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Group Actions on the Cubic Tree

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Abstract. It is known that every group which acts transitively on the ordered edges of the cubic tree Γ_3, with finite vertex stabilizer, is isomorphic to one of seven finitely presented subgroups of the full automorphism group of Γ_3 — one of which is the modular group. In this paper a complete answer is given for the question (raised by Djoković and Miller) as to whether two such subgroups which intersect in the modular group generate their free product with the modular group amalgamated.

Keywords: group actions, trees

1. Introduction

The cubic tree is the unique circuit-free connected graph all of whose vertices have degree 3. A copy of this graph (called Γ_3) may be constructed by using the modular group M, the group with presentation $M = \langle a, h \mid a^2 = h^3 = 1\rangle$, as follows: Let $H = \langle h\rangle$, take the left cosets of H in M as vertices, and join the cosets xH and yH by an edge if and only if $x^{-1}y \in HaH$. Thus the vertex H is joined to aH, haH, and h^2aH, while aH is joined to H, $ahaH$, and ah^2aH, and so on; in fact, the vertices of this graph are in one-to-one correspondence with all reduced words in a and h (and h^2) which, apart from the identity, end in a.

Note that elements of M induce automorphisms of Γ_3 by left multiplication, and, for example, multiplication by h may be viewed as a rotation about the vertex labeled H, while a may be thought of as a reflection, interchanging H with aH, the other neighbors of H with the other neighbors of aH, and so on. In particular, the action of M is transitive on the vertices of Γ_3 and is sharply transitive on its arcs (ordered edges); in other words, the action of M is arc-regular on Γ_3. Of course, the cubic tree has many more automorphisms than these. Indeed, given any path $(v_0, v_1, \ldots, v_{n-1}, v_n)$ of length n in Γ_3, there are automorphisms fixing each vertex v_i on this path and interchanging the other two vertices adjacent to v_n; it follows that Γ_3 is highly arc-transitive: its full automorphism group is transitive on paths of length n, for all $n \geq 0$.

Now clearly the stabilizer (in the full automorphism group) of any given vertex is infinite. On the other hand, there are subgroups which act transitively on the arcs of Γ_3 but which have a finite vertex stabilizer; for example, in the modular group M the stabilizer of the vertex labeled H is the subgroup H itself, which is of order 3. Up to isomorphism, however, there are only seven such subgroups; this was proved some years ago by Djoković and Miller in [4] by extending a theorem of Tutte concerning finite trivalent graphs with arc-transitive automorphism group (see [5] and [6]). Specifically, Djoković and Miller showed that if K is a group which acts transitively on the arcs of the cubic tree Γ_3 and if the stabilizer in K of a vertex of Γ_3 is finite, then K must be isomorphic to one of seven finitely presented subgroups of the full automorphism group of Γ_3 (one of which is the modular group). They also investigated relationships between these subgroups, describing them in terms of amalgams, and obtained results for the associated finite trivalent graphs – for example, a classification of those with vertex-primitive automorphism group.

At the end of [4] a number of open problems were listed. Djoković himself considered Problem 6 and one case of Problem 5 in [2], but he made some unfortunate errors, which he later resolved partially in [3]. Problems 1 to 3, questions concerning the existence of certain types of finite trivalent graphs, were answered affirmatively by Conder and Lorimer in [1].

In this paper a complete solution is given for Problem 5, the question of whether two arc-transitive subgroups of $\mathrm{Aut}(\Gamma_3)$, say, K and L, each with a finite vertex stabilizer and intersecting in the modular group, generate their free product $K \ast_M L$ with the modular group M amalgamated. (The answer turns out to be "no" if K or L acts transitively on paths of length 5 in Γ_3, and is "yes" in all other cases.) Problem 6 and related matters are discussed briefly in Section 5.

