



Universal Covers of Geometries of Far Away Type

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Abstract. The geometries studied in this paper are obtained from buildings of spherical type by removing all chambers at non-maximal distance from a given element or flag. I consider a number of special cases of the above construction chosen among those which most frequently appear in the literature, proving that the resulting geometry is always simply connected but for three cases of small rank defined over $GF(2)$ and $GF(4)$. I also compute the universal cover in those exceptional cases.

Keywords: buildings, universal covers, embeddings, binary codes

1. Introduction

Geometries obtained from buildings of spherical type by removing all elements at non-maximal distance from a given element or a flag, are met in the context of many interesting characterizations and classifications. Many of them also appear in connection with embeddings of buildings of spherical type, as affine expansions of some of those embeddings (see [16]). As shown in [16], the hull of an embedding corresponds to the universal cover of the expansion of that embedding. In particular, an embedding is its own hull if and only if its expansion is simply connected.

In this paper I consider a number of special cases of the construction sketched above, proving that nearly all of the geometries obtained in those cases are simply connected. It is likely that the same conclusion holds for more families of ‘far away’ geometries, different from those studied here. In fact, I have only considered those families that either are related to some of the embeddings discussed in [16] or include examples that have been investigated by some authors in some contexts. Actually, my selection misses one family which however, according to the above criteria, deserved to be studied, namely the case of the subgeometry of a building of type F_4 far from a given point or symp. I have not considered it simply because I couldn’t find the right way to treat it.

We follow [13] for basic notions and general results on geometries and Tits [18] for buildings. In particular, according to [13], we assume all geometries to be residually connected and firm, by definition.

We refer to chapters 8, 11 and 12 of [13] for m -covers, m -quotients and m -simple connectedness, but we are only interested in 2- and $(n - 1)$ -covers in this paper. We recall that the $(n - 1)$ -covers of a geometry of rank n are called *topological covers* in [13], but many authors simply call them *covers*. We too do so in this paper. Accordingly, in the sequel, the

universal cover of a geometry Γ of rank n is its universal $(n - 1)$ -cover and we say that Γ is *simply connected* if it is $(n - 1)$ -simply connected.

1.1. The geometry far from a flag

Suppose Δ is a thick building of connected spherical type and rank at least 2. It is well known that, given a flag $F \neq \emptyset$ and a chamber C of Δ , there is a unique chamber $C_F \supseteq F$ at minimal distance from C (Tits [18]). We denote the distance between C and C_F by $d(C, F)$. For every nonempty flag X , the *distance* $d(X, F)$ from X to F is the minimal distance $d(C, F)$ from F to a chamber $C \supseteq X$. We say that a flag X is *far* from F if $d(X, F)$ is maximal, compatibly with the types of F and X . We denote by $\text{Far}_\Delta(F)$ the substructure of Δ formed by the elements far from F , with the incidence relation inherited from Δ but rectified as follows: two elements x, y of $\text{Far}_\Delta(F)$ are incident in $\text{Far}_\Delta(F)$ if and only if they are incident in Δ and the flag $\{x, y\}$ is far from F .

As Δ is thick, the structure $\text{Far}_\Delta(F)$ is firm. It is known that $\text{Far}_\Delta(F)$ is residually connected (whence, it is a geometry) except for a few cases defined over $GF(2)$ (Blok and Brouwer [4]), but none of those exceptional cases will be met in this paper.

In the sequel we call $\text{Far}_\Delta(F)$ a *geometry of far away type*, also a *far away geometry*, for short. Before to state the results of this paper, we mention a few examples, focusing on simple connectedness.

Example 1.1 Suppose Δ is a non-degenerate projective geometry of dimension $n \geq 3$ and let A be a hyperplane or a point of Δ . Then $\text{Far}_\Delta(A)$ is an affine geometry or the dual of an affine geometry. In any case, $\text{Far}_\Delta(A)$ is simply connected.

Example 1.2 With Δ as in Example 1, let F be a point-hyperplane flag of Δ . Then $\text{Far}_\Delta(F)$ is an affine-dual-affine geometry as in Van Nypelseer [19]. It follows from the main result of [19] that $\text{Far}_\Delta(F)$ is simply connected.

Example 1.3 Let Δ be a thick polar space of rank $n > 2$ and p a point of Δ . Then $\text{Far}_\Delta(p)$ is an affine polar space (Cohen and Shult [7]). Affine polar spaces are simply connected (Pasini [12]; also Cuypers and Pasini [8] and [13, Proposition 12.50]). So, $\text{Far}_\Delta(p)$ is simply connected.

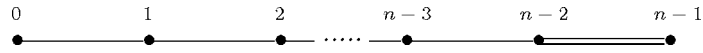
A similar conclusion holds when Δ is a building of type D_n and p is an element of Δ corresponding to a point of the polar space Π associated to Δ . Indeed, the subgeometry of Π corresponding to $\text{Far}_\Delta(p)$ is an affine polar space and, as recalled above, affine polar spaces are simply connected.

1.2. Main results

The geometries $\text{Far}_\Delta(A)$, $\text{Far}_\Delta(F)$ and $\text{Far}_\Delta(p)$ of Examples 1.1, 1.2 and 1.3 are simply connected. In this paper we obtain the same conclusion in a number of other cases. Explicitly, the following are the results we shall prove.

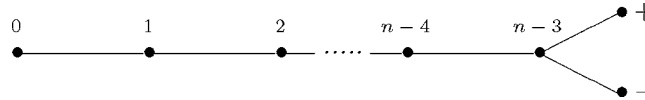
Theorem 1.1 *Let Δ be a non-degenerate projective geometry of dimension $n \geq 3$ and A an element of Δ other than a point or a hyperplane. Then $\text{Far}_\Delta(A)$ is simply connected.*

Theorem 1.2 *Let Δ be a thick polar space of rank $n \geq 3$ and A a maximal singular subspace of Δ . Namely $t(A) = n - 1$, where the nonnegative integers $0, 1, 2, \dots, n - 1$ are taken as types, as follows:*



Then $\text{Far}_\Delta(A)$ is simply connected, except when $n = 3$ and Δ is either the symplectic variety $\mathcal{S}(5, 2)$ of $PG(5, 2)$ or the hermitean variety $\mathcal{H}(5, 4)$ of $PG(5, 4)$.

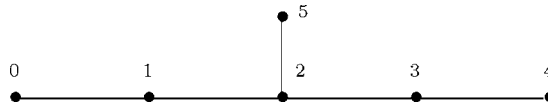
Theorem 1.3 *For $n \geq 4$, let Δ be a thick building of type D_n and A an element corresponding to a maximal singular subspace of the polar space associated to Δ . That is, $t(A) = +$ or $-$, where the nonnegative integers $0, 1, 2, \dots, n - 3$ and the symbols $+$ and $-$ are taken as types, as follows:*



Then $\text{Far}_\Delta(A)$ is simply connected.

Theorem 1.4 *For $n \geq 4$, let Δ be a thick building of type D_n and F a flag of Δ of type $\{+, -\}$, with types as in Theorem 1.4. Then $\text{Far}_\Delta(F)$ is simply connected except when $n = 4$ and Δ is defined over $GF(2)$.*

Theorem 1.5 *Let Δ be a thick building of type E_6 , with types $0, 1, 2, 3, 4, 5$ as follows:*



Then $\text{Far}_\Delta(p)$ is simply connected for every 0-element p .

A few small cases are not covered by the previous theorems: Theorem 1.2 misses the cases of $\Delta = \mathcal{S}(5, 2)$ and $\Delta = \mathcal{H}(5, 4)$. Theorem 1.5 misses the case of $\Delta = D_4(2)$, the building of type D_4 over $GF(2)$. The next theorem settles those exceptions.

Theorem 1.6 *With $\Delta = \mathcal{S}(5, 2)$, $\mathcal{H}(5, 4)$ or $D_4(2)$, let X be a singular plane of Δ in the first two cases and a $\{+, -\}$ -flag in the third case. In any case, let $\tilde{\Gamma}$ be the universal cover of $\Gamma := \text{Far}_\Delta(X)$. Then,*

- (1) *if $\Delta = \mathcal{S}(5, 2)$, then $\tilde{\Gamma}$ is a double cover;*
- (2) *if $\Delta = \mathcal{H}(5, 4)$, then $\tilde{\Gamma}$ is a 4-fold cover;*
- (3) *if $\Delta = D_4(2)$, then $\tilde{\Gamma}$ is a double cover.*

Claim (2) of the above theorem is new whereas (1) and (3) are known (Baumeister, Meixner and Pasini [2]). However, the proofs we will give of (1) and (3) are shorter than those of [2] and, we presume, more perspicuous.

We recall that the universal 2-cover and the universal cover of a geometry of rank 3 are the same thing, but in higher rank cases the former is possibly larger than the latter. Accordingly, a geometry of rank $n > 3$ might be simply connected without being 2-simply connected. However, nearly all of the geometries considered in this paper are 2-simply connected. Explicitly:

Corollary 1.7 *All far-away geometries considered in Theorems 1.1, 1.4 and 1.5 are 2-simply connected. The geometries of Theorem 1.2 are 2-simply connected provided that Δ is neither the symplectic variety $\mathcal{S}(2n - 1, 2)$ of $PG(2n - 1, 2)$ nor the hermitean variety $\mathcal{H}(2n - 1, 4)$ of $PG(2n - 1, 4)$. The geometries of Theorem 1.4 are 2-simply connected provided their underlying field is different from $GF(2)$.*

According to the above corollary, the following exceptional cases need a separate discussion:

- (1) $\Gamma = \text{Far}_\Delta(A)$ as in Theorem 1.2, but $\Delta = \mathcal{S}(2n - 1, 2)$;
- (2) $\Gamma = \text{Far}_\Delta(A)$ as in Theorem 1.2, but $\Delta = \mathcal{H}(2n - 1, 4)$;
- (3) $\Gamma = \text{Far}_\Delta(F)$ as in Theorem 1.4, but $\Delta = D_n(2)$.

A description of the universal 2-cover $\tilde{\Gamma}$ of Γ is known in cases (1) and (3) (Baumeister, Meixner and Pasini [2]). In particular, $\tilde{\Gamma}$ is a 2^k -fold cover with $k = 2^n - \binom{n+1}{2} - 1$ in case (1) and $k = 2^{n-1} - \binom{n}{2} - 1$ in case (3). I will also offer a construction of $\tilde{\Gamma}$ for cases (1) and (3) in Sections 9 and 10, which I think will clarify the descriptions given in [2]. A construction of $\tilde{\Gamma}$ for Γ as in case (2) is also given in Section 9. In principle, it is possible to exploit it to compute the size of $\tilde{\Gamma}$ but, regretfully, I have been able to accomplish that computation only for $n \leq 4$: When $n = 3$, $\tilde{\Gamma}$ is a 4-fold cover (Theorem 1.6(2)); when $n = 4$, $\tilde{\Gamma}$ is a 2^8 -fold cover.

As said at the beginning of this Introduction, I have not found the right approach to investigate $\text{Far}_\Delta(x)$ with Δ a thick building of type F_4 and x a point or a symp of Δ . I only mention a partial result, proved in [16] (Corollary 9.9):

Result 1.8 *Suppose Δ is of type $F_4(p)$ for a prime $p > 2$ and $\text{Res}(x) \cong \mathcal{S}(5, p)$. Then $\text{Far}_\Delta(x)$ is 2-simply connected.*

1.3. Organization of the paper

The rest of this introduction contains some references and the list of the diagrams of the geometries of far-away type considered in the previous theorems.

A few general results and constructions to be used in this paper are recalled in Section 2. Sections 3, 4, 5 and 7 contain the proofs of Theorems 1.1, 1.2, 1.4 and 1.5 respectively. We will split the proof of Theorem 1.4 in two parts. The first part forms Section 6 and deals with the generic case, where either the underlying field K of Δ is different from $GF(2)$ or $K = GF(2)$ but Δ has rank $n > 5$. The case of $\Delta = D_5(2)$ remains to consider, but we postpone its discussion till Section 10. The proof of Corollary 1.7 is given in Section 8. Claims (1) and (2) of Theorem 1.6 are proved in Section 9 and claim (3) is proved in Section 10. Universal 2-covers for the exceptional cases (1), (2) and (3) mentioned after Corollary 1.7 are discussed in Sections 9 and 10. The proof of Theorem 1.4 in the case of $\Delta = D_5(2)$, put aside in Section 6, will be obtained in Section 10 as a by-product of the informations we will collect in that section on the universal 2-cover of $\text{Far}_\Delta(F)$.

1.4. Remarks

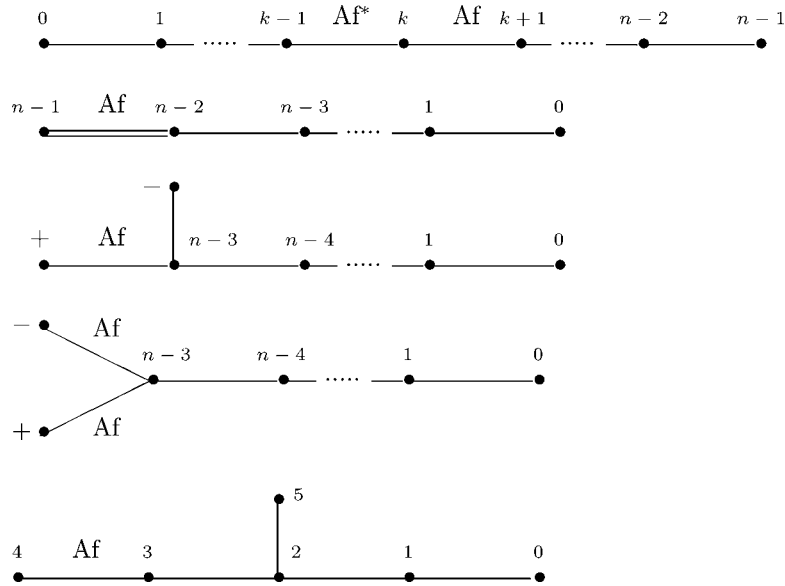
Remark 1.9 A few special cases of some of our theorems have been earlier discussed by a number of authors. For instance, Baumeister, Meixner and Pasini [2] consider the special case of Theorem 1.2 where Δ is the non-singular orthogonal quadric $Q(2n, 2)$ of $PG(2n, 2)$. Baumeister, Shpectorov and Stroth [3] consider the case of $\Delta = Q(2n, q)$ for any q (but their argument works as well for the general case of Theorem 1.2).

The special case of Theorem 1.4 with Δ defined over a finite field is implicit in Munemasa and Shpectorov [10] and Munemasa, Pasechnik and Shpectorov [11] (also in Hybrechts and Pasini [9] when the underlying field of Δ is $GF(2)$).

When $K = GF(q)$ with $q > 2$, the conclusion of Theorem 1.4 is contained in Baumeister and Stroth [1], who obtained that result group-theoretically, whereas the case of $q = 2$ is discussed by Baumeister, Meixner and Pasini [2].

Turning to $\text{Far}_\Delta(A)$ with $\Delta = \mathcal{H}(5, 4)$, let \mathcal{H} be the collinearity graph of the dual of $\Gamma := \text{Far}_\Delta(A)$, having the planes and the lines of Γ as vertices and edges. Then \mathcal{H} is isomorphic to the hermitean forms graph over $GF(4)$. A quadruple cover of \mathcal{H} is described by Brouwer, Cohen and Neumaier [5, p. 365]. That cover is in fact the collinearity graph of the dual of the universal cover of Γ . A double cover of \mathcal{H} is also described in [5]. It corresponds to a (non simply connected) double cover of Γ .

Remark 1.10 The far-away geometries considered in the previous theorems belong to the following diagrams, where $\bullet \xrightarrow{\text{Af}} \bullet$ stands for the class of affine planes, $\bullet \xrightarrow{\text{Af}^*} \bullet$ is the class of dual affine planes and $\bullet \xrightarrow{\text{Af}} \bullet$ represents the class of affine generalized quadrangles, which are geometries obtained from generalized quadrangles by removing a maximal full subquadrangle, an ovoid or the star of a point (the latter is always the case here). In all cases but Theorem 1.1 diagrams are given the orientation opposite to that used in the statements of the previous theorems. In the first picture, $k := n - 1 - d$ and $d = t(A)$ is the projective dimension of A .



