Journal of Algebraic Combinatorics, 17, 211–223, 2003

Poincaré Series of the Weyl Groups of the Elliptic Root Systems $A_1^{(1,1)}$, $A_1^{(1,1)^*}$ and $A_2^{(1,1)}$

TADAYOSHI TAKEBAYASHI takeba@shimizu.info.waseda.ac.jp
Department of Mathematical Science, School of Science and Engineering, Waseda University, Ohkubo Shinjuku-ku, Tokyo, 169-8555

Received February 8, 2001; Revised September 10, 2002

Abstract. We calculate the Poincaré series of the elliptic Weyl group $W(A_2^{(1,1)})$, which is the Weyl group of the elliptic root system of type $A_2^{(1,1)}$. The generators and relations of $W(A_2^{(1,1)})$ have been already given by K. Saito and the author.

Keywords: Poincaré series, elliptic root system, elliptic Weyl group

1. Introduction

Elliptic Weyl groups are the Weyl groups associated to the elliptic root systems introduced by K. Saito [5, 6], which are defined by a semi-positive definite inner product with 2-dimensional radical. The generators and their relations of elliptic Weyl groups were described from the viewpoint of a generalization of Coxeter groups by K. Saito and the author [7, 9]. The Poincaré series $W(t)$ of a group W with respect to a generator system is defined by

$$W(t) = \sum_{w\in W} t^{l(w)},$$

where t is an indeterminate and $l(w)$ is the length of a minimal expression of an element w in W in terms of the given generator system. If W is one of the finite or affine Weyl groups, it is known that

$$\sum_{w\in W} t^{l(w)} = \begin{cases} \displaystyle\prod_{i=1}^{n} \frac{1-t^{m_i+1}}{1-t} & \text{(W: finite)}, \\ \displaystyle\frac{1}{(1-t)^n}\prod_{i=1}^{n} \frac{1-t^{m_i+1}}{1-t} & \text{(W: affine)}, \end{cases}$$

where n is the rank and $m_1, \ldots, m_n$ are the exponents of W [1–4, 8]. The goal of the present article is to calculate the Poincaré series $W(t)$ of the elliptic Weyl groups W of types $A_1^{(1,1)}$, $A_1^{(1,1)^*}$ and $A_2^{(1,1)}$. In the cases of types $A_1^{(1,1)}$ and $A_1^{(1,1)*}$, although they have been already given by Wakimoto [10], we give a different proof from those and in the similar way we calculate the case of $A_2^{(1,1)}$. The result for $A_2^{(1,1)}$ is given by Theorem 3.7.

2. Poincaré series of the Weyl groups of types $A_1^{(1,1)}$ and $A_1^{(1,1)*}$

The generators and their relations of the elliptic Weyl group of type $A_1^{(1,1)}$ are given as follows [7, 9]:

Generators: w_i, w_i^* $(i = 0, 1)$.
Relations: $w_i^2 = w_i^{*2} = 1$ $(i = 0, 1)$, $w_0 w_0^* w_1 w_1^* = 1$.

The relation $w_0 w_0^* w_1 w_1^* = 1$ is rewritten as follows:

$$w_0^* w_1 = w_0 w_1^* (\Leftrightarrow w_1^* w_0 = w_1 w_0^*). \tag{2.1.1}$$

(It means that $w_i w_j^* = w_i^* w_j$ $(i \neq j)$.) We set $T := w_1 w_0$, $R := w_1^* w_1 = w_0 w_0^*$, then we easily see the following.

Lemma 2.1 *The elements T, R and w_1 generate the Weyl group of type $A_1^{(1,1)}$ and their fundamental relations are given by*;

$$TR = RT, \quad w_1 T = T^{-1} w_1, \quad w_1 R = R^{-1} w_1, \quad w_1^2 = 1.$$

From this, we have $W = \{R^m T^n w_1,\ R^m T^n,\ m, n \in \mathbb{Z}\}$. The elements T and w_1 generate a subgroup isomorphic to the affine Weyl group of type A_1, and all elements of that are classified to the following:

$$\{\text{(I)}\ T^n (n \geq 0), \quad \text{(II)}\ T^{-n} (n \geq 1), \quad \text{(III)}\ T^n w_1 (n \geq 0), \quad \text{(IV)}\ T^{-n} w_1 (n \geq 1)\}.$$

We multiply the elements $R^m (m \in \mathbb{Z})$ to the above elements from the left, and examine their minimal length in each case by using the following.

Lemma 2.2 *Let w be a minimal expression by w_0 and w_1. Then even if we attach $*$ to any letters of w, the length of w does not decrease.*

Proof: This is clear from the fact that a relation in w_i holds if and only if the relation in w_i^* obtained by attaching $*$ also holds. □

(I) $T^n = (w_1 w_0)^n \quad (n \geq 0)$

From the expression $R w_1 w_0 = w_1^* w_0$ and (2.1.1), we see that $R^k T^n = R^k (w_1 w_0)^n = (w_{11} w_{10})(w_{21} w_{20}) \cdots (w_{n1} w_{n0})$, for $0 \leq k \leq 2n$, where w_{i1} (resp. w_{i0}) is either w_1 or w_1^* (resp. w_0 or w_0^*) for all i, in such a way that $*$ is attached until the k-th letter. Further for $m \geq 1$, $R^{2n+m} T^n = R^m (R^{2n} T^n) = (w_1^* w_1)^m (w_1^* w_0^*)^n$, $R^{-m} T^n = (w_1 w_1^*)^m (w_1 w_0)^n$, and each length is $2n+2m$, so we get $\sharp\{R^kT^n,\ (n \geq 0,\ k \in \mathbb{Z}) \mid l(R^kT^n) = 2n\} = 2n+1$, and $\sharp\{R^kT^n,\ (n \geq 0,\ k \in \mathbb{Z}) \mid l(R^kT^n) = 2n+2m\} = 2$.