2. The Seven Groups

Unified presentations for the seven isomorphism types of subgroup of $\mathrm{Aut}(\Gamma_3)$ acting arc-transitively on Γ_3 with a finite vertex stabilizer are given in [1] as follows:

$$\begin{aligned}
G_1 &= \langle h, a \mid h^3 = a^2 = 1 \rangle, \\
G_2^1 &= \langle h, a, p \mid h^3 = a^2 = p^2 = 1, php = h^{-1}, ap = pa \rangle, \\
G_2^2 &= \langle h, a, p \mid h^3 = p^2 = 1, a^2 = p, php = h^{-1} \rangle, \\
G_3 &= \langle h, a, p, q \mid h^3 = a^2 = p^2 = q^2 = 1, pq = qp, ph = hp, \\
&\qquad qhq = h^{-1}, ap = qa \rangle, \\
G_4^1 &= \langle h, a, p, q, r \mid h^3 = a^2 = p^2 = q^2 = r^2 = 1, pq = qp, pr = rp, \\
&\qquad rq = pqr, h^{-1}ph = q, h^{-1}qh = pq, rhr = h^{-1}, \\
&\qquad ap = pa, aq = ra \rangle, \\
G_4^2 &= \langle h, a, p, q, r \mid h^3 = p^2 = q^2 = r^2 = 1, a^2 = p, pq = qp, pr = rp, \\
&\qquad rq = pqr, h^{-1}ph = q, h^{-1}qh = pq, rhr = h^{-1}, \\
&\qquad ap = pa, aq = ra \rangle,
\end{aligned}$$

$$\begin{aligned} G_5 \;=\; \langle h, a, p, q, r, s \,|\, & h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1, \\ & pq = qp,\ pr = rp,\ ps = sp,\ qr = rq,\ qs = sq, \\ & sr = pqrs,\ ph = hp,\ h^{-1}qh = r,\ h^{-1}rh = pqr, \\ & shs = h^{-1},\ ap = qa,\ ar = sa \rangle. \end{aligned}$$

These correspond, respectively, to the amalgams 1′, 2′, 2″, 3′, 4′, 4″, and 5′ given in [4] with, for example, the generators h, a, p, q, r, and s for our group G_5 satisfying exactly the same relations as do the elements ae, y, c, d, b, and e in the amalgam 5′.

Each of the groups G_n (for $n = 1, 3, 5$) and G_n^i (for $n = 2, 4$ and $i = 1, 2$) acts transitively on paths of the appropriate length n in Γ_3, with vertex stabilizer H generated by the element h when $n = 1$, by h and p when $n = 2$, by h, p, and q when $n = 3$, by h, p, q, and r when $n = 4$, and by h, p, q, r, and s when $n = 5$. These vertex stabilizers are isomorphic to C_3, S_3, $S_3 \times C_2$, S_4, and $S_4 \times C_2$, of orders 3, 6, 12, 24, and 48, respectively, and just as in Section 1, the vertices of Γ_3 may be taken as the left cosets of H in each case, with xH adjacent to yH if and only if $x^{-1}y \in HaH$.

In particular, the second generator a interchanges the vertex H with one of its neighbors, namely, aH, and this automorphism has order 2 in the cases of G_1, G_2^1, G_3, G_4^1, and G_5, but it has order 4 in the cases of G_2^2 and G_2^4. Moreover, in each of the groups G_2^1, G_3, G_4^1, and G_5, the subgroup generated by h and a is arc-regular on Γ_3 and is permutation isomorphic to G_1, the modular group.

Conversely, Djoković and Miller proved the following in [4]: every arc-regular subgroup M of Aut(Γ_3) is isomorphic to G_1, is contained in unique subgroups isomorphic to G_2^1 and G_3 (acting regularly on paths of lengths 2 and 3, respectively), and is contained in two subgroups isomorphic to G_4^1 (acting regularly on paths of length 4), each of which, in turn, is contained in a unique subgroup isomorphic to G_5 (acting regularly on paths of length 5). This situation is conveniently described by Figure 1, which is more or less a copy of Figure 5 in [4]. In fact, the single copy of G_2^1 is the normalizer of the given subgroup M in Aut(Γ_3), and conjugation by any of its elements not in M interchanges the two copies of G_4^1 as well as the two copies of G_5 (see the proof of Proposition 13 in [4]).