Remark 1.11 In the case considered in Theorem 1.1, if $k = n - 1 - t(A)$ as above, then the $\{0, 1, \dots, k - 1\}$ -truncation of $\text{Far}_\Delta(A)$ is an attenuated space. So, Theorem 1.1 implies that attenuated spaces of rank at least 3 are simply connected.

2. A selection of general results

2.1. Terminology and notation

As in [13], given an element x of a geometry Γ , we denote the type of x by $t(x)$, but we change the notation of [13] for residues, denoting the residue of x by $\text{Res}_\Gamma(x)$ (also $\text{Res}(x)$ if no ambiguity arises). The same notation will be used for flags. As in [13], given a subset $J \neq \emptyset$ of the type-set I of Γ , we denote by $\text{Tr}_J^-(\Gamma)$ the geometry obtained from Γ by removing all elements of type $j \in J$. We call $\text{Tr}_J^-(\Gamma)$ the J -truncation of Γ . When $J \subset I$, we set $\text{Tr}_J^+(\Gamma) := \text{Tr}_{I \setminus J}^-(\Gamma)$ and we call it the J -cotruncation of Γ .

As in [13], we denote by $\mathcal{D}(\Gamma)$ the diagram graph of Γ (also called basic diagram of Γ). Suppose that $\mathcal{D}(\Gamma)$ is connected. Denoted by I the set of types of Γ and given a type $0 \in I$, let $\text{fr}(0)$ be the neighbourhood of 0 in $\mathcal{D}(\Gamma)$. The 0 -point-line system $\mathcal{L}_0(\Gamma)$ of Γ is the point-line geometry having the 0 -elements of Γ as points and the flags of type $\text{fr}(0)$ as lines, with the incidence relation inherited from Γ . The collinearity graph of $\mathcal{L}_0(\Gamma)$ will be denoted by $\mathcal{G}_0(\Gamma)$.

2.2. A criterion for simple connectedness

Given a flag $F \neq \emptyset$ with $0 \notin t(F)$, $\text{Res}(F)$ is the direct sum of subgeometries corresponding to the connected components of the graph induced by $\mathcal{D}(\Gamma)$ on $I \setminus t(F)$. In particular, denoted

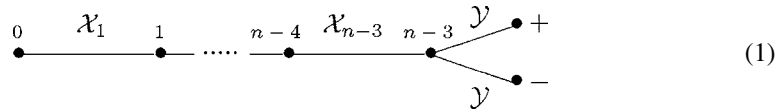
by I_0 the connected component of 0 in that induced subgraph, the I_0 -cotruncation of $\text{Res}(F)$ is a direct summand of $\text{Res}(F)$. We denote it by $\text{Res}_0(F)$. When $\text{fr}(0) \not\subseteq t(F)$, $\text{Res}_0(F)$ has rank at least 2 and we can consider its 0-point-line system $\mathcal{L}_0(\text{Res}_0(F))$. We denote the collinearity graph of $\mathcal{L}_0(\text{Res}_0(F))$ by $\mathcal{G}_0(F)$. When $\text{fr}(0) \subseteq t(F)$, then $\text{Res}_0(F)$ has rank 1 (in fact, it is the point-set of a line of $\mathcal{L}_0(\Gamma)$). In that case $\mathcal{G}_0(F)$ stands for the complete graph over the set $\text{Res}_0(F)$.

We say that a closed path of $\mathcal{G}_0(\Gamma)$ is *good* if it is a path of $\mathcal{G}_0(F)$ for some nonempty flag F . The following proposition, which immediately follows from [13, Theorem 12.64], is the main tool we will use in this paper:

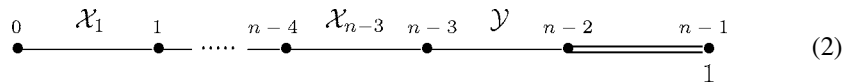
Proposition 2.1 *Suppose that no two lines of $\mathcal{L}_0(\Gamma)$ meet in more than one point. Then Γ is simply connected if and only if every closed path of $\mathcal{G}_0(\Gamma)$ splits in good closed paths.*

2.3. *Universal covers of shadow geometries*

The geometry $\mathcal{L}_0(\Gamma)$ coincides with the $\{0, 1\}$ -cotruncation of a geometry of the same rank as Γ but with a string as its diagram graph, usually called the *0-shadow geometry* of Γ and denoted by $\text{Sh}_0(\Gamma)$ (but we warn that the symbol $\text{Gr}_0(\Gamma)$ and the words *grassmann geometry* are used in [13] instead of $\text{Sh}_0(\Gamma)$ and *shadow geometry*). We are not going to recall the construction of $\text{Sh}_0(\Gamma)$ in general. We only consider the special case where Γ belongs to a diagram as follows, where $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{n-3}$ and \mathcal{Y} are classes of geometries of rank 2 different from generalized digons and 0, 1, $\dots, n - 3$, + and - are the types:



In this case the definition of $\text{Sh}_0(\Gamma)$ is a straightforward generalization of the construction of the polar space associated to a D_n -building. The elements of $\text{Sh}_0(\Gamma)$ are the elements and the $\{+, -\}$ -flags of Γ . The elements of Γ of type $i = 0, 1, \dots, n - 3$ keep their type in $\text{Sh}_0(\Gamma)$, those of type + and - form the class of $(n - 1)$ -elements of $\text{Sh}_0(\Gamma)$ and the $\{+, -\}$ -flags are given the type $n - 2$ as elements of $\text{Sh}_0(\Gamma)$. The incidence relation of $\text{Sh}_0(\Gamma)$ is inherited from Γ , except that two elements of type + and - are never incident in $\text{Sh}_0(\Gamma)$. The geometry $\text{Sh}_0(\Gamma)$ belongs to the following diagram:



Note that the residues of $\text{Sh}_0(\Gamma)$ of type $\{n - 2, n - 1\}$ are grids, as the order 1 at the right end of (2) reminds us.

The following are contained in [14, Theorems 7 and 9] (see also Rinauro [17]):

Proposition 2.2 *Let Γ be a geometry belonging to diagram (1). Then the universal cover of $\text{Sh}_0(\Gamma)$ is the 0-shadow geometry of the universal cover of Γ . Suppose furthermore that*

$\text{Sh}_0(\Gamma)$ admits the universal 2-cover. Then Γ also admits the universal 2-cover and the universal 2-cover of $\text{Sh}_0(\Gamma)$ is the 0-shadow geometry of the universal 2-cover of Γ .

Remark 2.3 If \mathcal{C} is the chamber system of a geometry Γ of rank $n > 3$, we do not know if the universal 2-cover of \mathcal{C} necessarily comes from a geometry. When it does, then that geometry is the universal 2-cover of Γ and we may say that Γ admits the universal 2-cover. This remark explains why the existence of the universal 2-cover of $\text{Sh}_0(\Gamma)$ is put as an hypothesis in the second part of Proposition 2.2. On the other hand, every geometry admits the universal cover. So, no hypothesis like that is needed in the first part. We could drop it from the second part too by rephrasing that statement in terms of chamber systems, but we prefer not to concern ourselves with them in this paper. Anyhow, all geometries considered in this paper admit the universal 2-cover. Indeed, for each of them, either we prove that it is 2-simply connected, or we construct its universal 2-cover as a geometry.

2.4. Expansions of $GF(2)$ -embeddings of matroids

In this section we discuss a special case of the theory of embeddings and expansions of [16], taking $GF(2)$ -vector spaces as codomains for the considered embeddings and assuming that the geometries to embed are finite dimensional simple matroids (also called *dimensional linear spaces*; see Buekenhout [6, Chapter 6]).

Henceforth \mathcal{M} is a given simple matroid of finite dimension $n \geq 1$, regarded as an n -tuple $\mathcal{M} = (P, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$, where P is the set of *points* and, for $i = 1, 2, \dots, n-1$, \mathcal{F}_i is the set of i -dimensional flats, also called i -flats for short. In particular, the 1-flats are the *lines* of \mathcal{M} . We set $\mathcal{L} := \mathcal{F}_1$ and $\mathcal{F} := \cup_{i=1}^{n-1} \mathcal{F}_i$.

Given a vector space V over $GF(2)$, a $GF(2)$ -embedding $\varepsilon : \mathcal{M} \rightarrow V$ of \mathcal{M} in V is an injective mapping $\varepsilon : P \rightarrow V \setminus \{0\}$ such that $\varepsilon(P)$ spans V and, for any two flats $X, Y \in \mathcal{F}$, we have $\langle \varepsilon(X) \rangle \subseteq \langle \varepsilon(Y) \rangle$ only if $X \subseteq Y$ (where, regarding X as a subset of P , we write $\varepsilon(X)$ for $\{\varepsilon(p)\}_{p \in X}$).

For $X \in \mathcal{F}$, we put $V_X := \langle \varepsilon(X) \rangle$. Also, for $x \in P$, we denote by V_x the 1-dimensional subspace of V spanned by $\varepsilon(x)$.

The *expansion* $\text{Exp}(\varepsilon)$ of ε is the geometry of rank $n+1$ defined as follows: The types of $\text{Exp}(\varepsilon)$ are the integers $0, 1, \dots, n$; the vectors of V are the elements of $\text{Exp}(\varepsilon)$ of type 0, which we also call *points* of $\text{Exp}(\varepsilon)$; the elements of type 1 are the cosets in V of the 1-dimensional subspaces V_x for $x \in P$ and the elements of type $j = 2, 3, \dots, n$ are the cosets of the subspaces V_X in V , for $X \in \mathcal{F}_{j-1}$; the incidence relation of $\text{Exp}(\varepsilon)$ is the natural one, namely symmetrized inclusion.

Clearly, $\text{Res}(v) \cong \mathcal{M}$ for every point $v \in V$ of $\text{Exp}(\varepsilon)$ and, given an element $W = v + V_X$ of $\text{Exp}(\varepsilon)$ of type $j > 1$, the $\{0, 1, \dots, j-1\}$ -cotruncation $\text{Res}_0(W)$ of $\text{Res}(W)$ is isomorphic to $\text{Exp}(\varepsilon_X)$, where $\varepsilon_X : \text{Res}_0(X) \rightarrow V_X$ is the restriction of ε to X . In particular, if all lines of \mathcal{M} have finite size $s > 2$, then the $\{0, 1\}$ -residues of $\text{Exp}(\varepsilon)$ are quotients of the point-graph of the s -dimensional cube; explicitly, if $d = \dim(V_L)$ for $L \in \mathcal{L}$, then $\text{Res}_0(v + V_L)$ is a 2^{s-d} -fold quotient of that graph. If every line of \mathcal{M} has just two points, then the $\{0, 1\}$ -residues of $\text{Exp}(\varepsilon)$ are ordinary quadrangles.

Example 2.1 Assume that the lines of \mathcal{M} have size 3. If $\dim(V_L) = 3$ for every $L \in \mathcal{L}$, then the $\{0, 1\}$ -residues of $\text{Exp}(\varepsilon)$ are isomorphic 3-dimensional cubes. If $\dim(V_L) = 2$ for every line L , then those residues are isomorphic to $AG(2, 2)$, which may be regarded as the quotient of the 3-cube by the antipodal relation.

Example 2.2 Suppose that, for every $L \in \mathcal{L}$, $|L| = 5$, $\dim(V_L) = 4$ and $\{V_x\}_{x \in L}$ is an ovoid of $PG(V_L) \cong PG(3, 2)$ (compare Section 9, Lemma 9.4). Then the $\{0, 1\}$ -residues of $\text{Exp}(\varepsilon)$ are isomorphic to the quotient of the 5-cube by the antipodal relation.

Let \widehat{V} be a $|P|$ -dimensional vector space over $GF(2)$ and ι a given bijection from P to a basis of \widehat{V} . Clearly, ι is a $GF(2)$ -embedding of \mathcal{M} . We call it the *free $GF(2)$ -embedding* of \mathcal{M} .

Every $GF(2)$ -embedding of \mathcal{M} is involved in the free embedding. More explicitly, given a $GF(2)$ -embedding $\varepsilon : \mathcal{M} \rightarrow V$, the mapping sending the image $\iota(x)$ of a point $x \in P$ to the vector $\varepsilon(x) \in V$, extends to a surjective linear transformation $\pi : \widehat{V} \rightarrow V$. As π maps $\iota(X)$ onto $\varepsilon(X)$, it maps the subspace $\widehat{V}_X := \langle \iota(X) \rangle$ of \widehat{V} onto the subspace V_X of V and, denoted by K the kernel of π and by K_X the kernel of the restriction of π to \widehat{V}_X , we have

$$K_X = K \cap \widehat{V}_X. \quad (1)$$

Henceforth we assume $n \geq 2$ and define:

$$\widetilde{K} := \langle K_X \rangle_{X \in \mathcal{F}}, \quad \widetilde{K}^{(1)} := \langle K_L \rangle_{L \in \mathcal{L}}, \quad \widetilde{V} := \widehat{V}/\widetilde{K}, \quad \widetilde{V}^{(1)} := \widehat{V}/\widetilde{K}^{(1)}.$$

We call \widetilde{K} and $\widetilde{K}^{(1)}$ the *local kernel* and the *1-local kernel* of ε . The subspace K will be called the *global kernel* of ε .

Clearly, $\widetilde{K}^{(1)} \leq \widetilde{K} \leq K$ and $\pi = \widetilde{\varphi}\widetilde{\pi} = \widetilde{\varphi}^{(1)}\widetilde{\pi}^{(1)}$ where $\widetilde{\pi}$ and $\widetilde{\pi}^{(1)}$ are the natural projections of \widehat{V} onto \widetilde{V} and $\widetilde{V}^{(1)}$, and $\widetilde{\varphi}$ and $\widetilde{\varphi}^{(1)}$ are the natural projections of \widetilde{V} and $\widetilde{V}^{(1)}$ onto V . Therefore,

$$\widetilde{K} \cap \widehat{V}_X = K_X \quad \text{for all } X \in \mathcal{F}, \quad (2)$$

$$\widetilde{K}^{(1)} \cap \widehat{V}_L = K_L \quad \text{for all } L \in \mathcal{L}. \quad (3)$$

Lemma 2.4 *Regarded \widetilde{V} , $\widetilde{V}^{(1)}$ and the subspaces V_X of V as additive groups and the posets*

$$\mathcal{A} := (\{V_X\}_{X \in \mathcal{F}} \cup \{V_x\}_{x \in P}, \subseteq), \quad \mathcal{A}^{(1)} := (\{V_L\}_{L \in \mathcal{L}} \cup \{V_x\}_{x \in P}, \subseteq)$$

as amalgams of groups, \widetilde{V} is the universal completion of \mathcal{A} and $\widetilde{V}^{(1)}$ is the universal completion of $\mathcal{A}^{(1)}$.

Proof: Let U be the universal completion of \mathcal{A} . As all groups V_X are generated by involutions, U is generated by involutions. Furthermore, as the points of \mathcal{M} are mutually collinear, any two generating involutions of U are contained in V_L for some $L \in \mathcal{L}$. So,

they commute in V_L . Hence U is an elementary abelian 2-group. It is now clear that U is a homomorphic image of \widehat{V} . The equality $U = \widetilde{V}$ follows from (2). By a similar argument, except for using (3) instead of (2), one can prove that $\widetilde{V}^{(1)}$ is the universal completion of $\mathcal{A}^{(1)}$. \square

We define the *hull* $\tilde{\varepsilon}$ and the *1-hull* $\tilde{\varepsilon}^{(1)}$ of ε as the compositions of the free embedding ι with the projections $\tilde{\pi} : \widehat{V} \rightarrow \widetilde{V}$ and $\tilde{\pi}^{(1)} : \widehat{V} \rightarrow \widetilde{V}^{(1)}$:

$$\tilde{\varepsilon} := \tilde{\pi} \iota, \quad \tilde{\varepsilon}^{(1)} := \tilde{\pi}^{(1)} \iota.$$

Both $\tilde{\varepsilon}$ and $\tilde{\varepsilon}^{(1)}$ are $GF(2)$ -embeddings and we have $\varepsilon = \tilde{\varphi} \tilde{\varepsilon} = \tilde{\varphi}^{(1)} \tilde{\varepsilon}^{(1)}$. Also, $\tilde{\pi} = \psi \tilde{\pi}^{(1)}$ and $\tilde{\varphi}^{(1)} = \tilde{\varphi} \psi$, where ψ is the natural projection of $\widetilde{V}^{(1)}$ onto \widetilde{V} . Hence $\tilde{\varepsilon} = \psi \tilde{\varepsilon}^{(1)}$.

