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Sparse Resultant under Vanishing Coefficients

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Abstract. The main question of this paper is: *What happens to the sparse* (*toric*) *resultant under vanishing coefficients*? More precisely, let $f_1, \dots, f_n$ be sparse Laurent polynomials with supports $\mathcal{A}_1, \dots, \mathcal{A}_n$ and let $\tilde{\mathcal{A}}_1 \supset \mathcal{A}_1$. Naturally a question arises: Is the sparse resultant of $f_1, f_2, \dots, f_n$ with respect to the supports $\tilde{\mathcal{A}}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ in any way related to the sparse resultant of $f_1, f_2, \dots, f_n$ with respect to the supports $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$? The main contribution of this paper is to provide an answer. The answer is important for applications with perturbed data where very small coefficients arise as well as when one computes resultants with respect to some fixed supports, not necessarily the supports of the f_i's, in order to speed up computations. This work extends some work by Sturmfels on sparse resultant under vanishing coefficients. We also state a corollary on the sparse resultant under powering of variables which generalizes a theorem for Dixon resultant by Kapur and Saxena. We also state a lemma of independent interest generalizing Pedersen's and Sturmfels' Poisson-type product formula.

Keywords: elimination theory, resultant, product formula, Newton polytope

1. Introduction

Resultants are of fundamental importance for solving systems of polynomial equations and therefore have been extensively studied (cf. [1, 3, 5, 6, 9, 10, 13, 16, 18–20, 22]). Recent research has focused on utilizing structure, naturally occurring in real life problems, of polynomials, for example, composition (cf. [7, 14, 15, 17, 21]) and sparsity (in the frame of toric algebra) (cf. [2, 4, 8, 11, 12, 23, 24]).

We ask: *What happens to the sparse* (*toric*) *resultant under vanishing coefficients*? That is, what is the sparse resultant of sparse Laurent polynomials $f_1, \dots, f_n$ assuming that some of the coefficients of f_1 are zero? More precisely, let $f_1, \dots, f_n$ be sparse Laurent polynomials with the supports $\mathcal{A}_1, \dots, \mathcal{A}_n$ and let $\tilde{\mathcal{A}}_1 \supset \mathcal{A}_1$. Naturally a question arises: Is the sparse resultant of $f_1, f_2, \dots, f_n$ with respect to the supports $\tilde{\mathcal{A}}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ in any way related to the sparse resultant of $f_1, f_2, \dots, f_n$ with respect to the supports $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$? The main contribution of this paper is to provide an answer: The sparse resultant of $f_1, f_2, \dots, f_n$ with respect to the supports $\tilde{\mathcal{A}}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ is some power of the sparse resultant of $f_1, f_2, \dots, f_n$ with respect to the supports $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ times a product of powers of sparse resultants of some parts of the f_i's. We also state a corollary (cf. Corollary 5) about the sparse resultant under powering of variables which is a generalization of a theorem for Dixon resultant shown by Kapur and Saxena using different techniques (cf. [17]). We also state a lemma (cf. Lemma 13) of independent interest generalizing Pedersen's and Sturmfels' Poisson-type product formula.

This result is important for applications where perturbed data with very small coefficients arise and these coefficients may tend to zero. For such cases, the main theorem, Theorem 1, gives information about the stability of the resultant. Furthermore, this result is important when one computes resultants with respect to some fixed supports, not necessarily the supports of the f_i's. This is sometimes done because for certain supports there are very efficient algorithms for resultant computation, consider for example the Dixon resultant (cf. e.g. [17]). Furthermore, we were motivated to work on sparse resultant under vanishing coefficients because we wanted to give an irreducible factorization of formula of [14]. For this purpose we used the main theorem, Theorem 1, of the present paper.

Theorem 1 extends a corollary by Sturmfels (cf. Corollary 4.2 of [25]) which essentially states that the sparse resultant of the Laurent polynomials $f_1, \dots, f_n$ with respect to their precise supports divides the sparse resultant of $f_1, \dots, f_n$ with respect to larger supports. This result, Theorem 1, also generalizes a lemma of [21], Lemma 9, for Macaulay resultant of dense polynomials under vanishing of leading forms.

We assume that the reader is familiar with the notions of sparse (toric) resultant, essential, integer lattice, fundamental simplex of an integer lattice, Newton polytope, primitive vector (i.e. a vector with integer coordinates whose gcd is one, cf. [8]), inward normal vector (cf. [8]), mixed volume (cf. [8, 12, 23, 25]). We let $\mathrm{Res}_{\mathcal{A}_1,\dots,\mathcal{A}_n}(\cdot)$ stand for sparse resultant with respect to the supports $\mathcal{A}_1, \dots, \mathcal{A}_n \subseteq \mathbb{Z}^{n-1}$, we let $\mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ stand for the integer sublattice of $\mathbb{Z}^{n-1}$ affinely generated by $\mathcal{A}_1, \dots, \mathcal{A}_n$ (in detail: the $\mathbb{Z}$-submodule of $\mathbb{Z}^{n-1}$ generated by the set of vectors of the form v_i, for $i = 1, \dots, n$, where v_i is any difference of two points in $\mathcal{A}_i$), we let $[L_1 : L_2]$ (where $L_2 \subseteq L_1$) stand for the quotient of the volumes of the fundamental simplices of the integer lattice L_2 and L_1 and we let $\mathcal{A}^\omega \subseteq \mathcal{A}$ stand for the set of vectors that lie in the face, with inward normal vector ω, of the convex hull of the bounded set $\mathcal{A}$. (In this definition the vector ω needs not to be primitive. However, in the following sections the vector ω will always be primitive.)

2. Main result

Let $f_1, \dots, f_n$ be sparse Laurent polynomials in the variables $x_1, \dots, x_{n-1}$ with non-empty supports $\mathcal{A}_1, \dots, \mathcal{A}_n$ and, for the sake of a simple presentation, with distinct symbolic coefficients.

Let $\tilde{\mathcal{A}}_1$ be a finite set with $\mathcal{A}_1 \subseteq \tilde{\mathcal{A}}_1 \subset \mathbb{Z}^{n-1}$ and let $(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)$ have a unique essential subset, not necessarily equal to $\{1, \dots, n\}$. We furthermore assume that this unique essential subset contains the index 1 (cf. Remarks 2 and 3).

Let $f^{\mathcal{A}}$ stand for the part, whose support is contained in the set $\mathcal{A}$, of the Laurent polynomial f and let $a_{\mathcal{A}}(\omega)$ stand for $-\min_v(\langle \omega, v\rangle)$, where $\langle \omega, v\rangle$ denotes the usual Euclidean inner product and v ranges over the convex hull of $\mathcal{A}$. Furthermore let H^ω stand for the lattice of all integer points contained in the (unique) hyperplane, passing through the origin, with normal vector ω. (So, throughout this paper, H is a constant symbol of a unary *function*. The symbol H does not stand for the unique hyperplane, passing through the origin, with normal vector ω.)

Now we are ready to state the main theorem.

Theorem 1 (Main theorem) *We have*

$$\begin{aligned}
&\mathrm{Res}_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(f_1, f_2, \ldots, f_n)\\
&\quad = \mathrm{Res}_{\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(f_1, f_2, \ldots, f_n)^{[\mathcal{L}(\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}\\
&\qquad \times \prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\left(f_2^{\mathcal{A}_2^{\omega}}, \ldots, f_n^{\mathcal{A}_n^{\omega}}\right)^{(\mathrm{a}_{\tilde{\mathcal{A}}_1}^{(\omega)}-\mathrm{a}_{\mathcal{A}_1}^{(\omega)})\,\frac{[\mathrm{H}^{\omega}:\mathcal{L}(\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega})]}{[\mathbb{Z}^{n-1}:\mathcal{L}(\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)]}},
\end{aligned}$$

where ω ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{A}_2 + \cdots + \mathcal{A}_n$. Furthermore this factorization is irreducible.

Remark 2 For the convenience of the reader we state the general definition of "essential" and explain how it is utilized in this paper.

Definition 4.1 of [24]: Suppose $C := (C_k)_{k\in K}$ is a $\#K$-tuple of polytopes in $\mathbb{R}^n$ or a $\#K$-tuple of finite subsets of $\mathbb{R}^n$, where K is a finite set and $\#K$ is the number of elements of K. We will allow any C_k to be empty and say that a nonempty subset $J \subseteq K$ is essential for C (or C has essential subset J) iff $C_j \neq \emptyset$ for all $j \in J$, $\dim(\sum_{j\in J} C_j) = \#J - 1$ and $\dim(\sum_{j\in J'} C_j) \geq \#J'$ for all nonempty proper subsets J' of J. (Note that K is $\{1, \ldots, n\}$ in [24]. We have replaced $\{1, \ldots, n\}$ by K because we want to allow any sets of indices.)