3. Restatement of the Problem

Problem 5 at the end of [4] can now be stated as follows: if K and L are two of the six subgroups in Figure 1 that properly contain the given subgroup M, with $K \cap L = M$, then is $\langle K, L\rangle$ the free product $K \ast_M L$ with the subgroup M amalgamated, or do there exist nontrivial elements of $K \ast_M M$ which act trivially on Γ_3?

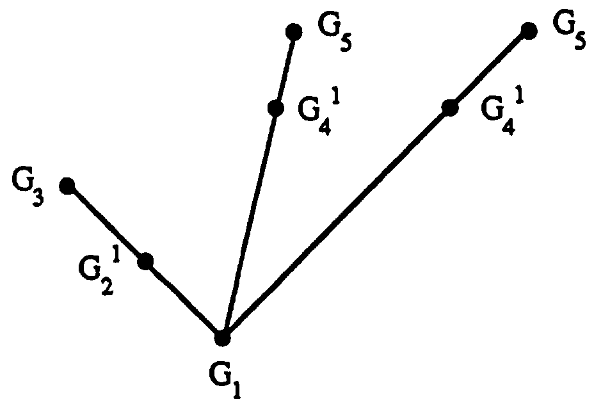

Figure 1.

Clearly, there are seven cases to consider, namely, where K and L are isomorphic in some order to G_2^1 and G_4^1, to G_2^1 and G_5, to G_3 and G_4^1, to G_3 and G_5, to G_4^1 and G_4^1 (distinct copies), to G_4^1 and G_5 (in distinct copies of G_5), or to G_5 and G_5 (distinct copies). As it turns out, we shall not have to deal with all of these because the solutions in some cases are easy consequences of the others; also, the case of G_2^1 and G_4^1 was settled by Djoković in [2] and [3]: these two subgroups do generate their free product with G_1 amalgamated.

Before we proceed to the solution (in Section 4) it is worthwhile to note that in each of the groups G_2^1, G_3, G_4^1, and G_5 as presented in Section 2, the relations provide a set of rewriting rules for words in the generators. For example, in G_5 they imply that $ph = hp$, $qh = hr$, $rh = hpqr$, and $sh = h^{-1}s$, while they also imply that $pa = aq$, $qa = ap$, $ra = as$, and $sa = ar$, so that every element of this group can be uniquely written in the form uv, where $u \in \langle h, a \rangle$ and $v \in \langle p, q, r, s \rangle$. Indeed, the same sort of decomposition occurs in all these groups, as explained in [1, Thm. 1.1].

4. Solution

We first deal with the case where K and L are distinct copies of G_5 containing the given subgroup M (isomorphic to G_1).

Let h, a, p, q, r, and s be generators for K that satisfy the relations for G_5 as given in Section 2, and such that $\langle h, a \rangle = M$. Also, let N be the normalizer of M in $\mathrm{Aut}(\Gamma_3)$, and choose P in N such that h, a, and P generate N and satisfy the relations given for G_2^1—with P in place of p. Then without loss of generality, the conjugates of h, a, p, q, r, and s by P may be taken as generators for L that satisfy the relations for G_5 because $P^{-1}KP = L$. In particular, these generators are h^{-1} ($= P^{-1}hP$), a ($= P^{-1}aP$), PpP, PqP, PrP, and PsP, where $P^{-1} = P$.