Proposition 2.5 *The geometries $\text{Exp}(\tilde{\varepsilon})$ and $\text{Exp}(\tilde{\varepsilon}^{(1)})$ are respectively the universal cover and the universal 2-cover of $\text{Exp}(\varepsilon)$. Their deck groups are isomorphic to K/\widetilde{K} and $K/\widetilde{K}^{(1)}$, respectively.*

Proof: The claims on $\text{Exp}(\tilde{\varepsilon})$ and K/\widetilde{K} follow from Theorem 3.3 of [16] and the fact that \widetilde{V} is the universal completion of \mathcal{A} (Lemma 2.4). The claims on $\text{Exp}(\tilde{\varepsilon}^{(1)})$ and $K/\widetilde{K}^{(1)}$ follow from Theorem 4.4 of [16], the fact that $\widetilde{V}^{(1)}$ is the universal completion of $\mathcal{A}^{(1)}$ and the well known fact that matroids are 2-simply connected. \square

Corollary 2.6 *The geometry $\text{Exp}(\varepsilon)$ is simply connected if and only if $K = \widetilde{K}$. It is 2-simply connected if and only if $K = \widetilde{K}^{(1)}$.*

(Trivial, by the second part of Proposition 2.5.) Corollary 2.6 is also helpful to compute \widetilde{K} in certain cases. Note first that, according to (1), for $X, Y \in \mathcal{F}$ we have $K_X \subseteq K_Y$ whenever $X \subseteq Y$. Therefore, in any case:

$$\widetilde{K} = \langle K_S \rangle_{S \in \mathcal{F}_{n-1}}. \quad (4)$$

Corollary 2.7 *Suppose that, for a given $k > 1$ and every $i = k, k+1, \dots, n-1$, all $\{0, 1, \dots, i\}$ -residues of $\text{Exp}(\varepsilon)$ are simply connected. Then*

$$\widetilde{K} = \langle K_S \rangle_{S \in \mathcal{F}_{k-1}}. \quad (5)$$

In particular, if all residues of $\text{Exp}(\varepsilon)$ of type $\{0, 1, \dots, i\}$ are simply connected for all $i = 2, 3, \dots, n-1$, then $\widetilde{K} = \widetilde{K}^{(1)}$.

Proof: As previously remarked, given an $(i+1)$ -element $W = v + V_X$ of $\text{Exp}(\varepsilon)$, the $\{0, 1, \dots, i\}$ -cotruncation of $\text{Res}(W)$ is isomorphic to the expansion of the restriction ε_X of ε to X . The conclusion follows by induction, applying (4) and Corollary 2.6 to K_X , which is the global kernel of ε_X . \square

Proposition 2.8 *We have $\tilde{\varepsilon}^{(1)} = \iota$ if and only if the set $\varepsilon(L)$ is linearly independent for every $L \in \mathcal{L}$.*

Proof: The set $\varepsilon(L)$ is independent if and only if $K_L = 0$. The conclusion immediately follows from this remark. \square

Corollary 2.9 *If all lines of \mathcal{M} have size 2, then $\tilde{\varepsilon}^{(1)} = \iota$.*

Proof: If $|L| = 2$, then $\varepsilon(L)$ only contains two vectors, whence it is independent. The conclusion follows from Proposition 2.8. \square

3. Proof of Theorem 1.1

Let $d = \dim(A)$. By assumption, $0 < d < n - 1$. We will apply Proposition 2.1 with the points of $\Gamma = \text{Far}_\Delta(A)$ taken as 0-elements. So, the lines of $\mathcal{L}_0(\Gamma)$ are the lines of Δ that miss A . We call them *lines* of Γ , for short.

Lemma 3.1 *For every point p and every line L of Γ , at most one of the points of L is non-collinear with p in Γ .*

Proof: If $p \in L$ there is nothing to prove. Suppose $p \notin L$ and let π be the plane of Δ spanned by p and L . As $L \cap A = \emptyset$, we have $|\pi \cap A| \leq 1$. Hence at most one of the lines of π through p meets A . \square

Lemma 3.2 *The graph $\mathcal{G}_0(\Gamma)$ has diameter 2 and every closed path of $\mathcal{G}_0(\Gamma)$ splits in triangles.*

Proof: Let (a, b, c, d) be a path of $\mathcal{G}_0(\Gamma)$ with $a \not\sim c$ and $b \not\sim d$, where \sim stands for the collinearity relation of Γ . Denoted by L the line of Γ through c and d , we have $L \setminus \{c, d\} \subseteq a \sim \cap b \sim$ by Lemma 3.1. However, $L \setminus \{c, d\} \neq \emptyset$, as $|L| \geq 3$. Thus, there exists a point collinear with all of a, b, c, d . Both claims of the corollary follow from this remark. \square

Lemma 3.3 *Every bad triangle of $\mathcal{G}_0(\Gamma)$ splits in good triangles.*

Proof: Every triangle of $\mathcal{G}_0(\Gamma)$ is contained in plane of Δ . Furthermore, if a plane of Δ contains a line of Γ , then it has at most one point in common with A . Thus, all planes containing triangles of $\mathcal{G}_0(A)$ meet A in at most a point.

When $n - d = 2$, a plane P of Δ belongs to Γ if and only if $P \cap A$ is a point. When $n - d > 2$, P belongs to Γ if and only if $P \cap A = \emptyset$. So, when $n - d = 2$ all triangles of $\mathcal{G}_0(\Gamma)$ are contained in planes of Γ , hence all of them are good.

Assume $n - d > 2$. Let $\{a, b, c\}$ be a bad triangle of $\mathcal{G}_0(\Gamma)$ and $P = \langle a, b, c \rangle$ the plane of Δ spanned by it. Then $P \cap A$ is a point, say p . However, as $n - d > 2$, P is contained in a 3-space S such $S \cap A = p$. Clearly, given a point $x \in S \setminus P$, each of the planes $\langle a, b, x \rangle$, $\langle b, c, x \rangle$ and $\langle a, c, x \rangle$ misses A . So, each of the triangles $\{a, b, x\}$, $\{b, c, x\}$ and $\{a, c, x\}$ is good. \square

Theorem 1.1 follows from Lemmas 3.2 and 3.3 via Proposition 2.1.

4. Proof of Theorem 1.2

As noticed in Remark 1.9, a proof of Theorem 1.2 is given by Baumeister, Shpectorov and Stroth [3]. However, as [3] has not yet appeared, we will prove that theorem here, for the sake of completeness. Our proof is in fact very similar to that of [3].

The elements of $\Gamma := \text{Far}_\Delta(A)$ are the points of Δ that do not belong to A and the singular subspaces S of Δ such that $S \cap A = \emptyset$. We keep for the elements of Γ the types they have in Δ and their usual names, as *point*, *line*, *plane*, *maximal singular subspace*. We denote by \sim the collinearity relation of Γ keeping the symbol \perp for the collinearity relation of Δ . If a, b are distinct collinear points of Δ , we denote by $\langle a, b \rangle$ the line of Δ through them.

4.1. Preliminaries

If $\{a, b, c\}$ is a triangle of $\mathcal{G}_0(\Gamma)$ not contained in a line, then $\{a, b, c\}$ is contained in a unique plane $\langle a, b, c \rangle$ of Δ . The triangle $\{a, b, c\}$ is good if and only if $\langle a, b, c \rangle \cap A = \emptyset$.

Lemma 4.1 *If $\{a, b, c\}$ is a bad triangle of $\mathcal{G}_0(\Gamma)$, then $\langle a, b, c \rangle \cap A$ is a point.*

Proof: The plane $\langle a, b, c \rangle$ contains three lines of Γ , namely three lines of Δ that miss A . Thus, $\langle a, b, c \rangle \cap A$ cannot contain any line. \square

Lemma 4.2 *Given in Γ a maximal singular subspace S and a point p , then p^\sim contains all points of $p^\perp \cap S$ but at most one.*

Proof: Let S' be the maximal singular subspace of Δ spanned by $\{p\} \cup (p^\perp \cap S)$. Then $A \cap (p^\perp \cap S) = \emptyset$, as $S \cap A = \emptyset$. However, $p^\perp \cap S$ is a hyperplane of S' . Hence $S' \cap A$ contains at most one point. Consequently, at most one of the lines of S' through p is missing in Γ . \square

Lemma 4.3 *The graph $\mathcal{G}_0(\Gamma)$ has diameter 2.*

Proof: Given two points a, b of Γ , let S be a maximal singular subspace of Γ containing b . If $a \in S$, then $a \sim b$. Otherwise, $a^\sim \cap S \neq \emptyset$ by Lemma 4.2. Picked a point $c \in a^\sim \cap S$, we have $a \sim c \sim b$. \square

Lemma 4.4 *Every closed path of $\mathcal{G}_0(\Gamma)$ splits in quadrangles and triangles.*

Proof: By Lemma 4.3, every closed path of $\mathcal{G}_0(\Gamma)$ splits in pentagons, quadrangles and triangles. We shall prove that every pentagon splits in quadrangles and triangles.

Given five points a_0, a_1, a_2, a_3, a_4 of Γ with $a_i \sim a_{i+1}$ for $0 \leq i \leq 4$ (indices computed modulo 5), pick a maximal singular subspace S of Γ on the line $\langle a_2, a_3 \rangle$. By Lemma 4.2,

$a_0^\sim \cap S$ contains at least one point, say b . Thus, the pentagon $\{a_0, a_1, a_2, a_3, a_4\}$ splits in $\{a_0, a_1, a_2, b\}$, $\{a_0, a_4, a_3, b\}$ and $\{a_2, a_3, b\}$. \square

So far, we have not distinguished between the cases of $n > 3$ and $n = 3$, but from now on we must discuss them separately.

4.2. The case of $n > 3$

Suppose Δ has rank $n > 3$.

Lemma 4.5 *Every quadrangle of $\mathcal{G}_0(\Gamma)$ splits in triangles.*

Proof: Given four points a_0, a_1, a_2, a_3 of Γ with $a_i \sim a_{i+1}$ for $i = 0, 1, 2, 3$ (indices computed modulo 4), pick a maximal singular subspace S of Γ containing $\langle a_1, a_2 \rangle$. The sets $a_0^\perp \cap S$ and $a_3^\perp \cap S$ are hyperplanes in the projective geometry $\text{Res}(S)$. Hence $a_0^\perp \cap a_3^\perp \cap S$ contains at least one line L , as $\dim(S) = n - 1 \geq 3$. Clearly, L belongs to Γ , as $S \in \Gamma$. By Lemma 4.2, for $i = 0, 3$ at most one point of L is missing in $a_i^\sim \cap S$. Hence L contains at least one point $b \in a_0^\sim \cap a_3^\sim$. Thus, $\{a_0, a_1, a_2, a_3\}$ splits in four triangles, namely the triangles $\{a_i, a_{i+1}, b\}$ for $i = 0, 1, 2, 3$. \square

Lemma 4.6 *Every bad triangle of $\mathcal{G}_0(\Gamma)$ splits in good triangles.*

Proof: Let $\{a, b, c\}$ be a bad triangle. By Lemma 4.1, $\langle a, b, c \rangle$ meets A in one point p . Note that p does not belong to any of the lines $\langle a, b \rangle$, $\langle b, c \rangle$ or $\langle c, a \rangle$ as all these lines belong to Γ . Let S be a maximal singular subspace of Γ containing $\langle b, c \rangle$. Then $a \notin S$. Consequently, $a^\perp \cap S$ is an $(n - 2)$ -dimensional subspace of S . However, only one point of $a^\perp \cap S$ is missing in a^\sim and, as $p \notin \Gamma$, that point is necessarily the intersection of $\langle b, c \rangle$ with the line $\langle a, p \rangle$ of Δ . As $\dim(a^\perp \cap S) = n - 2 \geq 2$, $a^\sim \cap S$ contains two lines L, M containing b and c respectively and meeting in a point d . Clearly, each of the triangles $\{a, b, d\}$, $\{a, c, d\}$ and $\{b, c, d\}$ is good. \square

The simple connectedness of Γ follows from Lemmas 4.4, 4.5 and 4.6 via Proposition 2.1.

4.3. The case of $n = 3$

In this subsection, $n = 3$ and Δ is neither $\mathcal{S}(5, 2)$ nor $\mathcal{H}(5, 4)$. Namely, every line of Δ belongs to at least four planes. Given a plane P of Δ and a point or a line X , we denote by $p_P(X)$ the plane of Δ containing $X \cup (X^\perp \cap P)$.

Lemma 4.7 *Given a line L of Γ , all but one of the planes of Δ on L belong to Γ . The missing plane meets A in one point.*

Proof: The plane $p_A(L)$ is the missing one. Indeed $L^\perp \cap A$ is a point and that point together with L span $p_A(L)$. \square

Lemma 4.8 *Given in Γ a line L and a point $p \notin L$, one of the following occurs:*

- (1) p and L are coplanar in Γ .
- (2) $p^\sim \cap L = L \setminus \{x\}$ for a point $x \in L$. In this case $p^\sim \cap P = L \setminus \{x\}$ for every plane $P \in \text{Res}_\Gamma(L)$.
- (3) $p^\sim \cap L = \{x\}$ for a point $x \in L$. In this case there exists at most one plane $P_0 \in \text{Res}_\Gamma(L)$ such that $p^\sim \cap P_0 = M \setminus \{x_0\}$ for a line M of P_0 on p and a point $x_0 \in M$. If P is any of the remaining planes of $\text{Res}_\Gamma(L)$, then $p^\sim \cap P$ is a line through x .
- (4) $p^\sim \cap L = \emptyset$. In this case $p^\perp \cap L$ is a point, say x , and for every plane $P \in \text{Res}_\Gamma(L)$, we have $p^\sim \cap P = M \setminus \{x\}$ for a line M of P on x .

Proof: If p and L are coplanar in Δ , then we have either (1) or (2) according to whether the plane of Δ containing L and p belongs to Γ or not (see Lemma 4.7 for the latter case). Suppose $p^\perp \cap L$ is a point, say x , and put $L' := \langle p, x \rangle$.

If $L' \in \Gamma$ then we are in case (3). Given two planes $P_1, P_2 \in \text{Res}_\Gamma(L)$, let $M_i := p^\perp \cap P_i$ and suppose that $p^\sim \cap P_i$ misses a point $x_i \in M_i$, for $i = 1, 2$. Let $L_i := \langle p, x_i \rangle$. As $x_i \not\sim p$, L_i meets A in a point p_i . Clearly, $p_1 \perp p_2$. Hence $x_1 \perp x_2$. Consequently, $L' \cup M_1 \cup M_2$ is contained in a singular subspace of Δ . However, this is impossible, as Δ has rank 3. We avoid this contradiction only assuming that things are as described in the second part of (3).

Finally, suppose $L' \notin \Gamma$. Then, for every plane $P \in \text{Res}_\Gamma(L)$, p and the line $p^\perp \cap P$ are as in case (2) and the situation is as described in (4). \square

Lemma 4.9 *Every quadrangle of $\mathcal{G}_0(\Gamma)$ splits in triangles.*

Proof: Let $\{a_0, a_1, a_2, a_3\}$ be a quadrangle of $\mathcal{G}_0(\Gamma)$, with $a_i \sim a_{i+1}$ and $a_i \not\sim a_{i+2}$ for every $i = 0, 1, 2, 3$ (indices computed modulo 4). Let L be the line of Γ through a_2 and a_3 . We have three cases to examine.

Case 1. $L \subset a_0^\perp \cap a_1^\perp$. Then both pairs (a_0, L) and (a_1, L) are as in case (2) of Lemma 4.8. Hence $a_0 \sim p \sim a_1$ for some point $p \in L$ and the quadrangle splits in the triangles $\{a_0, a_1, p\}$, $\{a_1, a_2, p\}$, $\{a_0, a_3, p\}$ and $\{a_2, a_3, p\}$ (the latter being contained in L).