Throughout this paper the sets C_j will be nonempty finite sets, that is, supports of some Laurent polynomials or supersets of their supports. Furthermore, it is easy to see that, for this special case, one can replace $\dim(\sum_{j\in J} C_j)$ in the definition of essential by the rank of $\mathcal{L}((C_j)_{j\in J})$ (as in [25]).

Remark 3 It is important to point out that in a particular degenerate case the definition of the sparse resultant in the main theorem is slightly different from the usual one. For degenerate cases where a strict subset $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ is uniquely essential for $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, we define

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n) := \mathrm{Res}_{\mathcal{A}_{i_1},\ldots,\mathcal{A}_{i_m}}\left(f_{i_1}, \ldots, f_{i_m}\right)^{\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}},$$

where the exponent $\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}$ is defined in the following paragraph, whereas usually one defines

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n) := \mathrm{Res}_{\mathcal{A}_{i_1},\ldots,\mathcal{A}_{i_m}}\left(f_{i_1}, \ldots, f_{i_m}\right).$$

The first definition allows us to handle the degenerate cases in a uniform and elegant way, whereas the second definition seems not to allow this.

In the following, we define the exponent $\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}$, where $\{1, \ldots, n\}$ has a unique (not necessarily strict) subset $\{i_1, \ldots, i_m\}$ essential for $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. If $m = n$ then we define $\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n} := 1$. Otherwise, let L be an integer lattice such that the integer lattice affinely generated by $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is the direct sum, as $\mathbb{Z}$-modules, of L and the integer lattice affinely generated by $\mathcal{A}_{i_1}, \ldots, \mathcal{A}_{i_m}$. Let π denote the projection onto L, which we naturally extend to the Laurent polynomials f_i. Then $\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}$ is defined to be the quotient of the mixed volume of the Newton polytopes of $\pi(f_{i_{m+1}}), \ldots, \pi(f_{i_n})$ and the volume of the fundamental parallelotope of L. It is easy to see that $\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}$ is well defined.

Note that this remark generalizes Remark 4 of [21].

Example 4 We illustrate Theorem 1 and Remark 3. Let

$$\begin{aligned}
f_1 &:= a_{100} + a_{120}\, x_1^2,\\
f_2 &:= a_{200} + a_{220}\, x_1^2 + a_{201}\, x_2 + a_{221}\, x_1^2\, x_2,\\
f_3 &:= a_{300} + a_{340}\, x_1^4 + a_{321}\, x_1^2\, x_2 + a_{302}\, x_2^2
\end{aligned}$$

and let

$$\tilde{\mathcal{A}}_1 \;:=\; \{(0,0),\ (2,0),\ (5,0)\}.$$

Observe that $n = 3$,

$$\begin{aligned}
&\mathcal{A}_1 = \{(0,0),\ (2,0)\}\,,\\
&\mathcal{A}_2 = \{(0,0),\ (2,0),\ (0,1),\ (2,1)\},\\
&\mathcal{A}_3 = \{(0,0),\ (4,0),\ (2,1),\ (0,2)\},\\
&[\mathcal{L}(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \mathcal{A}_3) : \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)] = 2,\\
&\mathrm{e}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3} = 1 \text{ (because } \{1,2,3\} \text{ is essential for } (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_2 + \mathcal{A}_3 = \{&(0,0),\ (4,0),\ (2,1),\ (0,2),\\
&(2,0),\ (6,0),\ (4,1),\ (2,2),\\
&(0,1),\ (4,1),\ (2,2),\ (0,3),\\
&(2,1),\ (6,1),\ (4,2),\ (2,3)\}.
\end{aligned}$$

The convex hull of $\mathcal{A}_2 + \mathcal{A}_3$ is shown in figure 1. It has five facets (edges) with primitive inward normal vectors

$$\begin{aligned}
\omega_1 &= (0,1),\\
\omega_2 &= (0,-1),\\
\omega_3 &= (1,0),\\
\omega_4 &= (-1,0),\\
\omega_5 &= (-1,-2).
\end{aligned}$$

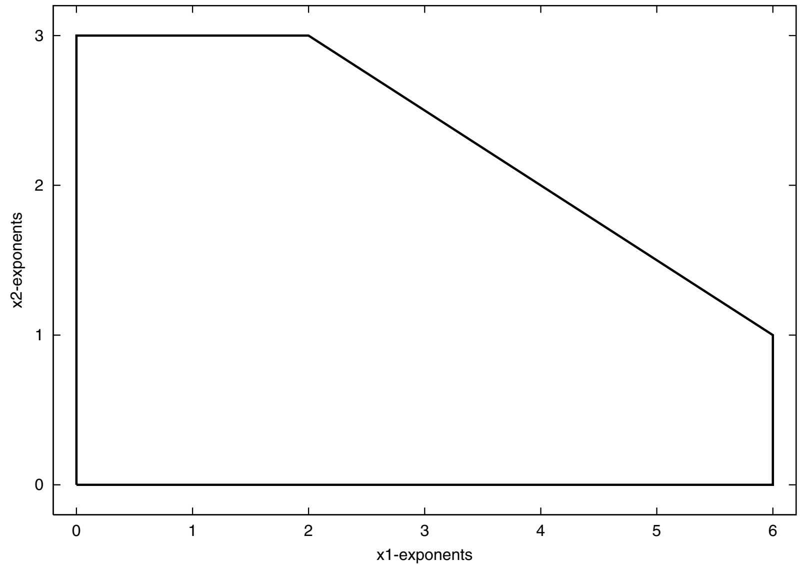

Figure 1. Convex hull of $\mathcal{A}_2 + \mathcal{A}_3$.

Observe that

$$
\begin{aligned}
&\mathrm{a}_{\tilde{\mathcal{A}}_1}(\omega_1) = 0, && \mathrm{a}_{\mathcal{A}_1}(\omega_1) = 0,\\
&\mathrm{a}_{\tilde{\mathcal{A}}_1}(\omega_2) = 0, && \mathrm{a}_{\mathcal{A}_1}(\omega_2) = 0,\\
&\mathrm{a}_{\tilde{\mathcal{A}}_1}(\omega_3) = 0, && \mathrm{a}_{\mathcal{A}_1}(\omega_3) = 0,\\
&\mathrm{a}_{\tilde{\mathcal{A}}_1}(\omega_4) = 5, && \mathrm{a}_{\mathcal{A}_1}(\omega_4) = 2,\\
&\mathrm{a}_{\tilde{\mathcal{A}}_1}(\omega_5) = 5, && \mathrm{a}_{\mathcal{A}_1}(\omega_5) = 2,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_2^{\omega_4} &= \{(2,0),\ (2,1)\},\\
\mathcal{A}_3^{\omega_4} &= \{(4,0)\},\\
\mathcal{A}_2^{\omega_5} &= \{(2,1)\},\\
\mathcal{A}_3^{\omega_5} &= \{(4,0),\ (2,1),\ (0,2)\},
\end{aligned}
$$

Furthermore observe that $\mathrm{e}_{\mathcal{A}_2^{\omega_4},\mathcal{A}_3^{\omega_4}} = 1$, $\mathrm{e}_{\mathcal{A}_2^{\omega_5},\mathcal{A}_3^{\omega_5}} = 2$. In order to compute $\mathrm{e}_{\mathcal{A}_2^{\omega_4},\mathcal{A}_3^{\omega_4}}$ and $\mathrm{e}_{\mathcal{A}_2^{\omega_5},\mathcal{A}_3^{\omega_5}}$ one proceeds very similarly. For the convenience of the reader we describe in derail how to compute $\mathrm{e}_{\mathcal{A}_2^{\omega_5},\mathcal{A}_3^{\omega_5}}$: The subset of $\{2, 3\}$ essential for $\{\mathcal{A}_2^{\omega_5}, \mathcal{A}_3^{\omega_5}\}$ is $\{2\}$, $\mathcal{L}(\mathcal{A}_2^{\omega_5}) = \{0\}$ and $\mathcal{L}(\mathcal{A}_2^{\omega_5}, \mathcal{A}_3^{\omega_5}) = \mathbb{Z}$. Therefore $L = \mathbb{Z}$ and $\mathcal{L}(\mathcal{A}_2^{\omega_5}, \mathcal{A}_3^{\omega_5})$ will be decomposed as $\mathbb{Z} \oplus \{0\}$. Therefore we let π map $f_3^{\omega_5}$ to $a_{340}\,x^2 + a_{321}\,x + a_{320}$ which implies that the mixed volume of $\pi(f_3^{\omega_5})$ is 2. Since the volume of the fundamental parallelotope of $\mathbb{Z}$ is 1, we get $\mathrm{e}_{\mathcal{A}_2^{\omega_5},\mathcal{A}_3^{\omega_5}} = \frac{2}{1} = 2$.

