\mathbb{F}_q) \cdot K_n$, the semidirect product of $GL_n(\mathbb{F}_q)$ and K_n. G_3 acts on K_n in a similar way as in (2.3). Then this action determines an association scheme. Thus, in oder to show that Qua$(n, q)/Q_0$ is isomorphic to Alt(n, q), it suffices to show that the action of $\overline{G_1}$ on $\overline{X_n}$ is equivalent to that of G_3 on K_n.

We define an isomorphism σ between $\overline{G_1}$ and G_3 by $\sigma\overline{(T, A)} = (T, A - A^{\mathrm{T}})$. We have the following commutative diagram:

$$\begin{array}{ccc} \overline{[X]} & \xrightarrow{\bar{\gamma}} & X - X^{\mathrm{T}} \\ (T, A) \downarrow & & \downarrow (T, A - A^{\mathrm{T}}) = \sigma(T, A) \\ \overline{[TXT^{\mathrm{T}} + A]} & \xrightarrow{\bar{\gamma}} & T(X - X^{\mathrm{T}})T^{\mathrm{T}} + (A - A^{\mathrm{T}}) \end{array}$$

So we complete the proof. □

If $(n, q) = (2, 2)$, Qua$(2, 2)$ and Sym(2,2) are isomorphic to the cube graph, which is bipartite and antipodal. Besides the association subschemes $(Q_B, \{R_0, R_1\})(B \in K_2)$, Qua$(2, 2)$ has 4 isomorphic subschemes given by the antipodal pairs. Thus the assertion (5) of the main theorem follows.

Here we assume that q is even. Sym(n, q) is not distance regular for $n > 2$. As pointed out in [3], Sym$(2, q)$ is distance regular and Sym$(3, q)$ contains a distance regular graph coming from a fusion scheme. The dual statements hold for Qua(n, q).

Remark 2 When assuming $GL(n, q)$ as an automorphism group, Propositions 5.1 and 5.4 follow from the representation theory of $GL(n, q)$.

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