Case 2. $a_0^\perp \cap L = a_3$ and $a_1^\perp \cap L = a_2$. Then we are in case (3) of Lemma 4.8. By Lemma 4.7, as Δ contains at least four planes on L , the geometry Γ contains at least three planes on L . According to (3) of Lemma 4.8, for at least one of those planes, say P , both $a_0^\sim \cap P$ and $a_1^\sim \cap P$ are lines. These lines meet in a point p . Thus, we can split $\{a_0, a_1, a_2, a_3\}$ in four triangles, namely $\{a_0, a_1, p\}$, $\{a_1, a_2, p\}$, $\{a_0, a_3, p\}$ and $\{a_2, a_3, p\}$.

Case 3. L is contained in a_i^\perp but not in a_j^\perp , for $(i, j) = (0, 1)$ or $(1, 0)$. Suppose $L \subset a_1^\perp$ and $L \cap a_0^\perp = a_3$, to fix ideas. As $a_1 \not\sim a_3$, the line $\langle a_1, a_3 \rangle$ meets A in a point p . As every line of Δ belongs to at least four planes, we can always take a plane P of Δ on p and a_0 , distinct from the plane containing $\{a_0, a_1, a_3\}$ and such that $P \cap A = p_P(a_1) \cap A = p$. If a'_1 is a point of $a_2^\perp \cap P$ different from p , then $a_3 \not\sim a'_1$, because $a_3^\perp \cap P = \langle a_0, p \rangle$. So, if we replace a_1 with a'_1 , we get a quadrangle as in Case 2. \square

Lemma 4.10 *Every bad triangle of $\mathcal{G}_0(\Gamma)$ splits in good triangles.*

Proof: Let $\{a_1, a_2, a_3\}$ be a bad triangle and p the point $\langle a_1, a_2, a_3 \rangle \cap A$ (compare Lemma 4.1). As every line of Δ belongs to at least four planes, we can take a plane $P \in \text{Res}_\Delta(p)$ with the property that $P \cap A = P \cap \langle a_1, a_2, a_3 \rangle = p$ and $p_A(a_i) \cap P = p$ for $i = 1, 2, 3$. Given a line L of P not through p , let P_0 be a plane of Γ on L , far from both A and $\langle a_1, a_2, a_3 \rangle$ in Δ . Suppose that, for $i = 1, 2$ or 3 , a point x of the line $a_i^\perp \cap P_0$ does not belong to a_i^\sim . Then the line $M := \langle a_i, x \rangle$ meets A in a point y . As both p and y belong to A , we have $p \perp y$. Hence $p \perp M$, as $a_i \perp p \perp y$. On the other hand, we also have $p \perp L$ and L meets the line $a_i^\perp \cap P_0$ in a point z .

Suppose $z = x$. Then $M \subset p_P(a_i)$, hence $y \in p_P(a_i)$. Thus, $p_P(a_i)$ contains at least two points of A , namely y and p . Consequently, $p_P(a_i) = p_A(a_i)$, hence $p_A(a_i) \cap P = M$, contrary to the assumption that $p_P(a_i) \cap A = p$. So, $z \neq x$. However, if so, p is collinear with two distinct lines of $p_{P_0}(a_i)$, namely M and the line $\langle a_i, z \rangle$. Hence $p \in p_{P_0}(a_i)$ (as Δ has rank 3). However this is impossible. Therefore, all points of $a_i^\perp \cap P_0$ are collinear with a_i in Γ , namely: $p_{P_0}(a_i) \in \Gamma$.

For $\{i, j, k\} = \{1, 2, 3\}$, put $L_i := \langle a_j, a_k \rangle$. Suppose that $p_{P_0}(L_i) \notin \Gamma$. Then $p_{P_0}(L_i) \cap A$ contains a point $x \in A$. We have $p \perp x$ and $x \notin L_i$. Moreover, $p \perp L_i$. Hence $p \in p_{P_0}(L_i)$, which is impossible. Therefore, $p_{P_0}(L_i) \in \Gamma$.

So far, we have proved that all planes $p_{P_0}(a_i)$ and $p_{P_0}(L_i)$ belong to Γ . Hence, with $b_i = L_i^\perp \cap P_0$, all triangles $\{b_i, a_j, a_k\}$ and $\{a_i, b_j, b_k\}$ are good. As the triangle $\{b_1, b_2, b_3\}$ is also good, we have decomposed $\{a_1, a_2, a_3\}$ in seven good triangles. \square

The conclusion of Theorem 1.2 follows from Lemmas 4.4, 4.9 and 4.10 via Proposition 2.1.

5. Proof of Theorem 1.3

Let Π be the polar space associated to Δ . The structure $\text{Far}_\Pi(A)$ can be defined in the same way as $\text{Far}_\Delta(A)$. However, as Π is non-thick, $\text{Far}_\Pi(A)$ is non-firm. So, $\text{Far}_\Pi(A)$ is not a geometry in the sense of [13]. Actually, most of the theory of [13] (including Theorem 12.64, rephrased as Proposition 2.1 in this paper), also holds for residually connected but non-firm incidence structures, but one should rewrite too many parts of [13] to show this with full evidence. So, we shall argue differently.

Let Φ be the $\{n - 1\}$ -truncation of $\text{Far}_\Pi(A)$. Then Φ is a geometry and the proof of Theorem 1.2 can be recycled to show that Φ is simply connected. Clearly, $\mathcal{G}_0(\Phi) = \mathcal{G}_0(\Gamma)$ where $\Gamma := \text{Far}_\Delta(A)$. All closed paths of $\mathcal{G}_0(\Gamma)$ that are good for Φ are also good for Γ . The simple connectedness of Γ follows from this remark and the simple connectedness of Φ , via Proposition 2.1. \square

6. Proof of Theorem 1.4 in the generic case

Troughout this section $\Gamma := \text{Far}_\Delta(F)$ where Δ is the building of type D_n defined over a given field K and F is a $\{+, -\}$ -flag as in Theorem 1.4, but we assume $n > 5$ when $K = GF(2)$.

6.1. Notation and preliminaries

As usual, we call the elements of Δ of type 0 and 1 *points* and *lines* respectively. The elements of Δ of type + or - are maximal singular subspaces of the polar space associated to Δ . Accordingly, we call them *maximal subspaces* of Δ . For an element X of Δ , let $\sigma(X)$ be the set of points (i.e. 0-elements) of Δ incident to X . We take the liberty to write X for $\sigma(X)$. So, given a set of points S or a point p , we will write $X \subseteq S$ for $\sigma(X) \subseteq S$ and $p \in X$ for $p \in \sigma(X)$.

We denote by U^+ and U^- the two maximal subspaces of Δ forming the flag F , with U^+ of type + and U^- of type -. We set $S := U^+ \cup U^-$ and $S_0 := U^+ \cap U^-$.

The geometry Γ can be described as follows: The 0-elements (points) of Γ are the points of Δ that do not belong to S ; for $i = 1, 2, \dots, n-3$, the i -elements of Γ are the i -elements of Δ that meet S trivially; the elements of Γ of type + and - (which we also call *maximal subspaces* of Γ) are the elements U of Δ of type + or - such that $|U \cap S| = 1$; the incidence relation is inherited from Δ , but for stating that two maximal subspaces X, Y are incident in Γ only if $X \cap Y \cap S = \emptyset$.

We recall that two maximal subspaces U, U' of Δ have the same type if and only if $U \cap U'$ has even codimension in any of U or U' . So, $\dim(U \cap U^+) + \dim(U \cap U^-)$ is odd for every maximal subspace U . In particular, every maximal subspace of Δ meets at least one of U^+ or U^- non-trivially and, if it meets S_0 non-trivially, then it has at least a line in common with at least one of U^+ and U^- . So, the maximal subspaces of Δ that belong to Γ are those which intersect S_0 trivially. Furthermore,

Lemma 6.1 *For every element X of Δ of type $i \leq n-3$ disjoint from both U^+ and U^- , for $\{\varepsilon, \eta\} = \{+, -\}$ there exists a maximal subspace U of Δ such that $U \supset X$, $U \cap U^\varepsilon = \emptyset$ and $|U \cap U^\eta| = 1$.*

Proof: As X is far from U^ε , it is contained in a maximal subspace U far from U^ε , namely such that $U \cap U^\varepsilon = \emptyset$. According to the above, $U \cap U^\eta$ is a point. \square

We now split our proof in two parts. We consider the case of $K \neq GF(2)$ first, obtaining the desired conclusion via Proposition 2.1 after a number of lemmas, as we have done in Sections 3 and 4. A different approach will be used in the case of $K = GF(2)$.

6.2. The case of $K \neq GF(2)$

In the sequel $K \neq GF(2)$. We denote by \sim the collinearity relation of Γ , keeping the symbol \perp for the collinearity relation of Δ . Also, given a clique X of the collinearity graph of Δ , we denote by $\langle X \rangle$ the singular subspace spanned by X in the polar space associated to Δ . In particular, if a, b are collinear points of Δ , then $\langle a, b \rangle$ is the line of Δ through them.

Lemma 6.2 *The graph $\mathcal{G}_0(\Gamma)$ has diameter 2.*

Proof: Given points a, b of Γ , let p be a point of $U^+ \setminus S_0$ non-collinear with any of a or b (such a point exists as $K \neq GF(2)$). Let U be a maximal subspace of Δ with $U \cap U^+ = \{p\}$

and $U \cap U^- = \emptyset$ (Lemma 6.1). Then U belongs to Γ . The point p , which is the unique point of $U \cap S$, belongs neither to $A := a^\perp \cap U$ nor to $B := b^\perp \cap U$. Put $V_a := \langle a, A \rangle$ and $V_b := \langle b, B \rangle$. As A is a hyperplane of V_a disjoint from S , the intersection $V_a \cap U^+$ is either a point or empty. Similarly, $V_a \cap U^-$ is either a point or trivial. So V_a belongs to Γ . Furthermore, one of $V_a \cap U^+$ and $V_a \cap U^-$ is a point and the other one is trivial. Consequently, exactly one of the lines of V_a through a meets S . Similarly, V_b belongs to Γ and exactly one of the lines of V_b through b meets S . Thus, and since $A \cap B$ is at least a line because $n \geq 4$, there is a point $d \in A \cap B$ joined with both a and b by lines avoiding S . \square

Lemma 6.3 *Every closed path of $\mathcal{G}_0(\Gamma)$ splits in quadrangles and triangles.*

Proof: By Lemma 6.2, every closed path of $\mathcal{G}_0(\Gamma)$ splits in pentagons, quadrangles or triangles. It remains to prove that every pentagon splits in quadrangles and triangles.

Let a, b_1, b_2, c_1, c_2 be points of Γ forming a pentagon of $\mathcal{G}_0(\Gamma)$, namely $a \sim b_i \sim c_i$ for $i = 1, 2$ and $c_1 \sim c_2$. Let U be a maximal subspace on $\langle c_1, c_2 \rangle$ disjoint from U^+ and meeting U^- in exactly one point, say p . Put $A := a^\perp \cap U$ and $V := \langle a, A \rangle$. As A is a hyperplane of both U and V and $U \cap U^+ = \emptyset$, the intersection $V \cap U^+$ is at most a point, whence $V \cap U^-$ is at most a line. Thus, the set of points $x \in A$ such that $\langle x, a \rangle \cap S \neq \emptyset$ is contained in the join of a point and a line of A . Consequently, there are points $x \in A$ such that $\langle x, a \rangle \cap S = \emptyset$. \square

Lemma 6.4 *Every quadrangle of $\mathcal{G}_0(\Gamma)$ splits in triangles.*

Proof: Let a_1, a_2, b_1, b_2 be points forming a quadrangle, with $a_i \sim b_j$ for $i, j = 1, 2$. We may assume that neither $a_1 \sim a_2$ nor $b_1 \sim b_2$, otherwise there is nothing to prove. We have three cases to consider.

Case 1 Suppose that $a_1 \not\sim a_2$ and $b_1 \not\sim b_2$. Let U be a maximal subspace on $\langle a_2, b_2 \rangle$ disjoint from U^+ and meeting U^- in exactly one point, say p (such a subspace exists by Lemma 6.1). Put $A := a_1^\perp \cap U$, $B := b_1^\perp \cap U$, $V := \langle a_1, A \rangle$ and $W := \langle b_1, B \rangle$. As A is a hyperplane of V and $V \cap U = A$, the maximal subspaces V and U have different type. So, $V \cap U^+$ is a point, say p_V , and $V \cap U^-$ is either a line or empty. Similarly, W and U have different type, $W \cap U^+$ is a point p_W and $W \cap U^-$ is either a line or empty. Clearly, W and V have the same type. Suppose $V \cap U^-$ is a line, say L_V . Then L_V meets $S_0 = U^+ \cap U^-$ in a point, as S_0 is a hyperplane in U^- . It also meets A in a point, since A is a hyperplane of V . That point must be the same as p , since p is the unique point of $U \cap U^-$. Hence $p \in L_V$. Furthermore, $p_V \in L_V$, as $L_V \cap S_0 \subseteq U^+ \cap V = p_V$. Similarly for W . Thus, the following are the cases that might occur, up to permuting V and W :

- (i) Both $L_V = V \cap U^-$ and $L_W = W \cap U^-$ are lines through p and both $p_V = L_V \cap S_0$ and $p_W = L_W \cap S_0$ are points different from p .
- (ii) $W \cap U^- = \emptyset$, $p_W = W \cap U^+$ is a point, $L_V = V \cap U^-$ is a line through p and $p_V = L_V \cap S_0$ is a point different from both p and p_W .
- (iii) $V \cap U^- = W \cap U^- = \emptyset$, whereas both $p_V = V \cap U^+$ and $p_W = W \cap U^-$ are points.

Clearly, $a_2 \in B$ and $b_2 \in A$. Therefore, $b_1 \notin V$ and $a_1 \notin W$, as $a_1 \not\perp a_2$ and $b_1 \not\perp b_2$. Hence $V \neq W$, the subspace $X := \langle a_1, b_1, A \cap B \rangle$ is maximal and $V \cap W = A \cap B$, of dimension $n - 3$. Also, X has type opposite to that of V and W , as it meets each of them in an $(n - 2)$ -dimensional subspace, namely $\langle A \cap B, a_1 \rangle$ and $\langle A \cap B, b_1 \rangle$. Consequently, X and U have the same type and $X \cap U = A \cap B$.

Assume (i). The lines L_V and L_W are distinct and span a plane $\langle L_V, L_W \rangle$ of Δ contained in U^- . Suppose first $n > 4$. Then a point $c \in A \cap B$ can always be chosen in such a way that each of the lines $\langle c, a_1 \rangle$ and $\langle c, b_1 \rangle$ misses $L_V \cup L_W$. Clearly, $c \neq p$. Furthermore, none of the lines $\langle a_2, c \rangle$ and $\langle b_2, c \rangle$ belongs to $A \cap B$. Hence each of them misses S , as $p \in A \cap B$ is the unique point of $U \cap S$. So, we have decomposed $\{a_1, b_1, a_2, b_2\}$ in four triangles, namely $\{a_1, b_1, c\}$, $\{b_1, a_2, c\}$, $\{a_2, b_2, c\}$ and $\{b_2, a_1, c\}$.

Suppose now that $n = 4$. Then $\dim(X) = 3$ and $A \cap B$ is a line. Furthermore, as X has type opposite to V and $V \cap U^-$ is a line, $X \cap U^-$ is either a plane or a point. If $X \cap U^-$ is a plane, that plane meets the line $\langle a_1, b_1 \rangle$ in a point, contrary to the fact that $\langle a_1, b_1 \rangle$ belongs to Γ . Therefore $X \cap U^-$ is a point, necessarily equal to p , as $p = A \cap B \cap U^-$. Chosen a point $c \neq p$ in the line $A \cap B$, we get a splitting of $\{a_1, b_1, a_2, b_2\}$ in four triangles $\{a_1, b_1, c\}$, $\{b_1, a_2, c\}$, $\{a_2, b_2, c\}$ and $\{b_2, a_1, c\}$ of $\mathcal{G}_0(\Gamma)$, as in the case of $n > 4$.