Finally observe that

$$\left[\mathrm{H}^{\omega_4} : \mathcal{L}\left(\mathcal{A}_2^{\omega_4}, \mathcal{A}_3^{\omega_4}\right)\right] = 1,$$
$$\left[\mathrm{H}^{\omega_5} : \mathcal{L}\left(\mathcal{A}_2^{\omega_5}, \mathcal{A}_3^{\omega_5}\right)\right] = 1$$

and

$$[\mathbb{Z}^2 : \mathcal{L}(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \mathcal{A}_3)] = 1.$$

Thus

$$\begin{aligned}\mathrm{Res}_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\mathcal{A}_3}(f_1, f_2, f_3) = {} & \mathrm{Res}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3}(f_2, f_2, f_3)^2 \\ & \times \mathrm{Res}_{\mathcal{A}_2^{\omega_3},\mathcal{A}_3^{\omega_3}}\left(f_2^{\omega_3}, f_3^{\omega_3}\right)^{(5-2)\cdot 1} \\ & \times \mathrm{Res}_{\mathcal{A}_2^{\omega_4},\mathcal{A}_3^{\omega_4}}\left(f_2^{\omega_4}, f_3^{\omega_4}\right)^{(5-2)\cdot 1}.\end{aligned}$$

In the following corollary we prove a formula for the sparse resultant under powering of variables. This corollary generalizes a theorem for Dixon resultant, shown by Kapur and Saxena (cf. [17]) using different techniques.

Corollary 5 *Let $\tilde{f}_i$ be obtained from f_i by replacing the variable x_j by $x_j^{d_j}$, where $d_j \in \mathbb{Z}$, for $j = 1, \ldots, n-1$, and let $\tilde{\mathcal{A}}_i$ be the set of all integer points contained in the Newton polytope of $\tilde{f}_i$. Then*

$$\mathrm{Res}_{\tilde{\mathcal{A}}_1,\ldots,\tilde{\mathcal{A}}_n}(\tilde{f}_1, \ldots, \tilde{f}_n) = \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n)^{|d_1\cdots d_{n-1}|\,[\mathcal{L}(\tilde{\mathcal{A}}_1,\ldots,\tilde{\mathcal{A}}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}.$$

Example 6 Let

$$f_1 := a_{100} + a_{124}\, x_1^2\, x_2^4,$$
$$f_2 := a_{200} + a_{266}\, x_1^6\, x_2^6,$$
$$f_3 := a_{300} + a_{342}\, x_1^4\, x_2^2$$

and $\tilde{f}_i$ be obtained from f_i by replacing x_1 by x_1^2 and x_2 by x_2^3.

Observe that $d_1 = 2,\ d_2 = 3$ and

$$\tilde{f}_1 = a_{100} + a_{124}\, x_1^4\, x_2^{12},$$
$$\tilde{f}_2 = a_{200} + a_{266}\, x_1^{12}\, x_2^{18},$$
$$\tilde{f}_3 = a_{300} + a_{342}\, x_1^8\, x_2^6.$$

Furthermore, observe that

$$\mathcal{A}_1 = \{(0, 0), (2, 4)\},$$
$$\mathcal{A}_2 = \{(0, 0), (6, 6)\},$$

$$\begin{aligned}
\mathcal{A}_3 &= \{(0,0),(4,2)\},\\
\tilde{\mathcal{A}}_1 &= \{(0,0),\ (1,3),\ (2,6),\ (4,12)\},\\
\tilde{\mathcal{A}}_2 &= \{(0,0),\ (2,3),\ (4,6),\ (6,9),\ (8,12),\ (10,15),\ (12,18)\},\\
\tilde{\mathcal{A}}_3 &= \{(0,0),\ (4,3),\ (8,6)\},
\end{aligned}$$

$\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ is spanned by $\{(2,4),\ (4,2)\}$ and that $\mathcal{L}(\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \tilde{\mathcal{A}}_3)$ is spanned by $\{(1,3),$ $(2,3)\}$. Thus the fundamental simplex of $\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ has volume (area) 6 and the fundamental simplex of $\mathcal{L}(\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \tilde{\mathcal{A}}_3)$ has volume (area) $\frac{3}{2}$ and therefore

$$[\mathcal{L}(\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \tilde{\mathcal{A}}_3) : \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)] = 4.$$

Thus

$$\mathrm{Res}_{\tilde{\mathcal{A}}_1,\tilde{\mathcal{A}}_2,\tilde{\mathcal{A}}_3}(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = \mathrm{Res}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3}(f_1, f_2, f_3)^{2\cdot 3\cdot 4}.$$

Proof (Corollary 5): Let $\mathcal{B}_i$ be the support of $\tilde{f}_i$. Since the convex hull of $\mathcal{B}_i$ equals the convex hull of $\tilde{\mathcal{A}}_i$, we have by Theorem 1

$$\mathrm{Res}_{\tilde{\mathcal{A}}_1,\ldots,\tilde{\mathcal{A}}_n}(\tilde{f}_1, \ldots, \tilde{f}_n) = \mathrm{Res}_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(\tilde{f}_1, \ldots, \tilde{f}_n)^P,$$

where P is

$$\begin{gathered}
[\mathcal{L}(\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \ldots, \tilde{\mathcal{A}}_n) : \mathcal{L}(\mathcal{B}_1, \tilde{\mathcal{A}}_2, \ldots, \tilde{\mathcal{A}}_n)]\\
[\mathcal{L}(\mathcal{B}_1, \tilde{\mathcal{A}}_2, \ldots, \tilde{\mathcal{A}}_n) : \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2, \tilde{\mathcal{A}}_3, \ldots, \tilde{\mathcal{A}}_n)]\\
\cdots\\
[\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}, \tilde{\mathcal{A}}_n) : \mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}, \mathcal{B}_n)].
\end{gathered}$$

Thus

$$P = [\mathcal{L}(\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_n) : \mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)].$$

By the construction of $\tilde{f}_i$, we have $\mathcal{B}_i = D\mathcal{A}_i$, where D is a diagonal matrix with diagonal entries $d_1, \ldots, d_{n-1}$. Therefore $w = |d_1 \cdots d_n| v$, where w and v, resp., is the volume of the fundamental simplex of $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ and $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, resp. Let $\tilde{v}$ be the volume of the fundamental simplex of $\mathcal{L}(\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_n)$. Then

$$\begin{aligned}
P &= \frac{w}{\tilde{v}} = \frac{|d_1 \cdots d_n| v}{\tilde{v}}\\
&= |d_1 \cdots d_n|\ [\mathcal{L}(\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_n) : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)].
\end{aligned}$$

Finally, note that

$$\mathrm{Res}_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(\tilde{f}_1, \ldots, \tilde{f}_n) = \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n).$$

Thus we have shown the corollary. □

$$\begin{tikzcd}[column sep=tiny]
\text{Theorem 8 ([23])} \arrow[dr] & & \text{Lemma 10} \arrow[dl] & & & \\
& \text{Lemma 11} \arrow[dr] & & \text{Lemma 12} \arrow[dl] & & \\
& & \text{Lemma 13} \arrow[dr] & & \text{Lemma 14} \arrow[dl] & \\
& & & \text{Theorem 1 (Main theorem)} & &
\end{tikzcd}$$

Figure 2. Dependency of the lemmas.

3. Proof of the main theorem

Before going into the details of the proof we describe its main structure. The proof is based on some generalization the Pedersen-Sturmfels product (cf. [23]). For the convenience of the reader we state this formula first (cf. Theorem 8 and Remark 9). In the following lemmas we generalize this product formula and then we prove the main theorem. The dependency of Theorem 1 on the lemmas and on the Pedersen-Sturmfels product is shown in figure 2.

Before listing the lemmas, we fix some notations.

Notation 7 We let

1. $\operatorname{sign}(r)$ denote the "sign" of a real number r, more precisely, $\operatorname{sign}(r) = -1$ if $r < 0$, $\operatorname{sign}(r) = 0$ if $r = 0$ and $\operatorname{sign}(r) = 1$ if $r > 0$.
2. $\operatorname{CH}(\mathcal{A}) \subset \mathbb{R}^{n-1}$ denote the convex hull of a bounded set $\mathcal{A} \subset \mathbb{Z}^{n-1}$.
3. $\operatorname{Vol}(P)$ denote the volume of some polytope P.
4. $\operatorname{Vol}_L(P)$ denote the normalized volume of some polytope P (not necessarily an L-lattice polytope), that is, the quotient between the volume of P and the volume of the fundamental simplex of the integer lattice L.
5. $\prod_\gamma f(\gamma)$, as in [23], denote the product, over the common roots γ with respect to some lattice of certain Laurent polynomials, of f evaluated at γ.

We state the Pedersen-Sturmfels product.