The case of (II) is similar to (I).

(III) $T^n w_1 = (w_1 w_0)^n w_1 \quad (n \geq 0)$

From $Rw_1 = w_1^*$ and (2.1.1), for $0 \leq k \leq 2n+1$, we have $R^kT^nw_1 = R^k(w_1w_0)^nw_1 = (w_{11}w_{10})\cdots(w_{n1}w_{n0})w_{n+1,1}$ where $w_{i1} \in \{w_1,\ w_1^*\}$ and $w_{i0} \in \{w_0,\ w_0^*\}$, so $\sharp\{R^kT^nw_1,\ (n \geq 0,\ k \in \mathbb{Z}) \mid l(R^kT^nw_1) = 2n+1\} = 2n+2$, and $\sharp\{R^kT^nw_1, (n \geq 0,\ k \in \mathbb{Z}) \mid l(R^kT^nw_1) = 2n+1+2m\} = \sharp\{R^{2n+1+m}T^nw_1,\ R^{-m}T^nw_1\} = 2$.

(IV) $T^{-n} w_1 = (w_0 w_1)^{n-1} w_0 \quad (n \geq 1)$

From $R^{-1}w_0 = w_0^*$, (2.1.1), and that for $m \geq 1$, $R^{-(2n-1)-m}T^{-n}w_1 = R^{-m}(w_0^*w_1^*)^{n-1}w_0^* = (w_0^*w_0)^m(w_0^*w_1^*)^{n-1}w_0^*$, $R^mT^{-n}w_1 = (w_0w_0^*)^m(w_0w_1)^{n-1}w_0$, we see that $\sharp\{R^kT^{-n}w_1,\ (n \geq 1,\ k \in \mathbb{Z}) \mid l(R^kT^{-n}w_1) = 2n-1\} = 2n$, and $\sharp\{R^kT^{-n}w_1,\ (n \geq 1,\ k \in \mathbb{Z}) \mid l(R^kT^{-n}w_1) = 2n+2m-1\} = 2$.

In the case of type $A_1^{(1,1)*}$, the generators and their relations are given as follows:

Generators: $w_0,\ w_1,\ w_1^*$.
Relations: $w_0^2 = w_1^2 = w_1^{*2} = (w_0w_1w_1^*)^2 = 1$.

This Weyl group is obtained from the Weyl group of type $A_1^{(1,1)}$ by removing one generator w_0^*, so we examine the case of type $A_1^{(1,1)*}$ similarly to the case of type $A_1^{(1,1)}$.

(I) $T^n = (w_1 w_0)^n \quad (n \geq 0)$

From $Rw_1 = w_1^*$, we have $R^nT^n = (w_1^*w_0)^n$, and for $m \geq 1$, $R^{n+m}T^n = R^m(w_1^*w_0)^n = (w_1^*w_1)^m(w_1^*w_0)^n$ and $R^{-m}T^n = (w_1w_1^*)^m(w_1w_0)^n$, so we get $\sharp\{R^kT^n, (n \geq 0, k \in \mathbb{Z}) \mid l(R^kT^n) = 2n\} = n+1$, and $\sharp\{R^kT^n, (n \geq 0, k \in \mathbb{Z}) \mid l(R^kT^n) = 2n+2m\} = 2$.

The case of (II) is similar to (I).

(III) $T^n w_1 = (w_1 w_0)^n w_1 \quad (n \geq 0)$

From $Rw_1 = w_1^*$, and $R^{n+1}(w_1w_0)^nw_1 = (w_1^*w_0)^nw_1^*$, we see that $\sharp\{R^kT^nw_1, (n \geq 0, k \in \mathbb{Z}) \mid l(R^kT^nw_1) = 2n+1\} = n+2$, and $\sharp\{R^kT^nw_1, k \in \mathbb{Z} \mid l(R^kT^nw_1) = 2n+1+2m\} = \sharp\{R^{n+1+m}T^nw_1, R^{-m}T^nw_1\} = 2$.

(IV) $T^{-n} w_1 = (w_0 w_1)^{n-1} w_0 \quad (n \geq 1)$

From $R^{-1}(w_0w_1) = w_0w_1^*$ and $R^{-(n-1)}(w_0w_1)^{n-1}w_0 = (w_0w_1^*)^{n-1}w_0$, we see that $\sharp\{R^kT^{-n}w_1,\ (n \geq 1,\ k \in \mathbb{Z}) \mid l(R^kT^{-n}w_1) = 2n-1\} = n$, and $\sharp\{R^kT^{-n}w_1,\ k \in \mathbb{Z} \mid l(R^kT^{-n}w_1) = 2n-1+2m\} = \sharp\{R^{-n+1-m}T^{-n}w_1,\ R^mT^{-n}w_1\} = 2$.

From the above argument, we obtain the following.