Suppose that (ii) occurs. As $L_V \cap A \cap B \subseteq V \cap U^- \cap A \cap B \subseteq V \cap U^- \cap W \subseteq U^- \cap W = \emptyset$, we have $L_V \cap A \cap B = \emptyset$. Consequently, $\langle a_1, x \rangle \cap S \neq \emptyset$ for at most one point $x \in A \cap B$. Similarly, $\langle b_2, y \rangle \cap S \neq \emptyset$ for at most one point $y \in A \cap B$ and $\langle b_1, z \rangle \cap S \neq \emptyset$ for at most one point $z \in A \cap B$. Furthermore, $\langle a_2, v \rangle \cap S = \emptyset$ for all points $v \in A \cap B$, as $p_W \notin B$. However, $|A \cap B| \geq 4$. (When $n = 4$, recall that $K \neq GF(2)$.) Therefore, a point $c \in A \cap B$ can be chosen in such a way that each of the lines $\langle a_i, c \rangle$ and $\langle b_i, c \rangle$ misses S . The required decomposition is obtained.

If (iii) holds then the conclusion follows by an argument as above. Things are even easier now and there is no need to recall that $K \neq GF(2)$. We leave the details for the reader.

Case 2. Suppose that $a_1 \perp a_2$ and $b_1 \perp b_2$. Then each of the lines $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ meets S in a point, as $a_1 \not\perp a_2$ and $b_1 \not\perp b_2$ by assumption. Hence the space $X := \langle a_1, b_1, a_2, b_2 \rangle$ meets S in a point (if X is a plane) or a line (when $\dim(X) = 3$). If X is a plane, then $c := \langle a_1, b_1 \rangle \cap \langle a_2, b_2 \rangle$ is a point of Γ . Thus, we can decompose $\{a_1, b_1, a_2, b_2\}$ in $\{a_1, b_1, c\}$, $\{b_1, a_2, c\}$, $\{a_2, b_2, c\}$ and $\{b_2, a_1, c\}$. On the other hand, when $\dim(X) = 3$ we can always pick a point $c \in X$ such that each of the lines $\langle a_1, c \rangle$, $\langle b_1, c \rangle$, $\langle a_2, c \rangle$ and $\langle b_2, c \rangle$ misses S . Again, we obtain a decomposition of $\{a_1, b_1, a_2, b_2\}$ in triangles.

Case 3. The case where only one of the relations $a_1 \perp a_2$ or $b_1 \perp b_2$ holds remains to consider. Suppose $a_1 \perp a_2$ but $b_1 \not\perp b_2$, to fix ideas. Then $p := \langle a_1, a_2 \rangle \cap S$ is a point.

Suppose first $p \in S \setminus S_0$. Without loss, we may assume that $p \in U^-$. So, $p \notin U^+$ as $p \notin S_0$. Pick a maximal subspace U of Δ containing $\langle a_1, a_2, b_2 \rangle$ and such that $U \cap U^- = p$. Then $U \cap U^+ = \emptyset$, as $\langle p, p^\perp \cap U^+ \rangle = U^-$. Given a point c of $(b_1^\perp \cap U) \setminus \langle a_1, a_2 \rangle$, we can decompose $\{a_1, b_1, a_2, b_2\}$ in $\{a_1, b_2, c\}$, $\{a_2, b_2, c\}$ and $\{a_1, b_1, a_2, c\}$, which are triangles

and quadrangles of $\mathcal{G}_0(\Gamma)$. (Recall that $U \cap S = p \in \langle a_1, a_2 \rangle$.) However, the quadrangle $\{a_1, b_1, a_2, c\}$ is as in Case 2. So, it admits a decomposition in triangles.

Suppose that $p \in S_0$. By Lemma 6.1 applied to $\text{Res}_\Delta(p)$, there exists a maximal subspace U of Δ containing $\langle a_1, a_2, b_2 \rangle$ and such that $U \cap U^+ = p$ and $L := U \cap U^-$ is a line through p . Put $B := U \cap b_1^\perp$ and $V := \langle b_1, B \rangle$. The line $\langle a_1, a_2 \rangle$ is contained in B and $B \cap S$ is either L or p . If $L \cap B = p$, then $V \cap U^- = p$ and $V \cap U^+$ is a line M through p with $M \cap U^- = p$. Indeed V and U belong to opposite families, U meets U^- in a line and B , which is a hyperplane of V , has only one point in common with U^- , namely p . In this case we can always pick a point $c \in B \setminus \langle a_1, a_2 \rangle$ in such a way that $\langle c, b_2 \rangle$ misses L and $\langle c, b_1 \rangle$ misses M . Thus, $\{a_1, b_1, a_2, b_2\}$ splits in the triangles $\{a_1, b_1, c\}$, $\{b_1, a_2, c\}$, $\{a_2, b_2, c\}$ and $\{b_2, a_1, c\}$.

Finally, suppose $L \subset B$. Then $V \cap U^-$ is a plane containing L , because U and V belong to different families. However, V contains lines of Γ and these lines miss the plane $V \cap U^-$. Therefore $\dim(V) \geq 4$. Furthermore, $V \cap S_0 = V \cap U^- \cap U^-$. So, $V \cap S$ is equal to the plane $V \cap U^-$. As $\dim(V) \geq 4$ and B is a hyperplane of V , the space B contains a point c that does not belong to the join of the planes $B \cap \langle V \cap U^-, b_1 \rangle$, $B \cap \langle V \cap U^-, a_1 \rangle$ and $B \cap \langle V \cap U^-, a_2 \rangle$. (When $n = 4$, recall that $K \neq GF(2)$.) For such a point c , each of the lines $\langle c, a_1 \rangle$, $\langle c, a_2 \rangle$, $\langle c, b_1 \rangle$ and $\langle c, b_2 \rangle$ misses S . Again, we have decomposed $\{a_1, b_1, a_2, b_2\}$ in triangles. \square

We say that a triangle $\{a, b, c\}$ of $\mathcal{G}_0(\Gamma)$ is *degenerate* if it is contained in a line of Γ . Clearly, all degenerate triangles are good.

Lemma 6.5 *We have $|\langle a, b, c \rangle \cap S_0| \leq 1$ for every non-degenerate triangle $\{a, b, c\}$ of $\mathcal{G}_0(\Gamma)$, with equality holding if and only if $\{a, b, c\}$ is bad.*

Proof: As the plane $\langle a, b, c \rangle$ contains lines that do not meet S , the intersection $\langle a, b, c \rangle \cap S_0$ is at most a point. Let U be a maximal singular subspace of Δ containing $\langle a, b, c \rangle$. Suppose that $\langle a, b, c \rangle \cap S_0 = \emptyset$. Then $\langle a, b, c \rangle$ is disjoint from at least one of U^+ and U^- . Consequently, it is contained in a maximal subspace of Δ that is disjoint from either U^+ or U^- and, therefore, belongs to Γ . In this case, $\{a, b, c\}$ is good. On the other hand, when $\langle a, b, c \rangle \cap S_0 \neq \emptyset$ none of the singular subspaces of Δ containing $\langle a, b, c \rangle$ is disjoint from any of U^+ or U^- . In that case $\{a, b, c\}$ is bad. \square

Lemma 6.6 *Every bad triangle of $\mathcal{G}_0(\Gamma)$ admits a decomposition in good triangles.*

Proof: Let $\{a, b, c\}$ be a bad triangle and $p := \langle a, b, c \rangle \cap S$. By Lemma 6.5, $p \in S_0$. By Lemma 6.1 in $\text{Res}_\Delta(p)$, we can choose a maximal subspace U of Δ containing $\langle a, b, c \rangle$ and such that $L := U \cap U^-$ is a line through p and $U \cap U^+ = p$. As $\dim(U) \geq 3$ and $K \neq GF(2)$, we can also pick a point $d \in U \setminus \langle a, b, c \rangle$ such that none of the planes $\langle d, a, b \rangle$, $\langle d, b, c \rangle$ and $\langle d, c, a \rangle$ contains p and each of the lines $\langle d, a \rangle$, $\langle d, b \rangle$ and $\langle d, c \rangle$ misses L . So, the triplets $\{d, a, b\}$, $\{d, b, c\}$ and $\{d, c, a\}$ are triangles of $\mathcal{G}_0(\Gamma)$. They are good, by Lemma 6.5. \square

The simple connectedness of Γ follows from Lemmas 6.3, 6.4 and 6.5 via Proposition 2.1.

6.3. The case of $K = GF(2)$ with $n > 5$

In this subsection, $K = GF(2)$ but $n > 5$. The proof we will give for this case makes no use of Proposition 2.1. We will recognize certain configurations in Γ that don't break when lifted to covers. Via them, we will be able to prove that the collinearity graph of the universal cover of Γ has diameter 2. The conclusion will follow.

Recall that, given a maximal subspace U of Δ , the mapping sending $X \in \text{Res}_\Delta(U)$ to the set $X \cap U$ is injective. In view of this, in the sequel we will freely indentify a flag $\{X, U\}$ with the intersection $X \cap U$ whenever this minor abuse will be convenient.

Lemma 6.7 *Given a line L of Γ and distinct points $a_1, a_2 \in L$, for $i = 1, 2$ let L_i be a line of Γ incident to a_i and distinct from L . Then one of the following occurs:*

- (1) *Some of the maximal subspaces of Γ contains all of L , L_1 and L_2 .*
- (2) *Modulo permuting L_1 and L_2 , there exist a maximal subspace U of Γ containing $L \cup L_1$ and planes A_1, A_2 of Γ such that $L_i \subset A_i$ for $i = 1, 2$, $A_1 \subset U$, $U \cap A_2$ is a line and A_1 has at least one point in common with the line $U \cap A_2$.*
- (3) *There exist a maximal subspace U of Γ containing L and planes A_1, A_2 of Γ such that $L_i \subset A_i$, $A_i \cap U$ is a line for $i = 1, 2$ and the lines $A_1 \cap U$ and $A_2 \cap U$ meet in a point.*

Proof: Let \mathcal{U} be the family of maximal subspaces of Δ containing L and disjoint from U^+ . Clearly, the elements of \mathcal{U} belong to Γ . Suppose some $U \in \mathcal{U}$ contains L_1 . If U also contains L_2 then we are in case (1).

Assuming $L_2 \not\subset U$, put $U' := \langle L_2^\perp \cap U, L_2 \rangle$. As $U \cap U^+ = \emptyset$ and $\{t(U), t(U')\} = \{+, -\}$, $U' \cap U^+$ is a point p_0 and $U' \cap U^-$ is either empty or a line through p_0 . However, $\dim(U') = n - 1 \geq 5$. Hence U' contains at least $m := 2^{n-2} - 4 \geq 12$ planes X_1, X_2, \dots, X_m on L_2 with no points in $S = U^+ \cup U^-$. As $U \cap U'$ is a hyperplane of U' , each of those planes meets $U \cap U'$ in a line. For $i = 1, 2, \dots, m$, put $M_i := X_i \cap U \cap U' = X_i \cap U$. Let V be a maximal subspace of Γ incident with L_1 and forming a $\{+, -\}$ -flag with U . The dual of $\text{Res}_\Gamma(U)$ is an affine geometry and we may regard $U \cap V$ as a point of that affine geometry. Therefore, for $i = 1, 2, \dots, m$ either $U \cap V \supset M_i$ or $V \cap U \cap M_i$ is a point. Suppose first that $U \cap V$ contains one of the lines M_i , say $U \cap V \supset M_1$ to fix ideas. Then both L_1 and M_1 belong to $\text{Res}_\Gamma(U, V)$. As $\text{Res}_\Gamma(U, V)$ is a projective geometry, any point of M_1 is coplanar with L_1 and we have case (2) (with $A_2 = X_1$).

Assume now that none of the lines M_i is contained in $U \cap V$. Then $p_i := M_i \cap U \cap V$ is a point. Suppose first $p_i = p_j$ for two distinct indices i, j . Then $p_i = p_j = a_2$, as $M_i \neq M_j$ and them both are incident to a_2 . Hence $a_2 \in U \cap V$. In $\text{Res}_\Gamma(U, V)$ we find a plane containing L_1 and $a_2 (= p_i = p_j)$ and we have case (2) with any of X_i or X_j taken as A_2 .

Suppose the points p_1, p_2, \dots, p_m are pairwise different. Denoted by \mathcal{A} the dual of $\text{Res}_\Gamma(U)$, we have $\mathcal{A} \cong AG(n-1, 2)$, $U \cap V$ is a point of \mathcal{A} and p_1, p_2, \dots, p_m appear as hyperplanes of \mathcal{A} on the point $U \cap V$. The line L_1 , regarded as an element of \mathcal{A} , is a subspace of codimension 2. As the hyperplanes p_1, p_2, \dots, p_m are mutually non-parallel, the space L_1 is non-parallel with at least $m - 3 \geq 9$ of them. Therefore, some p_i meets L_1 in a $(n-4)$ -space of \mathcal{A} . Namely, the point p_i and the line L_1 are coplanar in $\text{Res}_\Gamma(U)$, as in case (2).

If some members of \mathcal{U} contain L_2 , then we obtain the same conclusion as above, but permuting L_1 and L_2 . So, suppose that no member of \mathcal{U} contains any of L_1 or L_2 . Consequently, none of L_1 or L_2 is coplanar with L in Γ . Given $U \in \mathcal{U}$, for $i = 1, 2$ let V_i be the maximal subspace of Δ containing L_i and forming a $\{+, -\}$ -flag with U . (The element V_i is uniquely determined inside $\text{Res}_\Delta(a_i)$.) The subspace V_i meets $S = U^+ \cup U^-$ in either a point or a line. Consequently, for $i = 1, 2$ there exist at least $m = 2^{n-2} - 4$ planes $X_{i,1}, X_{i,2}, \dots, X_{i,m}$ of Γ contained in V_i , containing L_i and meeting U in a line. For $j = 1, 2, \dots, m$, put $M_{i,j} := X_{i,j} \cap U$. Clearly, $M_{i,j} \neq M_{i,h}$ if $j \neq h$, as $X_{i,j} \cap X_{i,h} = L_i \not\subseteq U$. Furthermore, $M_{i,j} \neq L$, as L_i and L are non-coplanar in Γ . Denoted by \mathcal{A}_i the dual of $\text{Res}_\Gamma(a_i, U)$, we have $\mathcal{A}_i \cong AG(n-2, 2)$ and $M_{i,1}, M_{i,2}, \dots, M_{i,m}$ are hyperplanes of \mathcal{A}_i containing the point $V_i \cap U$ of \mathcal{A}_i . At most one of those hyperplanes is parallel to L in \mathcal{A}_i . To fix ideas, suppose $M_{i,j}$ is non-parallel to L in \mathcal{A}_i for $j = 1, 2, \dots, m-1$ and let $Y_{i,j}$ be the meet of L and $M_{i,j}$ in \mathcal{A}_i . So, $Y_{i,j}$ is the plane of Γ containing $M_{i,j} \cup L$.

If $Y_{i,j} = Y_{i,h}$ for $j \neq h$ then $Y_{i,j}$ has at least two lines in common with $U \cap V_i$, namely $M_{i,j}$ and $M_{i,h}$. Therefore $Y_{i,j} \subset U \cap V_i$ and, consequently, $L \subset U \cap V_i$. Furthermore, $\{M_{i,j}, M_{i,h}, L\}$ is the triple of lines incident to $Y_{i,j}$ and a_i . Conversely, suppose $L \subset U \cap V_i$ and for every $j = 1, 2, \dots, m-1$, let Z_j be the 3-space of $\text{Res}_\Delta(V_i)$ containing $Y_{i,j}$ and L_i . As Z_j contains a plane of Γ , namely $Y_{i,j}$, Z_j has at most one point in S (recall that $V_i \cap S$ is either a point or a line). On the other hand, Z_j contains the plane of Δ spanned by L and L_i . By assumption, that plane does not belong to Γ , hence it contains a point of S . Thus, Z_j meets S in one point, which belongs to the plane spanned by L_i and L . Consequently, Z_j contains two planes of Γ on L_i . The plane $Y_{i,j}$ is one of those two. The other plane is $Y_{i,f_i(j)}$ for a uniquely determined index $f_i(j) \neq j$. Thus, one of the following holds:

- (*) $L \not\subseteq V_i$ and the planes $Y_{i,1}, Y_{i,2}, \dots, Y_{i,m-1}$ are pairwise distinct;
- (**) $L \subset V_i$, and $Y_{i,j} = Y_{i,f_i(j)}$ for an involutory fixed-point-free permutation f_i of $\{1, 2, \dots, m\}$.