Theorem 8 ([23]) *If $\{1, \ldots, n\}$ is essential for $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and furthermore $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \mathbb{Z}^{n-1}$, then*

$$\operatorname{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n) = \prod_\gamma f_1(\gamma) \prod_\omega \operatorname{Res}_{\mathcal{A}_2^\omega,\ldots,\mathcal{A}_n^\omega}\left(f_2^\omega, \ldots, f_n^\omega\right)^{\rho_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)},$$

where

$$\rho_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega) := \operatorname{sign}\left(\mathrm{a}_{\mathcal{A}_1}^{(\omega)}\right) \frac{\operatorname{Vol}_{\mathbb{Z}^{n-1}}\left(\operatorname{CH}\left(\mathcal{A}_1^\omega \cup \{0\}\right)\right)}{\operatorname{Vol}_{\mathcal{L}((\mathcal{A}_2+\cdots+\mathcal{A}_n)^\omega)}\left(\operatorname{CH}(\mathcal{A}_1)^\omega\right)},$$

γ ranges over the common zeros in $(K \setminus \{0\})^{n-1}$, with respect to the lattice $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, of $f_2, \ldots, f_n$, where K is the algebraic closure of the field generated by the complex

numbers and the symbolic coefficients of the f_i's, and ω ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{A}_2 + \cdots + \mathcal{A}_n$.

Remark 9 Firstly, note that in [23] the Pedersen-Sturmfels product did not consider the degenerate case where a strict subset of $\{2, \ldots, n\}$ is essential for $(\mathcal{A}_2^{\omega}, \ldots, \mathcal{A}_n^{\omega})$. However, it can be seen easily that the Pedersen-Sturmfels product also holds for these degenerate cases if we utilize the alternative definition of the sparse resultant given in Remark 3. One can adjust the proof of Theorem 1.1 of [23] in order to handle these cases. That is, one can easily show, similarly to the proof of Formula (6) of the present paper, a version of Proposition 7.1 of [23] for the alternatively defined sparse resultant. The rest of the proof of Theorem 1.1 of [23] remains unchanged and the version, given in Theorem 8 of the present paper, of the Pedersen-Sturmfels product follows.

Secondly, note that the presentation of the exponent $\rho_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)$ in Theorem 8 of the present paper is slightly different from the presentation in [23]. From the proof of Lemma 2.2 of [23] one can easily see that both presentations are equivalent. We chose this alternative presentation because it is more suitable for this paper.

Now we are ready to state the lemmas.

In the following lemma we study a generalized version $\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)$ of the exponent $\rho_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)$ of Pedersen's and Sturmfels' Theorem 8.

Lemma 10 *Let $\mathcal{B}_1, \ldots, \mathcal{B}_n \subset \mathbb{Z}^{n-1}$ be finite sets and furthermore let the map M : $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \to \mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ be a $\mathbb{Z}$-lattice isomorphism such that $\mathcal{B}_i = M(\mathcal{A}_i)$. Then*

$$\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega) = \delta_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(\nu),$$

where ω is a positive multiple of $M^{\mathrm{T}}(\nu)$, where M^{T} is the transpose of M, viewed as a $\mathbb{Q}$-linear map, and

$$\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega) := \operatorname{sign}\left(\mathrm{a}_{\mathcal{A}_1}^{(\omega)}\right) \frac{\mathrm{Vol}_{\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)}\left(\mathrm{CH}\left(\mathcal{A}_1^{\omega} \cup \{0\}\right)\right)}{\mathrm{Vol}_{\mathcal{L}((\mathcal{A}_2+\cdots+\mathcal{A}_n)^{\omega})}\left(\mathrm{CH}\left(\mathcal{A}_1\right)^{\omega}\right)}.$$

Proof: For $n = 1$, the lemma is trivial, so assume $n \geq 2$.

Let us first show that $M^{-1}(\mathrm{CH}\,(\mathcal{B}_1)^{\nu})$ is a face of $\mathrm{CH}\,(\mathcal{A}_1)$ with primitive inward normal vector ω that is a positive multiple of $M^{\mathrm{T}}(\nu)$. Firstly "$\subseteq$": Let $\langle \nu,\ y\rangle \geq -\mathrm{a}_{\mathcal{B}_1}(\nu)$ be an inequality defining a halfspace, with primitive inward normal vector $\nu \neq 0$, that supports the convex hull of $\mathcal{B}_1$. The inequality $\langle M^{\mathrm{T}}(\nu), x\rangle \geq -\mathrm{a}_{\mathcal{B}_1}(\nu)$ defines a halfspace with normal vector $M^{\mathrm{T}}(\nu) \neq 0$. By definition $M^{\mathrm{T}}(\nu)$ is an *inward* normal vector of this half space and the primitive inward normal vector ω is a positive multiple of $M^{\mathrm{T}}(\nu)$. Since $\langle \nu,\ y\rangle = \langle M^{\mathrm{T}}(\nu), M^{-1}(y)\rangle$ and $\mathrm{CH}\,(\mathcal{A}_1) = \mathrm{CH}(M^{-1}(\mathcal{B}_1)) = M^{-1}(\mathrm{CH}\,(\mathcal{B}_1))$, this halfspace contains $\mathrm{CH}\,(\mathcal{A}_1)$ and, since the points $M^{-1}(\mathrm{CH}\,(\mathcal{B}_1)^{\nu}) \subseteq \mathrm{CH}\,(\mathcal{A}_1)$ satisfy the equality, this halfspace supports $\mathrm{CH}\,(\mathcal{A}_1)$. Secondly "$\supseteq$": Take $x \in \mathrm{CH}\,(\mathcal{A}_1)$ such that $\langle M^{\mathrm{T}}(\nu), x\rangle = -\mathrm{a}_{\mathcal{B}_1}(\nu)$ and $M(x) \notin \mathrm{CH}\,(\mathcal{B}_1)^{\nu}$. Then $M(x)$ is contained in $M(\mathrm{CH}\,(\mathcal{A}_1)) = \mathrm{CH}\,(M(\mathcal{A}_1)) = \mathrm{CH}\,(\mathcal{B}_1)$ and $\langle \nu,\ M(x)\rangle = -\mathrm{a}_{\mathcal{B}_1}(\nu)$. Contradiction!

Next observe that the previous paragraph implies that

$$\operatorname{sign}(\mathrm{a}_{\mathcal{B}_1}(\nu)) = \operatorname{sign}(\mathrm{a}_{\mathcal{A}_1}^{(\omega)})$$

because $\mathrm{a}_{\mathcal{A}_1}^{(\omega)}$ is a certain positive multiple of $\mathrm{a}_{\mathcal{B}_1}(\nu)$.

Next we show that

$$\mathrm{Vol}_{\mathcal{L}(\mathcal{B}_1,\ldots,\mathcal{B}_n)}\left(\mathrm{CH}\left(\mathcal{B}_1^{\nu} \cup \{0\}\right)\right) = \mathrm{Vol}_{\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)}\left(\mathrm{CH}\left(\mathcal{A}_1^{\omega} \cup \{0\}\right)\right).$$

Let B be a basis for the lattice $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Since the mapping $M : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \to \mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ is a lattice isomorphism, $M(B)$ is a basis for $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$. Furthermore, let $\Delta_{\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)}$ and $\Delta_{\mathcal{L}(\mathcal{B}_1,\ldots,\mathcal{B}_n)}$, resp., denote the fundamental lattice simplex spanned by B and $M(B)$, resp. Then we have $\Delta_{\mathcal{L}(\mathcal{B}_1,\ldots,\mathcal{B}_n)} = M(\Delta_{\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)})$ and thus

$$\begin{aligned}\mathrm{Vol}_{\mathcal{L}(\mathcal{B}_1,\ldots,\mathcal{B}_n)}\left(\mathrm{CH}\left(\mathcal{B}_1^{\nu} \cup \{0\}\right)\right) &= \frac{\mathrm{Vol}\left(\mathrm{CH}\left(\mathcal{B}_1^{\nu} \cup \{0\}\right)\right)}{\mathrm{Vol}\left(\Delta_{\mathcal{L}(\mathcal{B}_1,\ldots,\mathcal{B}_n)}\right)} \\ &= \frac{\mathrm{Vol}\left(\mathrm{CH}\left(M(\mathcal{A}_1)^{\nu} \cup \{0\}\right)\right)}{\mathrm{Vol}\left(M(\Delta_{\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)})\right)}.\end{aligned}$$

Since

$$\mathrm{CH}\left(M(\mathcal{A}_1^{\omega}) \cup \{0\}\right) = M(\mathrm{CH}\left(\mathcal{A}_1^{\omega} \cup \{0\}\right)),$$

for some ω, we have by the substitution rule of integration

$$\mathrm{Vol}_{\mathcal{L}(\mathcal{B}_1,\ldots,\mathcal{B}_n)}\left(\mathrm{CH}\left(\mathcal{B}_1^{\nu} \cup \{0\}\right)\right) = \frac{\mathrm{Vol}\left(\mathrm{CH}\left(\mathcal{A}_1^{\omega} \cup \{0\}\right)\right)}{\mathrm{Vol}\left(\Delta_{\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)}\right)}.$$