$A_1^{(1,1)}$	$l(w)$ $(n \geq 1, m \geq 1)$	$\sharp$	$A_1^{(1,1)*}$	$l(w)$ $(n \geq 1, m \geq 1)$	$\sharp$
I	0	1	I	0	1
	$2n$	$2n+1$		$2n$	$n+1$
	$2m, 2(n+m)$	2		$2m, 2(n+m)$	2
II	$2n$	$2n+1$	II	$2n$	$n+1$
	$2(n+m)$	2		$2(n+m)$	2
III	$2n-1$	$2n$	III	$2n-1$	$n+1$
	$2(n+m)-1$	2		$2(n+m)-1$	2
IV	$2n-1$	$2n$	IV	$2n-1$	n
	$2(n+m)-1$	2		$2(n+m)-1$	2

Further from this, we obtain the following.

Proposition 2.3 ([10])
(i) *The number of the elements of* $W(A_1^{(1,1)})$ *and* $W(A_1^{(1,1)*})$ *of length n is given by*;

$$W\big(A_1^{(1,1)}\big):\ \ \sharp\{w \in W \mid l(w)=0\}=1,\quad \sharp\{w \in W \mid l(w)=n,\ (n \geq 1)\}=4n,$$
$$W\big(A_1^{(1,1)*}\big):\ \ \sharp\{w \in W \mid l(w)=0\}=1,\quad \sharp\{w \in W \mid l(w)=n,\ (n \geq 1)\}=3n.$$

(ii) *The Poincaré series of* $W(A_1^{(1,1)})$ *and* $W(A_1^{(1,1)*})$ *are given by*;

$$\sum_{w \in W(A_1^{(1,1)})} t^{l(w)} = \frac{(1+t)^2}{(1-t)^2}, \qquad \sum_{w \in W(A_1^{(1,1)*})} t^{l(w)} = \frac{1-t^3}{(1-t)^3}.$$

Proof: (i) For an integer $k \geq 2$, the number of pairs (m, n) satisfying $k = m + n (m \geq 1, n \geq 1)$ is equal to $k-1$, so in the case of type $A_1^{(1,1)}$, $\sharp\{w \in W \mid l(w) = 2n\} = (2n+1) \times 2 + 2 + 2 \times (n-1) \times 2 = 8n$, and $\sharp\{w \in W \mid l(w) = 2n-1\} = 2n \times 2 + 2 \times (n-1) \times 2 = 8n-4$, so we get the result. The case of type $A_1^{(1,1)*}$ is calculated similarly. Then (ii) is easily obtained from (i). □

3. Poincaré series of the Weyl group of type $A_2^{(1,1)}$

The elliptic Weyl group W of type $A_2^{(1,1)}$ is presented as follows [7, 9].

Generators: $w_i,\ w_i^* \quad (i = 0, 1, 2)$.

Relations: $w_i^2 = w_i^{*2} = 1 \quad (i = 0, 1, 2)$,

for $i \neq j$

$$w_i w_j w_i = w_j w_i w_j, \quad w_i^* w_j^* w_i^* = w_j^* w_i^* w_j^*,$$
$$w_i^* w_j w_i^* = w_j w_i^* w_j = w_i w_j^* w_i = w_j^* w_i w_j^*,$$

and $w_0 w_0^* w_1 w_1^* w_2 w_2^* = 1$.

We set $T_1 := w_0w_2w_0w_1$, $T_2 := w_0w_1w_0w_2$, $R_1 := w_1w_1^*$, and $R_2 := w_2w_2^*$, then we have the following.

Lemma 3.1

(i) *W is generated by $w_1, w_2, T_1, T_2, R_1, R_2$, and they satisfy the following fundamental relations*:

$$\begin{cases} w_iT_i = T_i^{-1}w_i \\ w_iR_i = R_i^{-1}w_i \\ w_iT_j = T_iT_jw_i \quad (i \neq j) \\ w_iR_j = R_iR_jw_i \quad (i \neq j). \end{cases}$$

(ii) $W = \{R_1^nR_2^mT_1^kT_2^lw, \quad (n, m, k, l \in \mathbb{Z}) \mid w = \mathrm{id}, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1\}$.

Proof: Let Φ be the elliptic root system of type $A_2^{(1,1)}$, then one has the expression [5]

$$\Phi = \{\pm(\epsilon_i - \epsilon_j) + nb + ma \mid 1 \le i < j \le 3,\ n, m \in \mathbb{Z}\},$$

with an inner product $\langle\,,\rangle$, which is a symmetric bilinear form given by

$$\langle\epsilon_i, \epsilon_j\rangle = \delta_{ij}, \quad \langle\epsilon_i, a\rangle = \langle\epsilon_i, b\rangle = \langle a, b\rangle = \langle a, a\rangle = \langle b, b\rangle = 0, \quad (1 \le i,\ j \le 3).$$