(Note that in case (**)) none of the lines $M_{i,j}$ is parallel to L in \mathcal{A}_i , as $m-1$ is odd.) Assume case (*), with $i = 1$ to fix ideas. Suppose that $Y_{1,j} \neq Y_{2,h}$ for any $j = 1, 2, \dots, m-1$ and $h = 1, 2, \dots$. Then we get at least $m-1 + m/2 \geq 2^{n-2} - 5 + 2^{n-3} - 2 = 2^{n-2} + 2^{n-3} - 7$ planes on L in $\text{Res}_\Gamma(U)$. But $\text{Res}_\Gamma(U)$ contains exactly $2^{n-2} - 2$ planes on L . Therefore, $2^{n-3} \leq 7$, which forces $n \leq 5$, contrary to the assumption $n > 5$. So, $Y_{1,j} = Y_{2,h}$ for suitable indices j and h , whence the lines $M_{1,j}$ and $M_{2,h}$ are coplanar and, therefore, they meet in a point. We have case (3) with $A_1 = X_{1,j}$ and $A_2 = X_{2,h}$.

Finally, suppose we have (**) for both $i = 1$ and $i = 2$. We may assume to have chosen indices in such a way that $f_1(i) = f_2(i) = i + m/2$ for $i = 1, 2, \dots, m/2$. So, $Y_{1,1}, \dots, Y_{1,m/2}$ and $Y_{2,1}, \dots, Y_{2,m/2}$ are the planes to consider. Note that $L \subset U \cap V_1 \cap V_2$, as $L \subset U \cap V_1$ and $L \subset U \cap V_2$. For $i = 1, 2$, there are exactly $2^{n-3} - 1$ planes on L in $\text{Res}_\Delta(U \cap V_i)$. As $m/2 \geq 2^{n-3} - 2$, at most one of those planes is missing in the family $\mathcal{Y}_i := \{Y_{i,j}\}_{j=1}^{m/2}$. However, as $n > 5$ by assumption, there are exactly $2^{n-4} - 1 \geq 3$ planes of Δ between $U \cap V_1 \cap V_2$ and L . By the above, at least $2^{n-4} - 2$ of them come from the family \mathcal{Y}_i . As $2 \cdot (2^{n-4} - 2) > 2^{n-4} - 1$, some of those planes belongs to $\mathcal{Y}_1 \cap \mathcal{Y}_2$. Suppose

$Y_{1,1} = Y_{2,1}$ is such a plane. In $\text{Res}_\Gamma(Y_{1,1})$ we see that the lines $M_{1,1}$ and $M_{2,1}$ meet in a point. Thus, we have case (3) with $A_1 = X_{1,1}$ and $A_2 = X_{2,1}$. \square

Let $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ be the universal covering of Γ .

Lemma 6.8 *The collinearity graph $\mathcal{G}_0(\tilde{\Gamma})$ of $\tilde{\Gamma}$ has diameter 2.*

Proof: Assuming that $\tilde{\Gamma}$ contains a pair of points at distance 3, we will show that those points have distance ≤ 2 . That contradiction will establish our Lemma.

Given two points \tilde{b}_1, \tilde{b}_2 of $\tilde{\Gamma}$ at distance 3, let $(\tilde{b}_1, \tilde{a}_1, \tilde{a}_2, \tilde{b}_2)$ be a path of $\mathcal{G}_0(\tilde{\Gamma})$ from \tilde{b}_1 to \tilde{b}_2 . For $i = 1, 2$ let \tilde{L}_i be the line through \tilde{b}_i and \tilde{a}_i and \tilde{L} the line through \tilde{a}_1 and \tilde{a}_2 . As φ is a covering, the line $L := \varphi(\tilde{L})$ is different from $L_i := \varphi(\tilde{L}_i)$ and the points $a_1 := \varphi(\tilde{a}_1)$ and $a_2 := \varphi(\tilde{a}_2)$ are different. Thus, a_1, a_2, L, L_1 and L_2 are as in the hypotheses of Lemma 6.7. Suppose case (1) of that lemma occurs and let U be a maximal subspace of Γ incident to L, L_1 and L_2 . As φ is a covering, U lifts via φ to a uniquely determined element \tilde{U} of $\tilde{\Gamma}$ on \tilde{L} . As φ isomorphically maps $\text{Res}(\tilde{L}_i)$ onto $\text{Res}_\Gamma(L_i)$ and $L_i \subset U$, the line \tilde{L}_i is incident to \tilde{U} . Thus, both \tilde{b}_1 and \tilde{b}_2 are incident to \tilde{U} . However, $\text{Res}(\tilde{U})$ is isomorphic to the dual of $AG(n - 1, 2)$ and the collinearity graph of a dual affine geometry has diameter 2, contrary to the hypothesis that \tilde{b}_1 and \tilde{b}_2 have distance 3.

So, case (1) of Lemma 6.7 cannot occur. Assume that (2) occurs, namely there exist a maximal subspace U of Γ containing $L \cup L_1$ and planes A_1, A_2 of Γ such that $L_i \subset A_i$ for $i = 1, 2, A_1 \subset U, M := U \cap A_2$ is a line and $A_1 \cap M$ contains a point p . Let $\tilde{U}, \tilde{A}_1, \tilde{A}_2$ and \tilde{M} be the preimages of U, A_1, A_2 and M in $\text{Res}(\tilde{L}), \text{Res}(\tilde{L}_1), \text{Res}(\tilde{L}_2)$ and $\text{Res}(\tilde{a}_2)$. As φ induces an isomorphism from $\text{Res}(\tilde{a}_i)$ to $\text{Res}_\Gamma(a_i)$, both \tilde{A}_1 and \tilde{M} are incident to \tilde{U} and \tilde{M} is incident to \tilde{A}_2 . Again, φ induces an isomorphism from $\text{Res}(\tilde{U})$ to $\text{Res}_\Gamma(U)$. Hence the point $p \in A_1 \cap M$ lifts via φ to a point \tilde{p} incident to \tilde{A}_1 and \tilde{M} , whence to \tilde{A}_2 too, as \tilde{M} is incident to \tilde{A}_2 . As we can see in $\text{Res}(\tilde{A}_1)$ and $\text{Res}(\tilde{A}_2)$, the point \tilde{p} is collinear with both \tilde{b}_1 and \tilde{b}_2 , contrary to the assumption that \tilde{b}_1 and \tilde{b}_2 have distance 3.

So, case (2) of Lemma 6.7 is also impossible. Case (3) remains to consider: there exist a maximal subspace U containing L and planes A_1, A_2 such that $L_i \subset A_i, M_i := A_i \cap U$ is a line and the lines M_1 and M_2 meet in a point p . The elements U, A_i and M_i lift to a maximal element \tilde{U} incident to \tilde{L} , a plane \tilde{A}_i incident to \tilde{L}_i and a line \tilde{M}_i incident to \tilde{a}_i and, as φ induces on $\text{Res}(\tilde{a}_i)$ an isomorphism to $\text{Res}_\Gamma(a_i)$, the line \tilde{M}_i is incident to both \tilde{A}_i and \tilde{U} . Similarly, the point p lifts to a unique point \tilde{p} of $\text{Res}(\tilde{U})$ incident to both \tilde{M}_1 and \tilde{M}_2 . For $i = 1, 2$ the points \tilde{p} and \tilde{b}_i are collinear in $\text{Res}(\tilde{A}_i)$. Once again, we have reached a contradiction. \square

We can now finish the proof. By way of contradiction, suppose φ is not an isomorphism. Then $\varphi(\tilde{a}_1) = \varphi(\tilde{a}_2)$ for distinct points \tilde{a}_1 and \tilde{a}_2 of $\tilde{\Gamma}$. The points \tilde{a}_1 and \tilde{a}_2 are distinct and non-collinear, as φ is a covering. However, by Lemma 6.8, there is a point \tilde{c} collinear with both \tilde{a}_1 and \tilde{a}_2 . For $i = 1, 2$ let \tilde{L}_i be the line through \tilde{a}_i and \tilde{c} . Clearly, $\tilde{L}_1 \neq \tilde{L}_2$. Hence the lines $L_1 := \varphi(\tilde{L}_1)$ and $L_2 := \varphi(\tilde{L}_2)$ are different, as φ induces an injective mapping on $\text{Res}(\tilde{c})$. Thus, we have two distinct lines L_1, L_2 of Γ with two distinct points

in common, namely $c := \varphi(\tilde{c})$ and $a := \varphi(\tilde{a}_1) = \varphi(\tilde{a}_2)$. This is impossible. Therefore φ is an isomorphism. The theorem is proved in this case too. \square

Remark 6.9 The hypothesis $K = GF(2)$ is not strictly necessary for the above, but we need K to be finite, in view of some arithmetical arguments used in the proof of Lemma 6.7. On the other hand, K was allowed to be infinite in Section 6.2.

7. Proof of Theorem 1.5

The basic idea of this proof resembles that used for Theorem 1.4 in the case of $K = GF(2)$: we shall prove that a certain subgraph of the incidence graph of the universal cover has a very small diameter and we will exploit that fact to show that the covering is an isomorphism.

Firstly, we state some terminology. Given Δ as in Theorem 1.5, we call the elements of Δ of type 0 and 1 *points* and *lines*, respectively, writing $x \perp y$ for two points x, y when they are collinear. With points and lines chosen as above, $\mathcal{G}_0(\Delta)$ is the collinearity graph of Δ . It is well known that any two points of Δ are incident with at least one common 4-element [13, Section 7.6.1]. Hence $\mathcal{G}_0(\Delta)$ has diameter 2.

The elements of Δ may be regarded as distinguished subspaces of the point-line system of Δ . Accordingly, given an element A of Δ and a point x or a subset X of the point-set of Δ , we write $x \in A$ if x is incident to A , $X \subseteq A$ if all points of X are incident to A , and so on. The claims gathered in the next lemma are straightforward.

Lemma 7.1 *All the following hold for a point p of Δ :*

- (1) *A point x of Δ is non-collinear with p if and only if there exists a unique 4-element of Δ incident to both x and p .*
- (2) *For $1 \leq i \leq 4$, an i -element X of Δ is far from p if and only if $X \cap p^\perp = \emptyset$.*
- (3) *A 5-element X of Δ is far from p if and only if $|X \cap p^\perp| = 1$.*

Given a point p of Δ , put $\Gamma := \text{Far}_\Delta(p)$. The next Lemma easily follows from Lemma 7.1.

Lemma 7.2 *All the following hold:*

- (1) *We have $\text{Res}_\Gamma(X) = \text{Res}_\Delta(X)$ for every 4-element X of Γ .*
- (2) *Given a point x of Γ , let A_x be the unique 4-element of Δ incident to both x and p (compare Lemma 7.1(1)). Then $\text{Res}_\Gamma(x)$ is the subgeometry of $\text{Res}_\Delta(x)$ far from A_x .*
- (3) *Given a 5-element X , let p_X be the unique point of X collinear with p (compare Lemma 7.1(3)). Then $\text{Res}_\Gamma(X)$ is the dual affine space obtained from $\text{Res}_\Delta(X)$ by removing p_X and all elements incident to it.*

Lemma 7.3 *Let $\tilde{\Gamma}$ be a geometry over the set of types $\{0, 1, 2, 3, 4, 5\}$ where the residues of the elements of type 0, 4 and 5 are as in Γ . Then for any two 4-elements of $\tilde{\Gamma}$ there exists a 0-element incident to them both.*

Proof: We shall prove that, given three 4-elements A, B, C of $\tilde{\Gamma}$, if there exist a 0-element a incident to both A and B and a 3-element S incident to both B and C , then there exists

a 0-element that is incident to both A and C . Once we have proved this, the conclusion follows by considering for every 4-element X a path of type $(4, 3, 4, \dots, 3, 4)$ from B to X in the incidence graph of $\tilde{\Gamma}$.

Let A, B, C, a, S be as above. By assumption, $\text{Res}(B)$ is a building of type D_5 . Hence there exists a 5-element $U \in \text{Res}(B)$ incident to both a and S . On the other hand, $\text{Res}(a)$ is isomorphic to the subgeometry of a building Δ_a of type D_5 far from a given point A_a of Δ_a . Denoted by \sim the collinearity relation of Δ_a and regarded U as a maximal singular subspace of the polar space Π_a associated to Δ_a , suppose first that $A \sim \cap U \subseteq A_a \sim$. Then the 3-space $X := A \sim \cap U = A_a \sim \cap U$ is contained in three maximal singular subspaces of Π_a , namely U , $\langle X, A \rangle$ and $\langle X, A_a \rangle$. This is impossible, as every singular 3-space of Π_a is contained in exactly two maximal singular subspaces. Hence $A \sim \cap U \not\subseteq A_a \sim$.

In Π_a , given a point $A' \in A \sim \cap U \setminus A_a \sim$, let S' be the line through A and A' . In the residue of A' in Π_a we find a maximal singular subspace U' incident to S' and intersecting U in a 3-space. Clearly, U' is a 1-element of $\tilde{\Gamma}$. We now turn to $\text{Res}(U)$, where both U' and C live. The dual of $\text{Res}(U)$ is an affine geometry and, in that affine geometry, C and U' are a point and a 3-space respectively. Hence there exists a hyperplane h of that affine geometry incident to both C and U' . The element h is in fact a 0-element of $\tilde{\Gamma}$, it is incident to C and, as it is incident to U' , it is incident to A , too. The claim is proved. \square

Now we can finish the proof of Theorem 1.5. Let $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ be a covering and suppose that $\varphi(A) = \varphi(B)$ for two 4-elements A, B of $\tilde{\Gamma}$. By Lemma 7.3, there exists a 0-element a incident to both A and B . As φ is a covering, its restriction to $\text{Res}(a)$ is injective. Hence $A = B$. So, φ is injective on the set of 4-elements. This forces φ to be an isomorphism. \square

8. Proof of Corollary 1.7

A geometry Γ of rank $n \geq 3$ is said to be *residually simply connected* if it is simply connected and, when $n > 3$, all residues of Γ of rank at least 3 are simply connected. Clearly, a residually simply connected geometry is also 2-simply connected. So, we only need to prove the following:

Lemma 8.1 *All far-away geometries considered in Theorems 1.1, 1.3 and 1.5 are residually simply connected. The geometries of Theorem 1.2 are residually connected provided that Δ is neither $\mathcal{S}(2n-1, 2)$ nor $\mathcal{H}(2n-1, 4)$. The geometries of Theorem 1.4 are residually connected provided that $\Delta \neq D_n(2)$.*

Proof: The main step of the proof is to show that the residues of the flags of the considered far-away geometry Γ are either far-away geometries of the same kind as Γ , but of lesser rank, or projective or affine geometries, or affine polar spaces (Section 1.1, Example 1.3), or affine-dual-affine geometries of rank 3 (Section 1.1, Example 1.2), or direct sums of geometries as above. Once this is proved, the conclusion follows. Indeed projective geometries, affine geometries, affine polar spaces, affine-dual-affine geometries and direct sums are simply connected whereas, when smaller far-away geometries of the same kind as Γ are involved, we can argue by induction.

We are not going to examine all cases one by one. We shall only consider one of them, just to show how to do, leaving the remaining cases for the reader.

Given Δ and F as in Theorem 1.4, let p be a point of $\Gamma = \text{Far}_\Delta(F)$ and U^+, U^- the two maximal subspaces of Δ forming F . As p belongs to Γ , $p \notin U^+ \cup U^-$. Put $\Delta_p := \text{Res}_\Delta(p)$ and $U_p^\varepsilon := \langle p, p^\perp \cap U^\varepsilon \rangle$ for $\varepsilon = +, -$. Then $F_p := \{U_p^+, U_p^-\}$ is a flag of Δ_p of type $\{+, -\}$ and $\text{Res}_\Gamma(p) \cong \text{Far}_{\Delta_p}(F_p)$. \square

9. Proof of claims (1) and (2) of Theorem 1.6

Throughout this section $\Gamma_{1,n} := \text{Far}_\Delta(A)$ for $\Delta = \mathcal{S}(2n - 1, 2)$ and $\Gamma_{2,n} := \text{Far}_\Delta(A)$ for $\Delta = \mathcal{H}(2n - 1, 4)$, with A an $(n - 1)$ -element of Δ in both cases and $n \geq 3$.