Now consider the following elements of the subgroup generated by K and L: $f_1 = (pPsP)^2$, $f_2 = (qPrP)^2$, $f_3 = (rPqP)^2$, $f_4 = (sPpP)^2$, $f_5 = (pqrPpqrP)^2$, $f_6 = (pqsPpqsP)^2$, $f_7 = (prsPprsP)^2$, and $f_8 = (qrsPqrsP)^2$. Using the relations for G_2^1 and G_5, we will show that each of these elements acts trivially on the cubic tree Γ_3. To do this, we use the fact that every vertex of Γ_3 corresponds to a left coset uH, where u is a reduced word in a, h, and h^2 (not ending in h or h^2) and H is the stabilizer in $\mathrm{Aut}(\Gamma_3)$ of the vertex fixed by h; in particular, this means an automorphism g acts trivially on Γ_3 if and only if $guH = uH$ for all $u \in \langle h, a\rangle$. Also, we note that H contains each of the elements f_i (for $1 \le i \le 8$) defined above, because p, q, r, s, and P are all in H. Moreover, the rewriting rules involving h and a yield the following: $f_1h = hf_1$, $f_2h = hf_3$, $f_3h = hf_5$, $f_4h = hf_4$, $f_5h = hf_2$, $f_6h = hf_7$, $f_7h = hf_8$, $f_8h = hf_6$, and $f_1a = af_2$, $f_2a = af_1$, $f_3a = af_4$, $f_4a = af_3$, $f_5a = af_6$, $f_6a = af_5$, $f_7a = af_7$, $f_8a = af_8$. (For example, $f_3h = hf_5$ since $rPqPh = rPqh^{-1}P = rPh^{-1}pqrP = rhPpqrP = hpqrPpqrP$.) Clearly, these imply that for $1 \le i \le 8$ and for every $u \in \langle h, a\rangle$ we have $f_iu = uf_j$ for some j, and thus $f_iuH = uH$. In other words, each f_i fixes every vertex of Γ_3, as claimed. Hence in this case K and L do not generate their free product with amalgamated intersection.

The same argument holds for the cases where K and L are isomorphic to G_2^1 and G_5 or to G_3 and G_5, because every copy of G_3 contains a copy of G_2^1 and (as noted earlier) conjugation by any element of the latter group not in the intersection M produces a second copy of G_5.

Also, a similar thing happens in the case where, say, K is isomorphic to G_5 and L is a copy of G_4^1 not contained in K: If h, a, p, q, r, s, and P are as before, then the elements h^{-1}, a, pq, qr, and ps satisfy the relations given for G_4^1 and generate the unique copy of G_4^1 in K that contains M, so their conjugates $h, a, PpqP, PqrP$, and $PpsP$ (by P) also satisfy those relations and generate L. But now, for example, using the fact obtained above that $(pPsP)^2 = 1$ in $\mathrm{Aut}(\Gamma_3)$, we find $p(PpsP)s(PpsP) = pPs(pPsPp)sP = PsPpP(PsP)sP = PsPpsPsP$, which has order 2, and therefore the element $(p(PpsP)s(PpsP))^2$ acts trivially also. Thus, again, K and L do not generate their free product with M amalgamated.

Next, we turn to the case where K is isomorphic to G_3 and L is isomorphic to G_4^1. Here the approach used above does not work because no nontrivial element of $K \underset{M}{*} L$ acts trivially on Γ_3. This time, let h, a, p, q, and r be generators for L that satisfy the relations for G_4^1 as given in Section 2, with $\langle h, a\rangle = M$, and let P and Q be elements of K such that h, a, P, and Q generate K and satisfy the relations for G_3: $h^3 = a^2 = P^2 = Q^2 = 1$, $PQ = QP$, $Ph = hP$, $QhQ = h^{-1}$, and $aP = Qa$. (Note that the p, q, r, and P in this case are not the same as they were earlier.) The two sets of relations together imply that every element g of $\langle K, L\rangle$ can be rewritten in the form $g = uv$, where $u \in \langle h, a\rangle$ and $v \in \langle P, Q, p, q, r\rangle$, and then, since each of P, Q, p, q, and r lies in the stabilizer H of the vertex of

Γ_3 labeled H, we have $gH = uvH = uH$ and $gaH = uvaH = ua(ava)H = uaH$. It follows that the only elements of $\langle K, L\rangle$ which fix both the vertex labeled H and its neighbor aH are those in $\langle P, Q, p, q, r\rangle$, and therefore any element of $\langle K, L\rangle$ which acted trivially on Γ_3 would have to be a word in P, Q, p, q, and r.