Finally we show that

$$\mathrm{Vol}_{\mathcal{L}((\mathcal{B}_2+\cdots+\mathcal{B}_n)^{\nu})}\left(\mathrm{CH}\left(\mathcal{B}_1\right)^{\nu}\right) = \mathrm{Vol}_{\mathcal{L}((\mathcal{A}_2+\cdots+\mathcal{A}_n)^{\omega})}\left(\mathrm{CH}\left(\mathcal{A}_1\right)^{\omega}\right).$$

We have already seen that $\mathrm{CH}(\mathcal{B}_1)^{\nu} = M(\mathrm{CH}(\mathcal{A}_1)^{\omega})$. Furthermore, we view the lattice $\mathcal{L}((\mathcal{A}_2 + \cdots + \mathcal{A}_n)^{\omega})$ and $\mathcal{L}((\mathcal{B}_2 + \cdots + \mathcal{B}_n)^{\nu})$, resp., as sublattices of $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$, resp. Then

$$M : \mathcal{L}((\mathcal{A}_2 + \cdots + \mathcal{A}_n)^{\omega}) \to \mathcal{L}((\mathcal{B}_2 + \cdots + \mathcal{B}_n)^{\nu})$$

is a affine lattice isomorphism and thus

$$\Delta_{\mathcal{L}((\mathcal{B}_2+\cdots+\mathcal{B}_n)^{\nu})} = M\left(\Delta_{\mathcal{L}((\mathcal{A}_2+\cdots+\mathcal{A}_n)^{\omega})}\right),$$

where $\Delta_{\mathcal{L}((\mathcal{B}_2+\cdots+\mathcal{B}_n)^{\nu})}$ and $\Delta_{\mathcal{L}((\mathcal{A}_2+\cdots+\mathcal{A}_n)^{\omega})}$, are the fundamental lattice simplices spanned by appropriate, similar to above, bases of the integer lattices $\mathcal{L}((\mathcal{B}_2 + \cdots + \mathcal{B}_n)^{\nu})$ and

$\mathcal{L}((\mathcal{A}_2 + \cdots + \mathcal{A}_n)^\omega)$. Since the map M restricted to the hyperplane with normal vector ω containing $\mathcal{L}((\mathcal{A}_2 + \cdots + \mathcal{A}_n)^\omega)$ is obviously injective, we have by the substitution rule of integration

$$\begin{aligned}\mathrm{Vol}_{\mathcal{L}((\mathcal{B}_2+\cdots+\mathcal{B}_n)^\nu)}(\mathrm{CH}\,(\mathcal{B}_1)^\nu) &= \frac{\mathrm{Vol}\,(\mathrm{CH}\,(\mathcal{B}_1)^\nu)}{\mathrm{Vol}\left(\Delta_{\mathcal{L}((\mathcal{B}_2+\cdots+\mathcal{B}_n)^\nu)}\right)}\\ &= \frac{\mathrm{Vol}\,(M(\mathrm{CH}\,(\mathcal{A}_1)^\omega))}{\mathrm{Vol}\left(M(\Delta_{\mathcal{L}((\mathcal{A}_2+\cdots+\mathcal{A}_n)^\omega)})\right)}\\ &= \frac{\mathrm{Vol}\,(\mathrm{CH}\,(\mathcal{A}_1)^\omega)}{\mathrm{Vol}\left(\Delta_{\mathcal{L}((\mathcal{A}_2+\cdots+\mathcal{A}_n)^\omega)}\right)}.\end{aligned}$$

Thus we have shown the lemma. □

Essentially, the following lemma contains the Poisson-type product formula for sparse resultant shown by Pedersen and Sturmfels. In [23] they show a formula assuming that the lattice generated by the supports of $f_1, \ldots, f_n$ is $\mathbb{Z}^{n-1}$. We remove this assumption.

Lemma 11 *If $\{1, \ldots, n\}$ is essential for $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, then*

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n) = \prod_\gamma f_1(\gamma) \prod_\omega \mathrm{Res}_{\mathcal{A}_2^\omega,\ldots,\mathcal{A}_n^\omega}\left(f_2^\omega, \ldots, f_n^\omega\right)^{\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)},$$

where γ ranges over the common zeros in $(K \setminus \{0\})^{n-1}$, with respect to the lattice $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, of $f_2, \ldots, f_n$, where K is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of the f_i's, δ is as defined in Lemma 10 *and ω ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{A}_2 + \cdots + \mathcal{A}_n$.*

Proof: Note that, since $\{1, \ldots, n\}$ is essential for $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, we have that $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a sublattice of $\mathbb{Z}^{n-1}$ of rank $n-1$. By mapping a basis of $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ onto the canonical basis of $\mathbb{Z}^{n-1}$ we construct a lattice isomorphism M from $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ to $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$, where $\mathcal{B}_i := M(\mathcal{A}_i)$. Furthermore we canonically extend M to Laurent polynomials with support in $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and let g_i stand for the image of f_i under M.

Note that $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n) = \mathrm{Res}_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(g_1, \ldots, g_n)$.

Furthermore, by the Poisson-type product formula of [23] (cf. Theorem 8), we have

$$\mathrm{Res}_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(g_1, \ldots, g_n) = \prod_\beta g_1(\beta) \prod_\nu \mathrm{Res}_{\mathcal{B}_2^\nu,\ldots,\mathcal{B}_n^\nu}\left(g_2^\nu, \ldots, g_n^\nu\right)^{\delta_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(\nu)},$$

where β ranges over the common zeros in $(K \setminus \{0\})^{n-1}$ of $g_2, \ldots, g_n$ with respect to $\mathcal{L}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$, where K is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of the g_i's and ν ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{B}_2 + \cdots + \mathcal{B}_n$. Since M is invertible and by Lemma 10, we have

$$\mathrm{Res}_{\mathcal{B}_1,\ldots,\mathcal{B}_n}(g_1,\ldots,g_n)=\prod_{\beta} g_1(\beta)\prod_{\omega}\mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\left(f_2^{\omega},\ldots,f_n^{\omega}\right)^{\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)},$$

where ω ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{A}_2+\cdots+\mathcal{A}_n$. Now, observe that by the construction (cf. [23]) of $\prod_{\beta} g_1(\beta)$, we have

$$\prod_{\beta} g_1(\beta)=\prod_{\gamma} f_1(\gamma),$$

where γ ranges over the common zeros in $(K\setminus\{0\})^{n-1}$ of $f_2,\ldots,f_n$ with respect to $\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$. Thus we have shown the lemma. $\square$

Next we rewrite the exponent $\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)$.

Lemma 12

$$\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)=\frac{\mathrm{a}_{\mathcal{A}_1}^{(\omega)}\left[\mathrm{H}^{\omega}:\mathcal{L}\left(\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}\right)\right]}{[\mathbb{Z}^{n-1}:\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]},$$

where $\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)$ is defined in Lemma 10.

Proof: Note that

$$\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)=\mathrm{sign}\left(\mathrm{a}_{\mathcal{A}_1}^{(\omega)}\right)\frac{\frac{\mathrm{Vol}\left(\mathrm{CH}\left(\mathcal{A}_1^{\omega}\cup\{0\}\right)\right)}{v}}{\frac{\mathrm{Vol}(\mathrm{CH}(\mathcal{A}_1)^{\omega})}{v^{\omega}}},$$

where v and v^{ω}, resp., is the volume of the fundamental simplex of the lattice generated by $\mathcal{A}_1,\ldots,\mathcal{A}_n$ and $(\mathcal{A}_2+\cdots+\mathcal{A}_n)^{\omega}$, resp. Note that

$$\mathrm{Vol}\left(\mathrm{CH}\left(\mathcal{A}_1^{\omega}\cup\{0\}\right)\right)=\frac{\mathrm{Vol}\left(\mathrm{CH}\left(\mathcal{A}_1\right)^{\omega}\right)\,d^{\omega}}{n-1},$$

where d^{ω} is the distance of the origin from the hyperplane supporting the convex hull $\mathrm{CH}\left(\mathcal{A}_1\right)^{\omega}$. Thus

$$\begin{aligned}\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)&=\mathrm{sign}\left(\mathrm{a}_{\mathcal{A}_1}^{(\omega)}\right)d^{\omega}v^{\omega}\frac{1}{(n-1)v}\\&=\mathrm{sign}\left(\mathrm{a}_{\mathcal{A}_1}^{(\omega)}\right)d^{\omega}(n-2)!\,h^{\omega}\frac{v^{\omega}}{h^{\omega}}\frac{1}{(n-1)!\,v}\\&=\frac{\mathrm{sign}\left(\mathrm{a}_{\mathcal{A}_1}^{(\omega)}\right)d^{\omega}(n-2)!\,h^{\omega}}{[\mathbb{Z}^{n-1}:\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}\frac{v^{\omega}}{h^{\omega}},\end{aligned}$$

where h^ω is the volume of the fundamental simplex of the lattice of all integer points contained in the hyperplane, passing through the origin, with normal vector ω.