9.1. Preliminaries

It is known [16, Lemmas 6.1 and 6.5] that the dual of $\Gamma_{1,n}$ is isomorphic to the expansion $\text{Exp}(\varepsilon_{1,n})$ of a $GF(2)$ -embedding $\varepsilon_{1,n} : PG(n - 1, 2) \rightarrow V(\binom{n+1}{2}, 2)$, called the *tensor embedding*, and the dual of $\Gamma_{2,n}$ is isomorphic to the expansion of a $GF(2)$ -embedding $\varepsilon_{2,n} : PG(n - 1, 4) \rightarrow V(n^2, 2)$, called the *twisted tensor embedding*. Referring the reader to [16, Section 6] for more details, we recall the definitions of $\varepsilon_{1,n}$ and $\varepsilon_{2,n}$.

Definition of $\varepsilon_{1,n}$ For a nonzero vector $v \in V = V(n, 2)$, $\varepsilon_{1,n}$ sends the point $\langle v \rangle$ of $PG(n - 1, 2)$ to the vector $v \otimes v$. These vectors span an $\binom{n+1}{2}$ -dimensional subspace V_1 of $V \otimes V$. In fact, if $\{e_i\}_{i=1}^n$ is a basis of V , then

$$\{e_i \otimes e_i\}_{i=1}^n \cup \{e_i \otimes e_j + e_j \otimes e_i\}_{i < j}$$

is a basis of V_1 . The additive group of V_1 can also be recovered inside the stabilizer of A in $S_{2n}(2) = \text{Aut}(\Delta)$ as the unipotent radical U of that parabolic subgroup. The group U acts regularly on the set of 3-elements of $\Gamma_{1,n}$. The isomorphism $\Gamma_{1,n} \cong \text{Exp}(\varepsilon_{1,n})$ is implicit in that fact.

Definition of $\varepsilon_{2,n}$ Given a basis $\{e_i\}_{i=1}^n$ of $V = V(n, 4)$, for every non-zero vector $v = \sum_{i=1}^n t_i e_i$ of V we put $\bar{v} := \sum_{i=1}^n t_i^2 e_i$ and $\varepsilon_{2,n}(\langle v \rangle) := v \otimes \bar{v}$. As $(tv) \otimes \overline{(tv)} = t^3 v \otimes \bar{v} = v \otimes \bar{v}$ for every non-zero scalar t , $\varepsilon_{2,n}$ is indeed a mapping from the set of points of $PG(n - 1, 4)$ to the set of non-zero vectors of $V \otimes V$. Regarded the latter as a $2n^2$ -dimensional $GF(2)$ -vector space, the vectors $v \otimes \bar{v}$ span an n^2 -dimensional subspace V_2 in it. Explicitly, given $\omega \in GF(4) \setminus GF(2)$, the vectors $e_{i,j} := e_i \otimes e_j$ and $f_{i,j} := \omega e_i \otimes e_j$ form a $GF(2)$ -basis of $V \otimes V$ and the set

$$\{e_{i,i}\}_{i=1}^n \cup \{e_{i,j} + e_{j,i}\}_{i < j} \cup \{e_{i,j} + f_{j,i} + f_{j,i}\}_{i < j}$$

is a basis of V_2 . The additive group of V_2 is isomorphic to the unipotent radical U of the stabilizer of A in $U_{2n}(2) = \text{Aut}(\Delta)$. The group U acts regularly on the set of 3-elements of $\Gamma_{2,n}$ and this fact implies that $\Gamma_{2,n} \cong \text{Exp}(\varepsilon_{2,n})$.

By Proposition 2.5 we immediately obtain the following, which in principle solves the problem of determining the universal 2-covers of $\Gamma_{1,n}$ and $\Gamma_{2,n}$:

Lemma 9.1 *For $i = 1, 2$, the universal 2-cover of $\Gamma_{i,n}$ is isomorphic to the expansion $\text{Exp}(\tilde{\varepsilon}_{i,n}^{(1)})$ of the 1-hull $\tilde{\varepsilon}_{i,n}^{(1)}$ of $\varepsilon_{i,n}$.*

As stated in Theorem 1.2, if $n > 3$ then $\Gamma_{i,n}$ is simply connected. Hence $\tilde{\varepsilon}_{i,n} = \varepsilon_{i,n}$ when $n > 3$. On the other hand, $\tilde{\varepsilon}_{i,n} = \tilde{\varepsilon}_{i,n}^{(1)}$ when $n = 3$ (recall that covers and 2-covers coincide in the rank 3 case). So, there is no need to study $\tilde{\varepsilon}_{i,n}$ here.

9.2. The universal 2-cover of $\Gamma_{1,n}$ and proof of claim (1) of Theorem 1.6

Proposition 9.2 *The universal 2-cover of $\Gamma_{1,n}$ is the expansion of the free embedding of $PG(n-1, 2)$ in $V(2^n-1, 2)$.*

Proof: Given a line L of $PG(n-1, 2)$, we have $L = \langle v_1, v_2 \rangle$ for two non-zero vectors v_1, v_2 of $V(n, 2)$. Hence

$$\varepsilon_{1,n}(L) = \{v_1 \otimes v_1, v_2 \otimes v_2, v_1 \otimes v_2 + v_2 \otimes v_1\}.$$

The vectors $v_1 \otimes v_1, v_2 \otimes v_2$ and $v_1 \otimes v_2 + v_2 \otimes v_1$ are independent. The conclusion follows from Lemma 9.1 and Proposition 2.8. \square

In view of Proposition 9.2 and since the codomain of $\varepsilon_{1,n}$ is $\binom{n+1}{2}$ -dimensional, the universal 2-cover of $\Gamma_{1,n}$ is a 2^k -fold cover where $k = 2^n - 1 - \binom{n+1}{2}$. In particular, the universal cover of $\Gamma_{1,3}$ is a double cover, as claimed in (1) of Theorem 1.6. \square

Remark 9.3 We can also describe the global kernel K of $\varepsilon_{1,n}$. By the above, $\dim(K) = 2^k$ with $k = 2^n - 1 - \binom{n+1}{2}$. In particular, $\dim(K) = 1$ when $n = 3$. In fact, if $n = 3$ then $K = \langle v_P \rangle$, where $v_P := \sum_{p \in P} \iota(p)$, P stands for the point-set of $PG(2, 2)$ and ι is the free embedding.

If $n > 4$, then $K = \langle v_S \rangle_{S \in \mathcal{P}}$ where \mathcal{P} is the collection of planes of $PG(n-1, 2)$ and $v_S := \sum_{p \in S} \iota(p)$. This follows by combining Corollary 2.6 with (4) of Section 2.4 when $n = 4$ and with Corollary 2.7 when $n > 4$.

9.3. Proof of claim (2) of Theorem 1.6

In view of Lemma 9.1, describing the universal 2-cover of $\Gamma_{2,n}$ amounts to compute the dimension of the 1-local kernel $\tilde{K}^{(1)}$ of $\varepsilon_{2,n}$, but this computation is not so easy as in the case of $\varepsilon_{1,n}$. We will accomplish it only when $n = 3$. We firstly describe K_L for L a line of $PG(n-1, 4)$.

Lemma 9.4 *The subspace $V_L = \langle \varepsilon_{2,n}(L) \rangle$ has dimension 4 and the five vectors of $\varepsilon_{2,n}(L)$ form an ovoid in $PG(V_L) \cong PG(3, 2)$.*

Proof: Given two points $p_\infty := \langle v_1 \rangle$ and $p_0 := \langle v_2 \rangle$ of L , the remaining three points of L are the 1-dimensional linear subspaces of $V(n, 4)$ spanned by the vectors $v_\lambda = v_2 + \lambda v_1$ for $\lambda \in GF(4) \setminus \{0\}$. With $p_\lambda := \langle v_\lambda \rangle$ and $\omega \in GF(4) \setminus GF(2)$ as in the definition of $\varepsilon_{2,n}$, we have:

$$\begin{aligned} \varepsilon_{2,n}(p_\infty) &= v_1 \otimes \bar{v}_1, \\ \varepsilon_{2,n}(p_0) &= v_2 \otimes \bar{v}_2, \\ \varepsilon_{2,n}(p_1) &= v_1 \otimes \bar{v}_1 + v_2 \otimes \bar{v}_2 + v_1 \otimes \bar{v}_2 + v_2 \otimes \bar{v}_1, \\ \varepsilon_{2,n}(p_\omega) &= v_1 \otimes \bar{v}_1 + v_2 \otimes \bar{v}_2 + v_2 \otimes \bar{v}_1 + \omega(v_1 \otimes \bar{v}_2 + v_2 \otimes \bar{v}_1), \\ \varepsilon_{2,n}(p_{1+\omega}) &= \varepsilon_{2,n}(p_\infty) + \varepsilon_{2,n}(p_0) + \varepsilon_{2,n}(p_1) + \varepsilon_{2,n}(p_\omega). \end{aligned}$$

Every 4-subset of the above quintuple of vectors is independent over $GF(2)$. The conclusion follows. \square

Corollary 9.5 $\dim(K_L) = 1$.

(Trivial, by Lemma 9.4.) We will give a more explicit description of K_L in a few lines, but we firstly state some notation. Denoted the point-set of $PG(n-1, 4)$ by P , for every point $p \in P$ we put $v_p := \iota(p)$ where, according to the notation of Section 2.4, ι denotes the free embedding of $PG(n-1, 4)$ in $V((4^n-1)/3, 2)$. Also, we put $v_X := \sum_{p \in X} v_p$ for $X \subseteq P$.

Lemma 9.6 $K_L = \langle v_L \rangle$.

Proof: As $\dim(K_L) = 1$, we have $K_L = \langle v_X \rangle$ for a suitable nonempty subset X of L . Furthermore, as $\varepsilon_{2,n}(L)$ is an ovoid of $PG(V_L)$, the subspaces $\langle v_p, v_X \rangle$ ($p \in L$) are 2-dimensional, pairwise distinct and no four of them are contained in a common 4-dimensional subspace. The unique choice of X that fulfils the above requirements is $X = L$. \square

Lemma 9.7 Let $n = 3$. Then $\dim(\tilde{K}^{(1)}) = 10$.

Proof: Pick a conic C and a dual conic C^* of $PG(2, 4)$, in such a way that all lines of C^* are exterior to C . Let \mathcal{T} be the bundle of lines of $PG(2, 4)$ tangent to C and put $\mathcal{B} := \mathcal{T} \cup C^*$. We claim that the set $B := \{v_L\}_{L \in \mathcal{B}}$ is a basis of $\tilde{K}^{(1)}$.

We firstly prove that B spans $\tilde{K}^{(1)}$. Let n be the nucleus of C and N the nuclear line of C^* . So, $C^* \cup \{N\}$ is the set of lines exterior to the hyperoval $C \cup \{n\}$. Every point of N belongs to exactly one line of C^* and, for every line $L \in C^*$, every point of L different from $L \cap N$ belongs to exactly one line of C^* different from L . Therefore,

$$v_N = \sum_{L \in C^*} v_L. \tag{1}$$

Given a secant line S of C , let $\{p_1, p_2\} = C \cap S$ and, for $i = 1, 2$, let T_i be the line through n and p_i . We can pick three lines $L_1, L_2, L_3 \in C^* \cup \{N\}$ in such a way that none of the

points $L_1 \cap L_2$, $L_2 \cap L_3$ and $L_3 \cap L_1$ belongs to $S \cup T_1 \cup T_2$. Then $\{S, T_1, T_2, L_1, L_2, L_3\}$ is a dual hyperoval. By the same argument used to prove (1) one can see that

$$v_S = v_{T_1} + v_{T_2} + v_{L_1} + v_{L_2} + v_{L_3}. \quad (2)$$

By (1) and (2), $\langle B \rangle = \tilde{K}^{(1)}$. It remains to prove that B is independent.

Suppose that $\sum_{L \in \mathcal{X}} v_L = 0$ for $\mathcal{X} \subseteq B$. As every line of \mathcal{T} contains one point that does not belong to any other line of B , we have $\mathcal{X} \subseteq C^*$. However, every line of C^* contains one point that does not belong to any other line of C^* . Hence $\mathcal{X} = \emptyset$. That is, B is independent. \square

We are now ready to finish the proof of claim (2) of Theorem 1.6. By the previous lemma, $\dim(\tilde{V}^{(1)}) = 11$ when $n = 3$. Since the codomain of $\varepsilon_{2,3}$ is 9-dimensional, $\text{Exp}(\tilde{\varepsilon}_{2,3}^{(1)})$ is a 4-fold cover of $\Gamma_{2,3}$. \square

Remark 9.8 Given a Fano subplane S of $PG(2, 4)$, the embedding induced by $\varepsilon_{2,3}$ on S is isomorphic to $\varepsilon_{1,3}$. Hence $\Gamma_{2,3}$ contains 120 copies of $\Gamma_{1,3}$, as many as the Fano subplanes of $PG(2, 4)$. By Remark 9.3, the global kernel K of $\varepsilon_{2,3}$ contains v_S for every Fano subplane S of $PG(2, 4)$. Hence $K \geq \langle v_S \rangle_{S \in \mathcal{P}}$, where \mathcal{P} stands for the collection of Fano subplanes of $PG(2, 4)$. In fact

$$(*)K = \langle v_S \rangle_{S \in \mathcal{P}}$$

but I am not going to prove this equality here. By combining Corollary 2.6 with Corollary 2.7 (or (4) of Section 2.4, when $n = 4$) we obtain that $(*)$ also holds for any n . (Needless to say, in the general case \mathcal{P} is the collection of Fano planes of $PG(n - 1, 4)$.)

9.4. The universal 2-cover of $\Gamma_{2,4}$

Lemma 9.9 *If $n = 4$, then $\dim(\tilde{K}^{(1)}) = 61$.*

Proof: Given two points p_1, p_2 of $PG(3, 4)$, let L_0 be the line joining them and, for $i = 1, 2$, pick six planes $S_{i,0}, S_{i,1}, \dots, S_{i,5}$ on p_i in such a way that they form a dual hyperoval in the star of p_i and L_0 is not contained in any of them. For $i, j = 1, 2, \dots, 5$, put $L_{i,j} := S_{1,i} \cap S_{2,j}$. Denoted by \mathcal{L}_1 the set of lines of $PG(3, 4)$ incident to p_1 and by $\mathcal{L}_{2,i}$ the set of lines of $S_{2,i}$ on p_2 , we set $\mathcal{L}_2 := \cup_{i=0}^5 \mathcal{L}_{2,i}$, $\mathcal{L}_3 := \cup \{L_{i,j}\}_{i,j=1}^5$, $\mathcal{B} := \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ and $B := \{v_L\}_{L \in \mathcal{B}}$.

Let \bar{B} be the set of lines L with $v_L \in \langle B \rangle$. For $k = 1, 2, \dots, 5$, $\mathcal{L}_{2,k} \cup \{L_{i,k}\}_{i=1}^5$ is a configuration of ten lines of $S_{2,k}$ as considered in the proof of Lemma 9.7. Hence \bar{B} contains all lines of $S_{2,k}$. Let L be a line of $PG(3, 4)$ skew with L_0 , and S the plane spanned by L and a . By the above, the five lines $S \cap S_{i,2}$ ($i = 1, 2, \dots, 5$) belong to \bar{B} . Furthermore, those five lines together with the five lines of S through p_1 form in S a configuration as in the proof of Lemma 9.7. Hence $L \in \bar{B}$. Thus, \bar{B} contains all lines skew with L_0 . It is now clear that \bar{B} contains all lines of $PG(3, 4)$, namely B spans $\tilde{K}^{(1)}$.