Let $F = \bigoplus_{1\le i<j\le 3}\mathbb{R}(\epsilon_i - \epsilon_j) \oplus \mathbb{R}b \oplus \mathbb{R}a$ be a real vector space. Let w_α be the reflection corresponding to the root α defined by $w_\alpha(x) = x- < x, \alpha^\vee > \alpha, \quad \forall x \in F$ with $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$. We set $\alpha_0 := \epsilon_3 - \epsilon_1 + b$, $\alpha_1 := \epsilon_1 - \epsilon_2$, $\alpha_2 := \epsilon_2 - \epsilon_3$ and $\alpha_i^* := \alpha_i + a$ $(i = 0, 1, 2)$. Then $w_i = w_{\alpha_i},\ w_i^* = w_{\alpha_i^*}$. We see that all reflections act on $\mathbb{R}b \oplus \mathbb{R}a$ as identity, and

$$\begin{cases} w_1(\epsilon_1) = \epsilon_2 \\ w_1(\epsilon_2) = \epsilon_1 \\ w_1(\epsilon_3) = \epsilon_3 \end{cases} \begin{cases} w_2(\epsilon_1) = \epsilon_1 \\ w_2(\epsilon_2) = \epsilon_3 \\ w_2(\epsilon_3) = \epsilon_2 \end{cases} \begin{cases} w_0(\epsilon_1) = \epsilon_3 + b \\ w_0(\epsilon_2) = \epsilon_2 \\ w_0(\epsilon_3) = \epsilon_1 - b \end{cases}$$

$$\begin{cases} w_1^*(\epsilon_1) = \epsilon_2 - a \\ w_1^*(\epsilon_2) = \epsilon_1 + a \\ w_1^*(\epsilon_3) = \epsilon_3 \end{cases} \begin{cases} w_2^*(\epsilon_1) = \epsilon_1 \\ w_2^*(\epsilon_2) = \epsilon_3 - a \\ w_2^*(\epsilon_3) = \epsilon_2 + a \end{cases} \begin{cases} w_0^*(\epsilon_1) = \epsilon_3 + a \\ w_0^*(\epsilon_2) = \epsilon_2 \\ w_0^*(\epsilon_3) = \epsilon_1 - a \end{cases}$$

From these, we have the following:

$$\begin{cases} T_1(\epsilon_1) = \epsilon_1 - b \\ T_1(\epsilon_2) = \epsilon_2 + b \\ T_1(\epsilon_3) = \epsilon_3 \end{cases} \begin{cases} T_2(\epsilon_1) = \epsilon_1 \\ T_2(\epsilon_2) = \epsilon_2 - b \\ T_2(\epsilon_3) = \epsilon_3 + b \end{cases} \begin{cases} R_1(\epsilon_1) = \epsilon_1 - a \\ R_1(\epsilon_2) = \epsilon_2 + a \\ R_1(\epsilon_3) = \epsilon_3 \end{cases} \begin{cases} R_2(\epsilon_1) = \epsilon_1 \\ R_2(\epsilon_2) = \epsilon_2 - a \\ R_2(\epsilon_3) = \epsilon_3 + a \end{cases}$$

From these actions, we have

$$w_0 = T_1T_2w_1w_2w_1, \quad w_0^* = R_1R_2T_1T_2w_1w_2w_1, \quad w_1^* = w_1R_1, \quad w_2^* = w_2R_2,$$

and from this, (i) is easily checked. (ii) follows from (i). □

We first consider minimal expressions of the elements $T_1^n T_2^m$ generated by $T_1 = w_0 w_2 w_0 w_1$, and $T_2 = w_0 w_1 w_0 w_2$, then by noting the following minimal expressions;

$$T_1 T_2 = w_0 w_1 w_2 w_1, \quad T_1 T_2^{-1} = (w_2 w_0 w_1)^2, \quad T_1 T_2^2 = (w_0 w_1 w_2)^2,$$

we have $T_1^n T_2^{n+i} = (0121)^n (0102)^i = (012)^2 (0121)^{n-1} (0102)^{i-1}$, and from this we obtain

$$T_1^n T_2^{n+i} (n \geq 1, i \geq 1) = \begin{cases} T_1^n T_2^{n+i} = (012)^{2i} (0121)^{n-i} & (1 \leq i < n, n \geq 2) \\ T_1^n T_2^{2n+i} = (0102)^i (012)^{2n} & (i \geq 0, n \geq 1) \end{cases}$$

where for brevity, we use 0, 1, 2, 0*, 1*, 2* for $w_0, w_1, w_2, w_0^*, w_1^*, w_2^*$, respectively. Further by considering minimal expressions of $T_1^n T_2^m w (w = w_1, w_2, w_1 w_2, w_2 w_1, w_1 w_2 w_1)$, we classify $T_1^n T_2^m (n, m \in \mathbb{Z})$ as follows.

$$T_1^n T_2^m (n, m \in \mathbb{Z}) = \begin{cases} T_1^n T_2^{n+i} = (012)^{2i} (0121)^{n-i} \quad (1 \leq i < n,\ n \geq 2) \quad (1 \leftrightarrow 2) \\ T_1^{-n} T_2^{-n-i} = (210)^{2i} (1210)^{n-i} \quad (1 \leq i \leq n,\ n \geq 1) \quad (1 \leftrightarrow 2) \\ T_1^n T_2^{2n+i} = (0102)^i (012)^{2n} \quad (i \geq 0,\ n \geq 1) \quad (1 \leftrightarrow 2) \\ T_1^{-n} T_2^{-2n-i} = (210)^{2n} (2010)^i \quad (i \geq 1,\ n \geq 0) \quad (1 \leftrightarrow 2) \\ T_1^{-n-i} T_2^n = (1020)^i (102)^{2n} \quad (i \geq 0,\ n \geq 1) \quad (1 \leftrightarrow 2) \\ T_1^{n+i} T_2^{-n} = (201)^{2n} (0201)^i \quad (i \geq 1,\ n \geq 0) \quad (1 \leftrightarrow 2) \\ T_1^n T_2^n = (0121)^n \quad (n \geq 1) \\ T_1^{-n} T_2^{-n} = (1210)^n \quad (n \geq 0), \end{cases} \tag{3.1.1}$$

where $(1 \leftrightarrow 2)$ means that we consider the element obtained by exchanging T_1 and T_2.