Now, by [8, p. 319], we have $\|\omega\| = (n-2)!\, h^\omega$, where $\|\omega\|$ stands for the Euclidean norm of ω. In detail: Cox, Little and O'Shea state, in the second-to-last formula on p. 319 of [8], the non-trivial fact that the volume of the fundamental parallelotope of an $(n-1)$-dimensional sublattice of $\mathbb{Z}^n$ equals the Euclidean length of the (unique up to sign) primitive normal vector of this sublattice. By replacing n by $n-1$, by considering that ω is assumed to be primitive as in Lemma 10 and since the volume of the fundamental parallelotope of the lattice of all integer points contained in the hyperplane, passing through the origin, with normal vector ω, is $(n-2)!$ times the volume of its fundamental simplex h^ω, the formula for $\|\omega\|$ follows.

Furthermore, it is easy to see that we have $\mathrm{sign}(\mathrm{a}_{\mathcal{A}_1}^{(\omega)}) d^\omega \, \|\omega\| = \mathrm{a}_{\mathcal{A}_1}^{(\omega)}$ and thus we have shown the lemma. □

Now we further generalize the Poisson-type product formula of Lemma 11. In the following lemma the set $\{1, \ldots, n\}$ is not necessarily the unique subset of $\{1, \ldots, n\}$ essential for $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

Lemma 13 *If the index* 1 *is contained in the unique subset of* $\{1, \ldots, n\}$ *essential for* $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, *then*

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n) = \prod_\gamma f_1(\gamma) \prod_\omega \mathrm{Res}_{\mathcal{A}_2^\omega,\ldots,\mathcal{A}_n^\omega}\big(f_2^\omega, \ldots, f_n^\omega\big)^{\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)},$$

where γ ranges over the common zeros in $(K \setminus \{0\})^{n-1}$, with respect to the lattice $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, of $f_2, \ldots, f_n$, where K is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of the f_i's, $\delta_{\ldots}(\omega)$ is as defined in Lemma 10 *and ω ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{A}_2 + \cdots + \mathcal{A}_n$.*

Proof: If $\{1, \ldots, n\}$ is the unique subset of $\{1, \ldots, n\}$ essential for the tuple $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ then the formula holds by Lemma 11.

Suppose, without loss of generality, $\{1, \ldots, k\}$ is the unique subset of $\{1, \ldots, n\}$ essential for $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Furthermore let $\mathcal{B}$ be the set of vertices of the standard simplex of $\mathbb{R}^{n-1}$ and g be a polynomial with distinct symbolic coefficients, distinct from all the other symbolic coefficients in this paper, with support $\mathcal{B}$. The overall strategy of the proof is as follows. We factorize

$$\mathrm{Res}_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(f_1\, g, f_2, \ldots, f_n)$$

in two different ways. One factorization (Step 1, Formula 4) is the right hand side of the lemma raised to some power times some factor and the second factorization (Step 2, Formula 6) is the left hand side of the lemma raised by the same power times the same factor. Thus the lemma follows up to some factor that is a certain root of unity (Step 3). Then we show that this root of unity is one (Step 4).

Now we carry out this strategy:

Step 1: Note that $f_1\, g$ has support $\mathcal{A}_1 + \mathcal{B}$. Furthermore note that the Newton polytope of $f_1\, g$ is $(n-1)$-dimensional and therefore $\{1, \ldots, n\}$ is essential for $(C_1, \ldots, C_n) = (\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)$. By Lemma 11 we have

$$\begin{aligned}&\operatorname{Res}_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(f_1\, g, f_2, \ldots, f_n)\\ &\quad = \prod_{\beta} f_1(\beta)\, g(\beta) \prod_{\omega} \operatorname{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\big(f_2^{\omega}, \ldots, f_n^{\omega}\big)^{\delta_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)}\end{aligned}$$

where β ranges over the common zeros in $(K\setminus\{0\})^{n-1}$, with respect to the lattice $\mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)$, of $f_2, \ldots, f_n$, where K is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of g and the f_i's, δ is as defined in Lemma 10 and ω ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{A}_2 + \cdots + \mathcal{A}_n$. Note that by Exercise 3, p. 318, of [8] the Newton polytope of $f_1\, g$ equals $\mathrm{CH}(\mathcal{A}_1) + \mathrm{CH}(\mathcal{B})$ and thus by Exercise 12, p. 325, of [8], we have $\mathrm{a}^{(\omega)}_{\mathcal{A}_1+\mathcal{B}} = \mathrm{a}^{(\omega)}_{\mathcal{A}_1} + \mathrm{a}^{(\omega)}_{\mathcal{B}}$ and therefore, by Lemma 12,

$$\begin{aligned}&\delta_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)\\ &\quad = \frac{\mathrm{a}^{(\omega)}_{\mathcal{A}_1+\mathcal{B}}\big[\mathrm{H}^{\omega} : \mathcal{L}\big(\mathcal{A}_2^{\omega}, \ldots, \mathcal{A}_n^{\omega}\big)\big]}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}\\ &\quad = \frac{\mathrm{a}^{(\omega)}_{\mathcal{A}_1}\big[\mathrm{H}^{\omega} : \mathcal{L}\big(\mathcal{A}_2^{\omega}, \ldots, \mathcal{A}_n^{\omega}\big)\big]}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]} + \frac{\mathrm{a}^{(\omega)}_{\mathcal{B}}\big[\mathrm{H}^{\omega} : \mathcal{L}\big(\mathcal{A}_2^{\omega}, \ldots, \mathcal{A}_n^{\omega}\big)\big]}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}.\end{aligned}$$

Furthermore

$$\begin{aligned}&\delta_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega) = \delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)[\mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n) : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)]\\ &\quad + \delta_{\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)[\mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n) : \mathcal{L}(\mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)],\end{aligned}$$

because

$$\begin{aligned}&[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]\\ &\quad = \frac{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)]}{[\mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n) : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)]}\\ &\quad = \frac{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}{[\mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n) : \mathcal{L}(\mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}.\end{aligned}$$

Thus, by Lemma 11 and by the construction of $\prod_{\beta} f_1(\beta)\, g(\beta)$ (cf. [23]),

$$\begin{aligned}&\operatorname{Res}_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(f_1\, g, f_2, \ldots, f_n)\\ &\quad = \prod_{\beta} f_1(\beta)\left(\prod_{\omega} \operatorname{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\big(f_2^{\omega}, \ldots, f_n^{\omega}\big)^{\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)\, I_1}\right)\end{aligned}$$

$$\times \prod_{\beta} g(\beta)\left(\prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\left(f_2^{\omega},\ldots,f_n^{\omega}\right)^{\delta_{\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)\,I_2}\right)$$

$$= \prod_{\beta} f_1(\beta)\left(\prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\left(f_2^{\omega},\ldots,f_n^{\omega}\right)^{\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)\,I_1}\right)$$

$$\times \left(\prod_{\beta''} g(\beta'')\prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\left(f_2^{\omega},\ldots,f_n^{\omega}\right)^{\delta_{\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)}\right)^{I_2}$$

$$= \prod_{\beta} f_1(\beta)\left(\prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\left(f_2^{\omega},\ldots,f_n^{\omega}\right)^{\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)\,I_1}\right)$$

$$\times \mathrm{Res}_{\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(g, f_2,\ldots,f_n)^{I_2}, \tag{1}$$

where β'' ranges over the common zeros in $(L\setminus\{0\})^{n-1}$, with respect to the lattice $\mathcal{L}(\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n)$, of $f_2,\ldots,f_n$, where L is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of g and $f_2,\ldots,f_n$ and where

$$I_1 = [\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]$$
$$I_2 = [\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n)].$$

Now we analyze Formula (1) further. Note that, since

$$\prod_{\gamma} f_1(\gamma) \;=\; \left(\prod_{\beta'} f_1(a)\beta'\right)^{\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}}, \tag{2}$$

where β' ranges over the common zeros in $(K'\setminus\{0\})^{n-1}$, with respect to the lattice $\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_k)$, of $f_2,\ldots,f_k$, where K' is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of $f_1,\ldots,f_k$, we have

$$\prod_{\beta} f_1(\beta) = \left(\prod_{\beta'} f_1(\beta')\right)^{\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}\,[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]},$$

$$= \left(\prod_{\gamma} f_1(\gamma)\right)^{[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}. \tag{3}$$

Thus Formula (1) yields the equality

$$\mathrm{Res}_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(f_1\,g, f_2,\ldots,f_n)$$
$$= \left(\prod_{\gamma} f_1(\gamma)\prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}(f_2^{\omega},\ldots,f_n^{\omega})^{\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)}\right)^{[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}$$
$$\times \mathrm{Res}_{\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n}(g, f_2,\ldots,f_n)^{[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n)]}. \tag{4}$$