Suppose that $\mathcal{X} \subseteq \mathcal{B}$ is such that $\sum_{L \in \mathcal{X}} v_L = 0$. The line L_0 has three points that do not belong to any other line of \mathcal{B} and every line of $\mathcal{L}_1 \setminus \{L_0\}$ contains one point that does not belong to any other line of \mathcal{L}_1 . It follows that $\mathcal{X} \cap \mathcal{L}_1 = \emptyset$. Similarly, every line of \mathcal{L}_2 contains one point that does not belong to any other line of $\mathcal{L}_2 \cup \mathcal{L}_3$. Hence $\mathcal{X} \subseteq \mathcal{L}_3$. Finally, every line of \mathcal{L}_3 contains a point that does not belong to any other line of \mathcal{L}_3 . Consequently, $\mathcal{X} = \emptyset$. Namely, \mathcal{B} is independent. As $|\mathcal{B}| = 61$, we have $\dim(\tilde{K}^{(1)}) \leq 61$. \square

Proposition 9.10 *The universal 2-cover of $\Gamma_{2,4}$ is a 2^8 -fold cover.*

Proof: By Lemma 9.9, $\text{Exp}(\tilde{\varepsilon}_{2,4}^{(1)})$, has exactly 2^{24} points. On the other hand, $\Gamma_{2,4}$ has 2^{16} points. The conclusion follows. \square

10. Proof of claim (3) of Theorem 1.6 and end of the proof of Theorem 1.4

In this section $\Gamma_n := \text{Far}_\Delta(F)$ for $\Delta = D_n(2)$ and F a $\{+, -\}$ -flag of Δ .

10.1. Preliminaries

It is known [15] that Γ_n admits a 2-quotient $\bar{\Gamma}_n$ with exactly 2^{n-1} elements of each of the two types $+$ and $-$, where every $(+)$ -element is incident to all $(-)$ -elements. That quotient is obtained by factorizing Γ_n over the center of the unipotent radical of the stabilizer of F in $O_{2n}^+(2)$. More geometrically, $\bar{\Gamma}_n = \Gamma_n / \Theta$ where Θ is the equivalence relation defined as follows: for $\{U^+, U^-\} = F$ and $S := U^+ \cup U^-$ as in Section 6, two elements X, Y of Γ_n correspond in Θ if and only if $X^\perp \cap S = Y^\perp \cap S$.

The first step of our investigation is to prove that the 0-shadow geometry of $\bar{\Gamma}_n$ is the expansion of a suitable $GF(2)$ -embedding, but we must describe that embedding first. Given a linear hyperplane H of $V = V(n, 2)$, the vectors of $V \setminus H$ and the cosets $v + X$ for $v \in V \setminus H$ and X a linear subspace of H form a copy of $AG(n - 1, 2)$. Thus, we have a $GF(2)$ -embedding of $AG(n - 1, 2)$ in $V(n, 2)$, which we call the *natural* $GF(2)$ -embedding of $AG(n - 1, 2)$. We denote it by v_n .

Lemma 10.1 *The 0-shadow geometry $\text{Sh}_0(\bar{\Gamma}_n)$ of $\bar{\Gamma}_n$ is isomorphic to the dual of the expansion $\text{Exp}(v_n)$ of the natural $GF(2)$ -embedding v_n of $AG(n - 1, 2)$.*

Proof: We firstly revisit the construction of the geometry $\bar{\Gamma}_n$, describing it as the gluing of two copies of $AG(n - 1, 2)$, as in [15].

Recall that, for $\xi \in \{+, -\}$, $\text{Res}_\Delta(U^\xi)$ can be regarded as a projective geometry $\mathcal{P}^\xi \cong PG(n-1, 2)$ with U^ξ as the set of points. The set $S_0 = U^+ \cap U^-$ is a hyperplane of \mathcal{P}^+ . Thus, the elements of \mathcal{P}^ξ that are not contained in S_0 form an affine geometry $\mathcal{A}^\xi \cong AG(n - 1, 2)$. The projective geometries \mathcal{P}^+ and \mathcal{P}^- induce on S_0 the same geometry $\mathcal{P}_0 (= \text{Res}_\Delta(F))$, which can be taken as the geometry at infinity of both \mathcal{A}^+ and \mathcal{A}^- . Accordingly, given an element X of \mathcal{A}^ξ of dimension $\dim(X) > 0$ and denoted by $\infty(X)$ its space at infinity (or its point at infinity, if $\dim(X) = 1$), we have $\infty(X) = X \cap S_0$. (Note that, when doing so, we are

regarding the points of S_0 as hyperplanes of \mathcal{P}_0 .) It is now clear that, for $i \leq n-3$, the classes of Θ on the set of i -elements of Γ_n bijectively correspond to the pairs $\{X^+, X^-\}$ where X^+ and X^- are $(n-2-i)$ -dimensional subspaces of \mathcal{A}^+ and \mathcal{A}^- with $\infty(X^+) = \infty(X^-)$. For $\xi \in \{+, -\}$, the classes of Θ on the set of ξ -elements of Γ_n bijectively correspond to the points of \mathcal{A}^η , where $\eta = \xi$ if n is odd and $\{\xi, \eta\} = \{+, -\}$ if otherwise. Two pairs $\{X^+, X^-\}$ and $\{Y^+, Y^-\}$ as above correspond to incident elements of $\bar{\Gamma}_n = \Gamma_n/\Theta$ if and only if X^ξ and Y^ξ are incident in \mathcal{A}^ξ for $\xi \in \{+, -\}$. For $\xi \in \{+, -\}$ a point $x \in \mathcal{A}^\xi$ and a pair $\{X^+, X^-\}$ correspond to incident elements of $\bar{\Gamma}_n$ if and only if $x \in X^\xi$. As all $(+)$ -elements of $\bar{\Gamma}_n$ are incident to all $(-)$ -elements, every pair $\{x^+, x^-\}$ with x^ξ a point of \mathcal{A}^ξ corresponds to a $\{+, -\}$ -flag of $\bar{\Gamma}_n$.

We shall now rephrase the above in a slightly different way. The sets U^+ and U^- are equipped with the right structures (namely, \mathcal{P}^+ and \mathcal{P}^-) and have the right intersection for we may regard them as hyperplanes of some projective geometry $\mathcal{P}_1 \cong PG(n, 2)$. Thus, \mathcal{P}^+ and \mathcal{P}^- are the projective geometries induced by \mathcal{P}_1 on U^+ and U^- , the set $S_0 = U^+ \cap U^-$ is an $(n-2)$ -dimensional subspace of \mathcal{P}_1 and \mathcal{P}_0 is the geometry induced by \mathcal{P}_1 on it. We keep for \mathcal{A}^+ and \mathcal{A}^- the meaning stated above, namely \mathcal{A}^ξ is the affine geometry obtained by removing S_0 from \mathcal{P}^ξ . As above, \mathcal{P}_0 is taken as the geometry at infinity of both \mathcal{A}^+ and \mathcal{A}^- .

Denoted by U the hyperplane of \mathcal{P}_1 containing S_0 but different from U^+ and U^- , let \mathcal{P} be the projective geometry induced by \mathcal{P}_1 on U , $\mathcal{A} \cong AG(n-1, 2)$ the complement of S_0 in \mathcal{P} and $\mathcal{A}_1 \cong AG(n, 2)$ the complement of U in \mathcal{P}_1 . For $j = 1, 2, \dots, n-2$, given a pair $\{X^+, X^-\}$ of j -dimensional subspaces of \mathcal{A}^+ and \mathcal{A}^- with $\infty(X^+) = \infty(X^-)$, the set $X := X^+ \cup X^-$ is a $(j+1)$ -dimensional subspace of \mathcal{A}_1 with $S_0 \not\subseteq \infty(X)$. Accordingly, $\infty(X)$ can be regarded as a j -dimensional subspace of \mathcal{A} . Thus, the elements of $\bar{\Gamma}_n$ of type $i = 0, 1, \dots, n-3$ bijectively correspond to the $(n-i-1)$ -dimensional subspaces of \mathcal{A}_1 with $(n-i-2)$ -dimensional subspaces of \mathcal{A} as their spaces at infinity. The set $S \setminus S_0 = (U^+ \cup U^-) \setminus S_0$ is the point-set of \mathcal{A}_1 . So, the points of \mathcal{A}_1 bijectively correspond to the elements of $\bar{\Gamma}_n$ of type $+$ and $-$. The lines of \mathcal{A}_1 with their point at infinity in \mathcal{A} are precisely those that meet both \mathcal{A}^+ and \mathcal{A}^- . So, those lines bijectively correspond to the $\{+, -\}$ -flags of $\bar{\Gamma}_n$.

Thus, we have produced a copy of the 0-shadow geometry of $\bar{\Gamma}_n$ inside \mathcal{A}_1 . The conclusion is now obvious. \square

Remark 10.2 One might object that $\text{Exp}(v_n)$ is isomorphic to the dual of $\text{Sh}_0(\bar{\Gamma}_n)$ rather than to $\text{Sh}_0(\bar{\Gamma}_n)$ itself, but the distinction between a geometry and its dual is not crucial here. We prefer not to insist on it.

As $\bar{\Gamma}_n$ is a 2-quotient of Γ_n , these two geometries have the same universal cover and the same universal 2-cover. Thus, by combining Lemma 10.1 with Proposition 2.5 and Proposition 2.2 we obtain the following:

Lemma 10.3 *The expansion of the hull of v_n is the 0-shadow geometry of the universal cover of Γ_n . The expansion of the 1-hull of v_n is the 0-shadow geometry of the universal 2-cover of Γ_n .*

10.2. The universal 2-cover of Γ_n and proof of claim (3) of Theorem 1.6

By Lemma 10.3 and Corollary 2.9 we immediately obtain the following:

Proposition 10.4 *The 0-shadow geometry of the universal 2-cover of Γ_n is the expansion $\text{Exp}(\iota)$ of the free embedding ι of $AG(n-1, 2)$ in $V(2^{n-1}, 2)$.*

$\text{Exp}(\iota)$ has $2^{2^{n-1}}$ points and the covering projection of $\text{Exp}(\iota)$ onto Γ_n maps them onto the elements of Γ_n of type $+$ and $-$. On the other hand, Γ_n has $2^{1+\binom{n}{2}}$ elements of type $+$ and $-$. Hence the universal 2-cover of Γ_n is a 2^k -fold cover, with $k = 2^{n-1} - \binom{n}{2} - 1$. In particular, the universal 2-cover of Γ_4 is a double cover.

Proof of Theorem 1.6(3): All residues of Γ_4 of rank 3 are simply connected (see Example 1.2 for those of type $\{+, 1, -\}$). Therefore, the universal cover and the universal 2-cover of Γ_4 coincide. As noticed above, the latter is a double cover. Claim (3) of Theorem 1.6 is proved. \square

 10.3. Revisiting $\text{Sh}_0(\Gamma_n)$ as an expansion

Let \bar{D}_n be the deck group of the projection from $\text{Exp}(\iota)$ onto $\text{Exp}(v_n)$. The 1-local kernel of v_n is trivial (Corollary 2.9). Hence \bar{D}_n is isomorphic to the global kernel K of v_n (Proposition 2.5). In fact, \bar{D}_n is the same thing as K , but viewed as acting by addition on $\widehat{V} = V(2^{n-1}, 2)$, as a group of translations. Accordingly, all subgroups of \bar{D}_n can be regarded as subspaces of \widehat{V} . In the sequel, we will freely switch from one to the other of these two points of view.

Every subgroup D of \bar{D}_n defines a 2-quotient $\text{Exp}(\iota)/D$ of $\text{Exp}(\iota)$. Denoted by P the point-set of $AG(n-1, 2)$, no two vectors of $\iota(P)$ belong to the same orbit of D . Indeed, if otherwise, the projection $\varphi : \text{Exp}(\iota) \rightarrow \text{Exp}(\iota)/D$ would send two distinct collinear points of $\text{Exp}(\iota)$ onto the same point of $\text{Exp}(\iota)/D$, contrary to the fact that φ is a 2-covering. Therefore, ι induces a $GF(2)$ -embedding ι/D of $AG(2^{n-1}, 2)$ in the quotient space \widehat{V}/D . Clearly, $\text{Exp}(\iota)/D = \text{Exp}(\iota/D)$. In particular, if $D_n \leq \bar{D}_n$ is the deck group of the projection of $\text{Exp}(\iota)$ onto $\text{Sh}_0(\Gamma_n)$, then

$$\text{Sh}_0(\Gamma_n) \cong \text{Exp}(\varepsilon_n) \tag{1}$$

where $\varepsilon_n := \iota/D_n$. As $\text{Exp}(\iota)$ is a $2^{2^{n-1} - \binom{n}{2} - 1}$ -fold cover of $\text{Sh}_0(\Gamma_n)$, we have

$$|D_n| = 2^{2^{n-1} - \binom{n}{2} - 1}. \tag{2}$$

Hence the codomain of ε_n is $(\binom{n}{2} + 1)$ -dimensional. Clearly, D_n is the global kernel of ε_n .

Remark 10.5 The geometry Γ_n is also an expansion, but in the more general sense of [16]: it is the expansion of the natural embedding of the line-grassmannian of $AG(n-1, 2)$ in the 2nd-exterior power of $V(n, 2)$ (see [16, Lemma 8.4]). However, as line-grassmannians are not matroids, the handy machinery set up in Section 2.4 does not work for expansions like this. That is the reason why we have switched from Γ_n to $\text{Sh}_0(\Gamma_n)$.

10.4. End of the proof of Theorem 1.4

We keep the notation of Section 10.3.

Lemma 10.6 *We have $D_4 = \langle v_P \rangle$, where $v_P := \sum_{p \in P} \iota(p)$ and P stands for the point-set of $AG(3, 2)$.*

Proof: By (2) of Section 10.3, $\dim(D_4) = 1$. Hence $D_4 = \langle v_X \rangle$, where $v_X = \sum_{p \in X} \iota(p)$ for a suitable nonempty subset X of P . On the other hand, as $\text{Exp}(\iota)/D_4 \cong \text{Sh}_0(\Gamma_4)$ is flag-transitive, the normalizer N of D_4 in $\text{Aut}(\text{Exp}(\iota))$ is flag-transitive on $\text{Exp}(\iota)$ (see [13, Theorem 12.59]). In particular, N induces on $\iota(P)$ a transitive permutation group N_P . However, as N normalizes D_4 and D_4 is spanned by v_X , N_P is forced to stabilize X . Hence $X = P$. \square

Lemma 10.7 *We have $D_5 = \langle v_S \rangle_{S \in \mathcal{P}}$, where $v_S := \sum_{p \in S} \iota(p)$ and \mathcal{P} is the collection of planes of $AG(4, 2)$.*

Proof: By Lemma 10.6, $K_S = \langle v_S \rangle$ for every $S \in \mathcal{P}$. Hence D_5 , which is the global kernel of ε_5 , contains $W := \langle v_S \rangle_{S \in \mathcal{P}}$. However, $\dim(D_5) = 5$ by (2) of Section 10.3. So, we only need to prove that $\dim(W) \geq 5$.

Given a minimal spanning set $\{p_i^\infty\}_{i=1}^4$ of the space at infinity of $AG(4, 2)$ and a point p of $AG(4, 2)$, for $k = 1, 2, 3, 4$ let S_k be the 3-space of $AG(4, 2)$ containing p and with $\langle p_i^\infty \rangle_{i \neq k}$ as the plane at infinity. Let S_5 be the 3-space of $AG(4, 2)$ parallel to S_1 . By a routine computation one can check that the vectors $v_{S_1}, v_{S_2}, v_{S_3}, v_{S_4}$ and v_{S_5} are linearly independent. Hence $\dim(W) \geq 5$. \square

We can now finish the proof of Theorem 1.4. By Lemma 10.7, the local kernel and the global kernel of ε_5 coincide. Hence $\text{Exp}(\varepsilon_5)$ is simply connected, by Corollary 2.6. As $\text{Sh}_0(\Gamma_5) \cong \text{Exp}(\varepsilon_5)$ (Section 10.3, (1)), $\text{Sh}_0(\Gamma_5)$ is simply connected, too. Therefore Γ_5 is simply connected, by the first part of Proposition 2.2.

Remark 10.8 The equality $D_n = \langle v_S \rangle_{S \in \mathcal{P}}$, with \mathcal{P} the collection of planes of $AG(n-1, 2)$, also holds when $n > 5$, as it follows from Lemma 10.7 and Corollaries 2.6 and 2.7. We refer to Baumeister, Meixner and Pasini [2] for more information.

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