Assume now that $\langle K, L\rangle$ contains such an element, say, $g = X_1y_1X_2y_2\cdots X_my_m$, with $1 \neq X_i \in \langle P, Q\rangle$ and $1 \neq y_i \in \langle p, q, r\rangle$ for $1 \leq i \leq m$. Because $\langle P, Q\rangle \cong C_2\times C_2$ there are three possibilities for each Y_i, namely, P, Q, and PQ, and because $\langle p, q, r\rangle$ is dihedral of order 8 there are seven possibilities for each X_i, namely, p, q, r, pq, pr, qr, and pqr. Also, since g fixes every vertex of Γ_3, we know the same is true of ugu^{-1} for all $u \in \langle K, L\rangle$. We shall prove, however, by induction on m, that for any nontrivial element y' distinct from y_m in $\langle p, q, r\rangle$, there is an element u in M such that $ug = uX_1y_1X_2y_2\cdots X_my_m = X_1y_1X_2y_2\cdots X_{m-1}y_{m-1}X_my'u'$ for some u' in M; it then follows that $y_m^{-1}y'u'u^{-1}(= g^{-1}ugu^{-1})$ fixes every vertex, a contradiction, and this shows that no such element g can be found.

When $m = 1$ take $u = (haha)^k$ if $X_1 = P$, or $u = (h^{-1}ah^{-1}a)^k$ if $X_1 = Q$, or $u = (h^{-1}aha)^k$ if $X_1 = PQ$, with $k = 1, 2, 3, 4, 5$, or 6, chosen so that $uX_1y_1 = X_1y'u'$, where y' is the given element distinct from y_1 in $\langle p, q, r\rangle$. This works because in each case $uX_1 = X_1(hah^{-1}a)^k$, and the element $hah^{-1}a$ induces a 7-cycle by left multiplication on the left cosets of M that contain nontrivial elements of $\langle p, q, r\rangle$: $hah^{-1}ap = rh^{-1}aha \in rM$, and, similarly, $hah^{-1}ar \in pqrM$, $hah^{-1}apqr \in prM$, $hah^{-1}apr \in pqM$, $hah^{-1}apq \in qrM$, $hah^{-1}aqr \in qM$, and $hah^{-1}aq \in pM$.

For the rest of the proof we make the (stronger) inductive hypothesis that for any choice of nontrivial elements $y_1', y_2', \ldots, y_{m-1}'$ from $\langle p, q, r\rangle$, there exists an element w in M such that $wX_1y_1X_2y_2\cdots X_{m-1}y_{m-1} = X_1y_1'X_2y_2'\cdots X_{m-1}y_{m-1}'w'$ for some w' in M. Use of this hypothesis allows us to make specific choices for the elements $y_1, y_2, \ldots, y_{m-1}$, depending on which of P, Q, and PQ the X_m happens to be.

Take the following elements of M: $u_1 = hahahah^{-1}a$, $u_2 = hahah^{-1}aha$, $u_3 = hah^{-1}ahaha$, $u_4 = h^{-1}ahahaha$, $u^5 = hah^{-1}ah^{-1}ah^{-1}a$, $u_6 = h^{-1}ahah^{-1}ah^{-1}a$, $u_7 = h^{-1}ah^{-1}ahah^{-1}a$, and $u_8 = h^{-1}ah^{-1}ah^{-1}aha$. All of these are conjugates of u_1 or u_1^{-1} in M, and, in fact, the relations in K yield the following (some of which are redundant):

$$
\begin{array}{lll}
u_1P = Pu_3, & u_1Q = Qu_6, & u_1PQ = PQu_8,\\
u_2P = Pu_5, & u_2Q = Qu_4, & u_2PQ = PQu_7,\\
u_3P = Pu_1, & u_3Q = Qu_8, & u_3PQ = PQu_6,\\
u_4P = Pu_7, & u_4Q = Qu_2, & u_4PQ = PQu_5,\\
u_5P = Pu_2, & u_5Q = Qu_7, & u_5PQ = PQu_4,\\
u_6P = Pu_8, & u_6Q = Qu_1, & u_6PQ = PQu_3,\\
u_7P = Pu_4, & u_7Q = Qu_5, & u_7PQ = PQu_2,\\
u_8P = Pu_6, & u_8Q = Qu_3, & u_8PQ = PQu_1.
\end{array}
$$