Step 2: Therefore we conclude from Formula 4 by Lemma 11 that

$$\begin{aligned}
&\mathrm{Res}_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\dots,\mathcal{A}_n}(f_1\, g, f_2, \dots, f_n)\\
&\quad = \mathrm{Res}_{\mathcal{A}_1,\dots,\mathcal{A}_n}(f_1, \dots, f_n)^{[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\dots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\dots,\mathcal{A}_n)]}\\
&\qquad \times \mathrm{Res}_{\mathcal{B},\mathcal{A}_2,\dots,\mathcal{A}_n}(g, f_2, \dots, f_n)^{[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\dots,\mathcal{A}_n):\mathcal{L}(\mathcal{B},\mathcal{A}_2,\dots,\mathcal{A}_n)]} \times Q,
\end{aligned}$$

where Q is a rational function depending only on the coefficients of the Laurent polynomials $f_2, \dots, f_n$. Now we show that Q is 1. First we observe that Q is a polynomial: Obviously $Q = \frac{R}{S}$, where R and S are relatively prime polynomials. Since the left hand side of the previous equality is a polynomial, the denominator S divides either

$$\mathrm{Res}_{\mathcal{A}_1,\dots,\mathcal{A}_n}(f_1, \dots, f_n)$$

or

$$\mathrm{Res}_{\mathcal{B},\mathcal{A}_2,\dots,\mathcal{A}_n}(g, f_2, \dots, f_n).$$

Since these two resultants are irreducible and depend on either f_1 or g, the denominator S, which does not depend on f_1 or g, is a constant. Next we show that the total degree of Q in the coefficients of $f_2, \dots, f_{n-1}$ and f_n, resp., is zero. That is, we show that, for $i \geq 2$,

$$\begin{aligned}
&\frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{A}_1+\mathcal{B})\,, \mathrm{CH}\,(\mathcal{A}_2), \dots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \dots, \mathrm{CH}\,(\mathcal{A}_n))}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1+\mathcal{B}, \mathcal{A}_2, \dots, \mathcal{A}_n)]}\\
&\quad = \frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{M}_1), \dots, \mathrm{CH}\,(\mathcal{M}_{i-1}), \mathrm{CH}\,(\mathcal{M}_{i+1}), \dots, \mathrm{CH}\,(\mathcal{M}_k))\ \mathrm{e}_{\mathcal{A}_1,\dots,\mathcal{A}_n}\ I_1}{[\mathbb{Z}^{k-1} : \mathcal{L}(\mathcal{M}_1, \dots, \mathcal{M}_k)]}\\
&\qquad + \frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{B})\,, \mathrm{CH}\,(\mathcal{A}_2), \dots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \dots, \mathrm{CH}\,(\mathcal{A}_n))\ I_2}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{B}, \mathcal{A}_2, \dots, \mathcal{A}_n)]},
\end{aligned} \tag{5}$$

where $M : \mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_k) \to \mathbb{Z}^{k-1}$ is a lattice embedding, $\mathcal{M}_i$ stands for $M(\mathcal{A}_i)$ and $\mathrm{MV}(\cdot)$ stands for the mixed volume (cf. [8]). Now, equality (5) can be proved as follows: By Exercise 3, p. 318, and by Exercise 12, p. 325, of [8] and by the multilinearity of the mixed volume

$$\begin{aligned}
&\frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{A}_1+\mathcal{B})\,, \mathrm{CH}\,(\mathcal{A}_2), \dots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \dots, \mathrm{CH}\,(\mathcal{A}_n))}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1+\mathcal{B}, \mathcal{A}_2, \dots, \mathcal{A}_n)]}\\
&\quad = \frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{A}_1)\,, \mathrm{CH}\,(\mathcal{A}_2), \dots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \dots, \mathrm{CH}\,(\mathcal{A}_n))}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1+\mathcal{B}, \mathcal{A}_2, \dots, \mathcal{A}_n)]}\\
&\qquad + \frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{B})\,, \mathrm{CH}\,(\mathcal{A}_2), \dots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \dots, \mathrm{CH}\,(\mathcal{A}_n))}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1+\mathcal{B}, \mathcal{A}_2, \dots, \mathcal{A}_n)]}.
\end{aligned}$$

In order to further rewrite this equality we apply Bernshtein's theorem (cf. [8, 12]:

$$\frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{A}_1)\,, \mathrm{CH}\,(\mathcal{A}_2), \dots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \dots, \mathrm{CH}\,(\mathcal{A}_n))}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)]}$$

is the number of roots with respect to the lattice $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of a system of Laurent polynomial equations with symbolic coefficients with supports $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, \ldots, \mathcal{A}_n$, where obviously the roots are defined over the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of the Laurent polynomials. Similarly one can interpret the first summand in the right hand side of Formula (5). Thus we see by the definition of e that

$$\frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{A}_1)\,, \mathrm{CH}\,(\mathcal{A}_2), \ldots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \ldots, \mathrm{CH}\,(\mathcal{A}_n))}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}$$
$$= \frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{M}_1), \ldots, \mathrm{CH}\,(\mathcal{M}_{i-1}), \mathrm{CH}\,(\mathcal{M}_{i+1}), \ldots, \mathrm{CH}\,(\mathcal{M}_k))\ \mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}\ I_1}{[\mathbb{Z}^{k-1} : \mathcal{L}(\mathcal{M}_1, \ldots, \mathcal{M}_k)]}$$

and we also see easily that

$$\frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{B})\,, \mathrm{CH}\,(\mathcal{A}_2), \ldots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \ldots, \mathrm{CH}\,(\mathcal{A}_n))}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}$$
$$= \frac{\mathrm{MV}(\mathrm{CH}\,(\mathcal{B})\,, \mathrm{CH}\,(\mathcal{A}_2), \ldots, \mathrm{CH}\,(\mathcal{A}_{i-1}), \mathrm{CH}\,(\mathcal{A}_{i+1}), \ldots, \mathrm{CH}\,(\mathcal{A}_n))\ I_2}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}.$$

Thus we have shown equality (5) and therefore Q is a constant. Now, by the construction of Q and as we have seen in the beginning of Step 2, we have that Q is a polynomial, obviously, of the form $T_1^{\alpha_1} \cdots T_m^{\alpha_m}$, where m is the number of its distinct polynomial prime factors T_j and the α_j's are non-negative integers. Since Q is also a constant, we have $\alpha_j = 0$ and thus $Q = 1$. Thus we have shown that

$$\begin{aligned}&\mathrm{Res}_{\mathcal{A}_1+\mathcal{B},\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1\, g, f_2, \ldots, f_n)\\ &\quad = \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n)^{[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}\\ &\qquad \mathrm{Res}_{\mathcal{B},\mathcal{A}_1,\ldots,\mathcal{A}_n}(g, f_2, \ldots, f_n)^{[\mathcal{L}(\mathcal{A}_1+\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{B},\mathcal{A}_2,\ldots,\mathcal{A}_n)]}.\end{aligned} \tag{6}$$

Step 3: Since the right hand side of Formula (4) equals the right hand side of Formula (6), the formula of the lemma holds up to a certain constant factor σ, namely an $[\mathcal{L}(\mathcal{A}_1 + \mathcal{B}, \mathcal{A}_2, \ldots, \mathcal{A}_n) : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)]$-th root of unity.

Step 4: Now, we show that $\sigma = 1$. By Lemma 11 and Remark 3, we have

$$\begin{aligned}&\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n)\\ &\quad = \left(\prod_{\beta'} f_1(\beta') \prod_{\nu} \mathrm{Res}_{M(\mathcal{A}_2)^{\nu},\ldots,M(\mathcal{A}_k)^{\nu}}(M(f_2)^{\nu}, \ldots, M(f_k)^{\nu})^{\delta_{M(\mathcal{A}_1),\ldots,M(\mathcal{A}_k)}(\nu)}\right)^{\mathrm{e}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}},\end{aligned}$$

where the vector ν ranges over the primitive inward normal vectors of the facets of the convex hull of $M(\mathcal{A}_2) + \cdots + M(\mathcal{A}_k)$, where $M : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_k) \to \mathbb{Z}^{k-1}$ is some lattice isomorphism which we naturally extend to the f_i's. Since this factorization of $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1,\ldots,f_n)$ equals the factorization we have derived previously, and because of Formula 2, we have $\sigma = 1$.

Thus we have shown the lemma. $\square$

Lemma 14 *The set* $\{1,\ldots,n\}$ *has a unique subset essential for* $\{\mathcal{A}_1,\ldots,\mathcal{A}_n\}$ *and this unique essential subset contains the index* 1.

Proof: Let $\dim(L)$ denote the rank of an integer lattice L and let $\#I$ denote the cardinality of a set I.