Similarly to the case of type $A_1^{(1,1)}$, we use the following.

Lemma 3.2 *Let w be a minimal expression by w_0, w_1 and w_2. Then even if we attach $*$ to any letters of w, the length of that does not decrease.*

In each case we multiply $R_1^k R_2^l$ from the left, and examine their minimal length. For $1 \leq i < n$, $T_1^n T_2^{n+i} = (012)^{2i} (0121)^{n-i}$, by noting the expressions:

$$\begin{cases} 0^*12012 = (R_1 R_2)\, 012012 \\ 01^*2012 = R_2\, 012012 \\ 012^*012 = (R_1 R_2)\, 012012 \\ 0120^*12 = R_2\, 012012 \\ 01201^*2 = (R_1 R_2)\, 012012 \\ 012012^* = R_2\, 012012 \end{cases} \qquad \begin{cases} 0^*121 = (R_1 R_2)\, 0121 \\ 01^*21 = R_2\, 0121 \\ 012^*1 = (R_1 R_2)\, 0121 \\ 0121^* = R_1\, 0121, \end{cases}$$

we consider how many R_1, R_2 and $R_1 R_2$ can be contained in $(012)^{2i} (0121)^{n-i}$ by attaching $*$ to arbitrary letters. From the above, $(012)^2$ can contain $3 \times R_1 R_2$ and $3 \times R_2$, and 0121 can contain $2 \times R_1R_2,\ 1 \times R_1,\ 1 \times R_2$, so by the relation, $(012)^2 R_j = R_j(012)^2$ $(j = 1, 2)$, we see that $(012)^{2i}(0121)^{n-i}$ can contain $(n-i) \times R_1,\ (n+2i) \times R_2$ and $(2n+i) \times R_1R_2$.

Lemma 3.3 *For* $1 \le i < n$

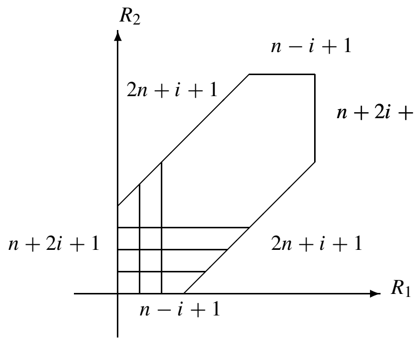
$$\begin{aligned} &R_1^k R_2^l (R_1R_2)^m T_1^n T_2^{n+i} \\ &\quad = R_1^k R_2^l (R_1R_2)^m (012)^{2i}(0121)^{n-i} \\ &\quad = (w_{10}w_{11}w_{12}) \cdots (w_{2i,0}w_{2i,1}w_{2i,2})(w'_{10}w'_{11}w'_{12}w''_{11}) \\ &\qquad \cdots (w'_{n-i,0}w'_{n-i,1}w'_{n-i,2}w''_{n-i,1}) \end{aligned}$$

where w_{ij}, *and* $w'_{ij} = w_j, w_j^*(j = 0, 1, 2)$ *and* $w''_{i1} = w_1, w_1^*$, *for any* $0 \le k \le n-i, 0 \le l \le n+2i, 0 \le m \le 2n+i$.

We count the number

$$\begin{aligned} &\sharp\big\{R_1^k R_2^l T_1^n T_2^{n+i}, (1 \le i < n, n \ge 2, k, l \in \mathbb{Z}) \bigm| l\big(R_1^k R_2^l T_1^n T_2^{n+i}\big) \\ &\quad = l\big(T_1^n T_2^{n+i}\big) = 4n + 2i\big\}. \end{aligned}$$

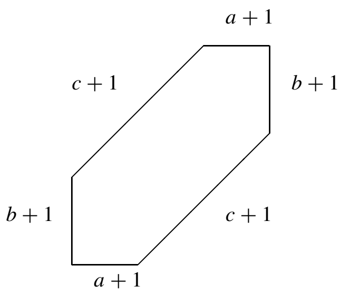
For the purpose we use the following figure:



then the number is equal to the number of the vertices of the lattices, where $n-i+1, n+2i+1,$ and $2n+i+1$ are the number of vertices on each edge.

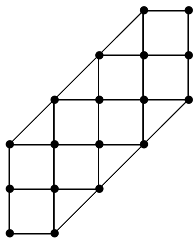
Then we use the following.

Lemma 3.4



In the left figure, the number of the vertices of the lattices is $ab + bc + ca + a + b + c + 1$.

(For example, the case of $a = 1, b = 2, c = 3$)



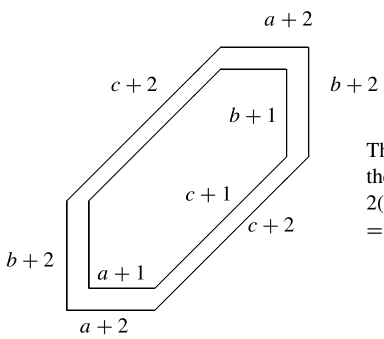
$\sharp\,\{\text{all vertices}\} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 + 1 + 2 + 3 + 1 = 18.$

By multiplying $R_1^{\pm 1},\ R_2^{\pm 1}$, and $(R_1 R_2)^{\pm 1}$, $(R_1 = w_1 w_1^*, R_2 = w_2 w_2^*, R_1 R_2 = w_0^* w_0)$, we obtain the elements whose length are $4n + 2i + 2$, and actually we have only to multiply to the boundary in the figure, and iterating this procedure we get the following.