We proceed in three steps: We show that $\{1,\ldots,n\}$ has a subset essential for $(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ (Step 1). Then we show that the index 1 is contained in any essential subset (Step 2). Finally we show that there is only one essential subset (Step 3). Now we carry out this strategy:

Step 1: Obviously we have $n > n-1 \geq \dim\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$. Consider the set $S := \{I \subseteq \{1,\ldots,n\} \mid I \neq \emptyset \text{ and } \#I > \dim\mathcal{L}((\mathcal{A}_i)_{i\in I})\}$. The set S is nonempty and finite and therefore contains a minimal element J. For all proper subsets K of J we have $\dim\mathcal{L}((\mathcal{A}_k)_{k\in K}) \geq \#K$. Now fix a proper subset $K \subset J$ with cardinality $\#K = \#J - 1$. Then $\#J > \dim\mathcal{L}((\mathcal{A}_j)_{j\in J}) \geq \dim\mathcal{L}((\mathcal{A}_k)_{k\in K}) \geq \#J - 1$. Therefore $\dim\mathcal{L}((\mathcal{A}_j)_{j\in J}) = \#J - 1$. Thus $\{1,\ldots,n\}$ has a subset essential for $(\mathcal{A}_1,\ldots,\mathcal{A}_n)$.

Step 2: The index 1 is contained in any essential subset of $\{\mathcal{A}_1,\ldots,\mathcal{A}_n\}$; otherwise $\{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\}$ has a subset being essential but not containing the index 1 which is a contradiction!

Step 3 (*uniqueness*): Take two different subsets I and J, resp., of $\{1,\ldots,n\}$ essential for $\{\mathcal{A}_1,\ldots,\mathcal{A}_n\}$ which contain the index 1. Let $d_1 := \dim\mathcal{L}((\mathcal{A}_i)_{i\in I})$ and $d_2 := \dim\mathcal{L}((\mathcal{A}_j)_{j\in J})$. Thus $\#I = d_1 + 1$ and $\#J = d_2 + 1$. Furthermore let $K := I \cap J$, whose cardinality is $\#K \geq 1$. We show that

$$\dim\mathcal{L}\big((\mathcal{A}_i)_{i\in(I\cup J)\setminus\{1\}}\big) < \#((I \cup J)\setminus\{1\}).$$

Note that, since I and J are essential, as in Step 1, we have

$$\dim\mathcal{L}\big((\mathcal{A}_i)_{i\in I\setminus\{1\}}\big) = \dim\mathcal{L}((\mathcal{A}_i)_{i\in I})) = d_1$$

and

$$\dim\mathcal{L}\big((\mathcal{A}_j)_{j\in J\setminus\{1\}}\big) = \dim\mathcal{L}((\mathcal{A}_j)_{j\in J}) = d_2.$$

Therefore $\dim\mathcal{L}((\mathcal{A}_i)_{i\in(I\cup J)\setminus\{1\}}) = \dim\mathcal{L}((\mathcal{A}_i)_{i\in I\cup J})$ which is at most

$$\dim\mathcal{L}((\mathcal{A}_i)_{i\in I}) + \dim\mathcal{L}((\mathcal{A}_j)_{j\in J}) - \dim\mathcal{L}((\mathcal{A}_k)_{k\in K}).$$

Since K is a proper subset of the essential I, we have that $\dim\mathcal{L}((\mathcal{A}_k)_{k\in K}) \geq \#K$ and therefore

$$\dim\mathcal{L}\big((\mathcal{A}_i)_{i\in(I\cup J)\setminus\{1\}}\big) \leq d_1 + d_2 - \#K.$$

Furthermore, note that $d_1 + d_2 - \#K < d_1 + d_2 + 1 - \#K = \#((I \cup J)\setminus\{1\})$.

Next observe, as in Step 1, that $(I \cup J)\backslash\{1\}$ contains a subset which is essential for $(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$. This is a contradiction because $(I \cup J)\backslash\{1\}$ does not contain the index 1!

Thus we have shown the lemma. $\square$

Now we are ready to prove the main theorem.

Proof (Theorem 1): Let $\tilde{f}_1$ be a Laurent polynomial with distinct symbolic coefficients with support $\tilde{\mathcal{A}}_1$. By Lemma 13 we have

$$\begin{aligned}&\mathrm{Res}_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(\tilde{f}_1, f_2, \ldots, f_n)\\ &\quad= \prod_{\gamma} \tilde{f}_1(\gamma) \prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^\omega,\ldots,\mathcal{A}_n^\omega}\big(f_2^\omega, \ldots, f_n^\omega\big)^{\delta_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)},\end{aligned}$$

where γ ranges over the common zeros in $(\tilde{K}\setminus\{0\})^{n-1}$, with respect to the lattice $\mathcal{L}(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$, of $f_2, \ldots, f_n$, where $\tilde{K}$ is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of $\tilde{f}_1$ and the f_i's, and ω ranges over the primitive inward normal vectors of the facets of the convex hull of $\mathcal{A}_2 + \cdots + \mathcal{A}_n$. Since the symbolic coefficients of $\tilde{f}_1$ are algebraically independent from the symbolic coefficients of $f_2, \ldots, f_n$, Lemma 13 is stable under specialization of $\tilde{f}_1$ and we have

$$\begin{aligned}&\mathrm{Res}_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n)\\ &\quad= \prod_{\gamma} f_1(\gamma) \prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^\omega,\ldots,\mathcal{A}_n^\omega}\big(f_2^\omega, \ldots, f_n^\omega\big)^{\delta_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)},\end{aligned}$$

where γ ranges over the common zeros in $(K\setminus\{0\})^{n-1}$, where K is the algebraic closure of the field generated by the complex numbers and the symbolic coefficients of f_1 and the remaining f_i's and ω is as before.

Furthermore, as in [23] by the construction of $\prod_\gamma f_1(\gamma)$, we have

$$\prod_{\gamma} f_1(\gamma) = \prod_{\gamma'} f_1(\gamma')^{[\mathcal{L}(\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}, \tag{7}$$

where γ' ranges over the common zeros in $(K\setminus\{0\})^{n-1}$ of $f_2, \ldots, f_n$ with respect to $\mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

By Lemmas 13 and 14, we have

$$\prod_{\gamma'} f_1(\gamma') = \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n) \prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^\omega,\ldots,\mathcal{A}_n^\omega}\big(f_2^\omega, \ldots, f_n^\omega\big)^{-\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)}$$

and thus

$$\begin{aligned}\prod_{\gamma} f_1(\gamma) &= \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n)^{[\mathcal{L}(\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}\\ &\qquad\times \prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^\omega,\ldots,\mathcal{A}_n^\omega}\big(f_2^\omega, \ldots, f_n^\omega\big)^{-\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)\,[\mathcal{L}(\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}.\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathrm{Res}_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(\tilde{f}_1, f_2, \ldots, f_n) \\
&\quad = \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1, \ldots, f_n)^{[\mathcal{L}(\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]} \\
&\qquad \times \prod_{\omega} \mathrm{Res}_{\mathcal{A}_2^{\omega},\ldots,\mathcal{A}_n^{\omega}}\left(f_2^{\omega}, \ldots, f_n^{\omega}\right)^{\delta_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega)-\delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)[\mathcal{L}(\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n):\mathcal{L}(\mathcal{A}_1,\ldots,\mathcal{A}_n)]}.
\end{aligned} \tag{8}$$

Note that

$$\frac{[\mathcal{L}(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)]}{[\mathbb{Z}^{n-1} : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)]} = \frac{1}{[\mathbb{Z}^{n-1} : \mathcal{L}(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}$$

and therefore by Lemma 12

$$\begin{aligned}
&\delta_{\tilde{\mathcal{A}}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}(\omega) - \delta_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(\omega)[\mathcal{L}(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) : \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_n)] \\
&\quad = \frac{\left(\mathrm{a}_{\tilde{\mathcal{A}}_1}^{(\omega)} - \mathrm{a}_{\mathcal{A}_1}^{(\omega)}\right)\left[\mathrm{H}^{\omega} : \mathcal{L}\left(\mathcal{A}_2^{\omega}, \ldots, \mathcal{A}_n^{\omega}\right)\right]}{[\mathbb{Z}^{n-1} : \mathcal{L}(\tilde{\mathcal{A}}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)]}.
\end{aligned}$$

Therefore we have shown the equality of the main theorem.

Since the Laurent polynomials in the main theorem have symbolic coefficients and the sparse resultants on the right hand side of the equality are defined with respect to the precise supports of these Laurent polynomials, the factorization on the right hand side is irreducible.

Thus we have shown the main theorem. □

4. Conclusion

In this paper we studied sparse resultant under vanishing of coefficients. The sparse resultant of some Laurent polynomials f_i with respect to any supports is some power of the sparse resultant of the f_i's with respect to their precise supports times a product of powers of sparse resultants of some parts of the f_i's. This result is important for applications where perturbed data with very small coefficients arise as well as when one computes resultants with respect to some fixed supports, not necessarily the supports of the f_i's, in order to speed up computations. This work extended some work by Sturmfels on sparse resultant under vanishing coefficients (cf. Corollary 4.2 of [25]).

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