Lemma 3.5

$$\sharp\left\{R_1^m R_2^l T_1^n T_2^{n+i},\ (1 \leq i < n,\ n \geq 2,\ m, l \in \mathbb{Z}) \,\middle|\, l\left(R_1^m R_2^l T_1^n T_2^{n+i}\right) = 4n + 2i + 2k,\ (k \geq 1)\right\} = 8n + 4i + 6k.$$

Proof:



The number of the vertices of the boundary of the outside is
$2(a + 1 + b + 1 + c + 1) = 2(a + b + c) + 6$
$= \sharp\,\{\text{the boundary of the figure of the previous element}\} + 6.$

□

Next we consider the elements $T_1^n T_2^{n+i} w$, for $w = w_1, w_2, w_1 w_2, w_2 w_1$, and $w_1 w_2 w_1$, then we have the following:

$$\begin{cases} T_1^n T_2^{n+i} = (012)^{2i}(0121)^{n-i} \\ T_1^n T_2^{n+i} 1 = (012)^{2i}(0121)^{n-i-1}012 \\ T_1^n T_2^{n+i} 2 = (012)^{2i}(0121)^{n-i-1}021 \\ T_1^n T_2^{n+i} 12 = (012)^{2i}(0121)^{n-i-1}01 \\ T_1^n T_2^{n+i} 21 = (012)^{2i}(0121)^{n-i-1}02 \\ T_1^n T_2^{n+i} 121 = (012)^{2i}(0121)^{n-i-1}0. \end{cases}$$

In the similar method to the case of $T_1^n T_2^{n+i}$, in this case and for other cases we count how many $R_1^{\pm 1}$, $R_2^{\pm 1}$ and $(R_1 R_2)^{\pm 1}$ can be contained in a minimal expression. By the figure of the number of $R_1^{\pm 1}$, $R_2^{\pm 1}$ and $(R_1 R_2)^{\pm 1}$, we count the number of a minimal expression of the elements of the Weyl group and that of increasing length by 2, which is equal to $\sharp$(the boundary of the figure of the previous element) + 6. In the sequal, we examine the number of the vertices on each edge of the figure in a minimal expression, first we have

$$\begin{cases} 2^*10210 = R_2^{-1}\ 210210 \\ 21^*0210 = (R_1 R_2)^{-1}\ 210210 \\ 210^*210 = R_2^{-1}\ 210210 \\ 2102^*10 = (R_1 R_2)^{-1}\ 210210 \\ 21021^*0 = R_2^{-1}\ 210210 \\ 210210^* = (R_1 R_2)^{-1}\ 210210 \end{cases} \qquad \begin{cases} 1^*210 = R_1^{-1}\ 1210 \\ 12^*10 = (R_1 R_2)^{-1}\ 1210 \\ 121^*0 = R_2^{-1}\ 1210 \\ 1210^* = (R_1 R_2)^{-1}\ 1210 \end{cases}$$

$$\begin{cases} 0^*102 = (R_1 R_2)\ 0102 \\ 01^*02 = R_2\ 0102 \\ 010^*2 = R_1^{-1}\ 0102 \\ 0102^* = R_2\ 0102 \end{cases} \qquad \begin{cases} 2^*010 = R_2^{-1}\ 2010 \\ 20^*10 = R_1\ 2010 \\ 201^*0 = R_2^{-1}\ 2010 \\ 2010^* = (R_1 R_2)^{-1}\ 2010 \end{cases}$$

$$\begin{cases} 1^*020 = R_1^{-1}\ 1020 \\ 10^*20 = R_2\ 1020 \\ 102^*0 = R_1^{-1}\ 1020 \\ 1020^* = (R_1 R_2)^{-1}\ 1020 \end{cases} \qquad \begin{cases} 1^*02102 = R_1^{-1}\ 102102 \\ 10^*2102 = R_2\ 102102 \\ 102^*102 = R_1^{-1}\ 102102 \\ 1021^*02 = R_2\ 102102 \\ 10210^*2 = R_1^{-1}\ 102102 \\ 102102^* = R_2\ 102102 \end{cases}$$

From these and (3.1.1), we obtain the following eight tables.

(I) $\mathbf{T_1^n T_2^{n+i} = (012)^{2i}(0121)^{n-i}}$ $\mathbf{(1 \le i < n,\ n \ge 2)}$			
$(012)^{2i}(0121)^{n-i}w$	$\sharp R_1^{\pm 1}$	$\sharp R_2^{\pm 1}$	$\sharp (R_1 R_2)^{\pm 1}$
$(012)^{2i}(0121)^{n-i}$	$n-i$	$n+2i$	$2n+i$
$(012)^{2i}(0121)^{n-i-1}012$	$n-i-1$	$n+2i$	$2n+i$
$(012)^{2i}(0121)^{n-i-1}021$	$n-i$	$n+2i-1$	$2n+i$
$(012)^{2i}(0121)^{n-i-1}01$	$n-i-1$	$n+2i$	$2n+i-1$
$(012)^{2i}(0121)^{n-i-1}02$	$n-i$	$n+2i-1$	$2n+i-1$
$(012)^{2i}(0121)^{n-i-1}0$	$n-i-1$	$n+2i-1$	$2n+i-1$

(II) $\mathbf{T_1^{-n}T_2^{-n-i} = (210)^{2i}(1210)^{n-i}\ (1 \le i \le n,\ n \ge 1)}$			
$(210)^{2i}(1210)^{n-i}$	$n-i$	$n+2i$	$2n+i$
$(210)^{2i}(1210)^{n-i}1$	$n-i+1$	$n+2i$	$2n+i$
$(210)^{2i}(1210)^{n-i}2$	$n-i$	$n+2i+1$	$2n+i$
$(210)^{2i}(1210)^{n-i}12$	$n-i+1$	$n+2i$	$2n+i+1$
$(210)^{2i}(1210)^{n-i}21$	$n-i$	$n+2i+1$	$2n+i+1$
$(210)^{2i}(1210)^{n-i}121$	$n-i+1$	$n+2i+1$	$2n+i+1$

(III) $\mathbf{T_1^{n}T_2^{2n+i} = (0102)^{i}(012)^{2n}\ (i \ge 0,\ n \ge 1)}$			
$(0102)^{i}(012)^{2n}$	i	$3n+2i$	$3n+i$
$(0102)^{i}(012)^{2n}1$	$i+1$	$3n+2i$	$3n+i$
$(0102)^{i}(012)^{2n-2}01201$	i	$3n+2i-1$	$3n+i$
$(0102)^{i}(012)^{2n-2}012021$	$i+1$	$3n+2i$	$3n+i-1$
$(0102)^{i}(012)^{2n-2}0120$	i	$3n+2i-1$	$3n+i-1$
$(0102)^{i}(012)^{2n-2}01202$	$i+1$	$3n+2i-1$	$3n+i-1$

(IV) $\mathbf{T_1^{-n}T_2^{-2n-i} = (210)^{2n}(2010)^{i}\ (i \ge 1,\ n \ge 0)}$			
$(210)^{2n}(2010)^{i}$	i	$3n+2i$	$3n+i$
$(210)^{2n}(2010)^{i-1}210$	$i-1$	$3n+2i$	$3n+i$
$(210)^{2n}(2010)^{i}2$	i	$3n+2i+1$	$3n+i$
$(210)^{2n}(2010)^{i-1}2102$	$i-1$	$3n+2i$	$3n+i+1$
$(210)^{2n}(2010)^{i}21$	i	$3n+2i+1$	$3n+i+1$
$(210)^{2n}(2010)^{i-1}21021$	$i-1$	$3n+2i+1$	$3n+i+1$

(V) $\mathbf{T_1^{-n-i}T_2^{n} = (1020)^{i}(102)^{2n}\ (i \ge 0,\ n \ge 1)}$			
$(1020)^{i}(102)^{2n}$	$3n+2i$	$3n+i$	i
$(1020)^{i}(102)^{2n}1$	$3n+2i+1$	$3n+i$	i
$(1020)^{i}(102)^{2n-2}10210$	$3n+2i$	$3n+i-1$	i
$(1020)^{i}(102)^{2n}12$	$3n+2i+1$	$3n+i$	$i+1$
$(1020)^{i}(102)^{2n-2}102101$	$3n+2i$	$3n+i-1$	$i+1$
$(1020)^{i}(102)^{2n-2}1021012$	$3n+2i+1$	$3n+i-1$	$i+1$

(VI) $\mathbf{T_1^{n+i}T_2^{-n} = (201)^{2n}(0201)^{i}\ (i \ge 1,\ n \ge 0)}$			
$(201)^{2n}(0201)^{i}$	$3n+2i$	$3n+i$	i
$(201)^{2n}(0201)^{i-1}202$	$3n+2i-1$	$3n+i$	i
$(201)^{2n}(0201)^{i}2$	$3n+2i$	$3n+i+1$	i
$(201)^{2n}(0201)^{i-1}20$	$3n+2i-1$	$3n+i$	$i-1$
$(201)^{2n}(0201)^{i-1}2012$	$3n+2i$	$3n+i+1$	$i-1$
$(201)^{2n}(0201)^{i-1}201$	$3n+2i-1$	$3n+i+1$	$i-1$

(VII) $\mathbf{T_1^n T_2^n = (0121)^n\ (n \geq 1)}$			
$(0121)^n$	n	n	$2n$
$(0121)^{n-1}012$	$n-1$	n	$2n$
$(0121)^{n-1}021$	n	$n-1$	$2n$
$(0121)^{n-1}01$	$n-1$	n	$2n-1$
$(0121)^{n-1}02$	n	$n-1$	$2n-1$
$(0121)^{n-1}0$	$n-1$	$n-1$	$2n-1$

(VIII) $\mathbf{T_1^{-n} T_2^{-n} = (1210)^n\ (n \geq 0)}$			
$(1210)^n$	n	n	$2n$
$(1210)^n 1$	$n+1$	n	$2n$
$(1210)^n 2$	n	$n+1$	$2n$
$(1210)^n 12$	$n+1$	n	$2n+1$
$(1210)^n 21$	n	$n+1$	$2n+1$
$(1210)^n 121$	$n+1$	$n+1$	$2n+1$

We explain how to read the above tables, by using **(I)**. In the element $(012)^{2i}(0121)^{n-i}w$, w runs the elements $\{id,\ 012,\ 021,\ 01,\ 02,\ 0\}$. The row of $\sharp R_1^{\pm 1}$ denotes the number of $R_1^{\pm 1}$, for example, in the case $(012)^{2i}(0121)^{n-i}$, $\sharp R_1 = n-i$. Therefore the third line in **(I)** means that in the type $(012)^i(0121)^{n-i}$, the number of the elements such that $l(w) = 3 \times 2i + 4 \times (n-i) = 4n+2i$, is equal to $\sharp$ { all vertices in the figure of $\sharp R_1 = n-i,\ \sharp R_2 = n+2i,\ \sharp(R_1R_2) = 2n+i$ }. From all tables, we find the following.

Lemma 3.6
(i) *By the suitable rearrangements of rows and columns, all tables are rewritten as*

$\sharp R_1^{\pm 1}, \sharp R_2^{\pm 1}, \sharp (R_1R_2)^{\pm 1}$		
a	b	c
a	$b+1$	c
a	$b+1$	$c+1$
$a+1$	b	c
$a+1$	b	$c+1$
$a+1$	$b+1$	$c+1$

and

	a	b	c
I	$n-i-1$	$n+2i-1$	$2n+i-1$
II	$n-i$	$n+2i$	$2n+i$
III	$3n+i-1$	i	$3n+2i-1$
IV	$3n+i$	$i-1$	$3n+2i$
V	i	$3n+i-1$	$3n+2i$
VI	$i-1$	$3n+i$	$3n+2i-1$
VII	$n-1$	$n-1$	$2n-1$
VIII	n	n	$2n$

(ii) *In all eight tables, we see that the minimal length $l(w)$ of each element w is equal to the sum of $\sharp R_1^{\pm 1}$, $\sharp R_2^{\pm 1}$, and $\sharp (R_1R_2)^{\pm 1}$, that is, $l(w) = \sharp R_1^{\pm 1} + \sharp R_2^{\pm 1} + \sharp (R_1R_2)^{\pm 1}$.*

From this lemma, we obtain the main result.

Theorem 3.7 *The Poincaré series of the Weyl group of type $A_2^{(1,1)}$ is given by*

$$\sum_{w\in W} t^{l(w)} = \frac{1+4t+17t^2+19t^3+17t^4+4t^5+t^6}{(1-t)^4(1+t)^2}$$
$$= \frac{(1+t+t^2)(1+3t+13t^2+3t^3+t^4)}{(1-t)^4(1+t)^2}.$$

Proof: We set $w(a,b,c) := (ab+bc+ca+a+b+c+1)t^{a+b+c} + \sum_{k=1}^{\infty}\{2(a+b+c)+6k\}t^{a+b+c+2k}$, and $W(a,b,c) := w(a,b,c) + w(a,b+1,c) + w(a,b+1,c+1) + w(a+1,b,c) + w(a+1,b,c+1) + w(a+1,b+1,c+1)$. Then the Poincaré series is calculated as follows:

$$\begin{aligned}\sum_{w\in W} t^{l(w)} = 2\Bigg\{ &\sum_{n=2}^{\infty}\sum_{i=1}^{n-1} W(n-i-1, n+2i-1, 2n+i-1)\\ &+ \sum_{n=1}^{\infty}\sum_{i=1}^{n} W(n-i, n+2i, 2n+i)\\ &+ \sum_{n=1}^{\infty}\sum_{i=0}^{\infty} W(3n+i-1, i, 3n+2i-1)\\ &+ \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} W(3n+i, i-1, 3n+2i)\\ &+ \sum_{n=1}^{\infty}\sum_{i=0}^{\infty} W(i, 3n+i-1, 3n+2i)\\ &+ \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} W(i-1, 3n+i, 3n+2i-1)\Bigg\}\\ &+ \sum_{n=1}^{\infty} W(n-1, n-1, 2n-1) + \sum_{n=0}^{\infty} W(n, n, 2n).\end{aligned}$$

By using Mathematica, we obtain the desired result. □

References

1. N. Bourbaki *Groupes et algebrès de Lie*, Ch. 4–6, Hermann, Paris, 1968; Mason, Paris, 1981.
2. J.E. Humphreys, "Reflection groups and Coxeter groups," *Cambridge Studies in Advanced Math*. Cambridge University Press, 1990, vol. 29.
3. N. Iwahori and H. Matsumoto, "On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups," *Publ. Math. I.H.E.S*. **25** (1965), 5–48.
4. I.G. Macdonald, "The Poincaré series of a Coxeter group," *Math. Ann*. **199** (1972), 161–174.
5. K. Saito, "Extended affine root systems I," *Publ. RIMS, Kyoto Univ*. **21** (1985), 75–179.

6. K. Saito, "Extended affine root systems II," *Publ. RIMS, Kyoto Univ*. **26** (1990), 15–78.
7. K. Saito and T. Takebayashi, "Extended affine root systems III," *Publ. RIMS, Kyoto Univ*. **33** (1997), 301–329.
8. L. Solomon, "The orders of the finite Chevalley groups," *Journal of Algebra* **3** (1966), 376–393.
9. T. Takebayashi "Relations of the Weyl groups of extended affine root systems $A_l^{(1,1)}$, $B_l^{(1,1)}$, $C_l^{(1,1)}$, $D_l^{(1,1)}$," *Proc. Japan Acad*. **71**(6) (1995), A123–124.
10. M. Wakimoto, "Poincaré series of the Weyl group of elliptic Lie algebras $A_1^{(1,1)}$ and $A_1^{(1,1)*}$," q-alg/9705025.