Similarly, the relations in L give also $u_1pr = pru_7$, $u_2pqr = pqru_6$, $u_3pq = pqu_5$, $u_4p = pu_8$, $u_5pq = pqu_3$, $u_6qr = qru_2$, $u_7pr = pru_1$, and $u_8p = pu_4$. Thus for each of these elements u_i $(1 \leq i \leq 8)$ and for every X chosen from $\{P, Q, PQ\}$ there is a nontrivial element y in $\langle p, q, r\rangle$ such that $u_iXy = Xyu_j$ for some j. The same is also true of powers of the u_i; moreover, for every X in $\{P, Q, PQ\}$ and for every element y in $\langle p, q, r\rangle$ we find $u_i^3Xy = Xyz$ for some $z \in M$. On the other hand, y can always be chosen so that the element z does not have this property, indeed, so that $zP = Pz'$, where z' induces a 7-cycle on the left cosets of M that contain nontrivial elements of $\langle p, q, r\rangle$. For example, $u_7^3Pq = Pu_4^3q = Pqz$, where $z = h^{-1}ahah^{-1}ahahah^{-1}ah^{-1}ah^{-1}ahahahah^{-1}a$, but then $z'p = qz''$ with z'' in M and, similarly, $z'q \in qrM$, $z'qr \in pqM$, $z'pq \in prM$, $z'pr \in pqrM$, $z'pqr \in rM$, and $z'r \in pM$. The same situation occurs when P is replaced by Q or PQ. In fact, in all cases y can be chosen so that $P^{-1}zP$, $Q^{-1}zQ$, and $(PQ)^{-1}zPQ$ act as 7-cycles in this way: when $u_1^3X = Xu_k^3$ and $k = 1, 2, 3, 4, 5, 6, 7, 8$, respectively, taking $y = p, r, q, pr, p, q, r, pq$ will do the trick.

It follows that if we take $u = u_1^3$, say, then (if $m \geq 2$) the elements $y_1, y_2, \cdots, y_{m-2}$ can be chosen such that $uX_1y_1X_2y_2\cdots X_{m-2}y_{m-2} = X_1y_1X_2y_2\cdots X_{m-2}y_{m-2}u_j^3$ for some j, and then y_{m-1} can be chosen so that $u_j^3X_{m-1}y_{m-1}X_m = X_{m-1}y_{m-1}zX_m = X_{m-1}y_{m-1}X_mz'$, where the element $z' \in M$ induces a 7-cycle on the left cosets of M that contain nontrivial elements of $\langle p, q, r\rangle$ in the manner illustrated previously. In particular, given any nontrivial y' in $\langle p, q, r\rangle$ we know $(z')^ky_m \in y'M$ for some k, and then (finally) we obtain

$$\begin{aligned} u^kX_1y_1X_2y_2\cdots X_my_m &= X_1y_1X_2y_2\cdots X_{m-2}y_{m-2}u_j^{3k}X_{m-1}y_{m-1}X_m \\ &= X_1y_1X_2y_2\cdots X_{m-1}y_{m-1}X_m(z')^ky_m \\ &= X_1y_1X_2y_2\cdots X_{m-1}y_{m-1}X_my'u', \end{aligned}$$

as required. This is sufficient to complete the induction and thereby also the proof that no nontrivial element of $\langle K, L\rangle$ can act trivially on Γ_3. In particular, $\langle K, L\rangle$ is isomorphic to $K \mathop{*}_M L$.

We are left with just two cases to consider, namely, one where K and L are copies of G_2^1 and G_4^1 and the other where K and L are distinct copies of G_4^1 (intersecting in $M \cong G_1$). Both of these cases are covered by the one above, because the unique copy N of G_2^1 that contains M is itself contained in a copy of G_3, and, further as N is the normalizer of M in $\mathrm{Aut}(\Gamma_3)$, two copies of G_4^1 lie inside the subgroup generated by N and any single copy of G_4^1 (which intersects N in M). Thus in both remaining cases the subgroups K and L generate their free product with the modular group M amalgamated.

Incidentally, the case of G_2^1 and G_4^1 was solved earlier by Djoković in [2] and [3] by using a similar method (albeit in a quite different form) to the one above. This case may also be handled by choosing h, a, p, q, r, P, and Q as in the case of G_3 and G_4^1 and proving by induction that for all $m \geq 1$ the group $M = \langle h, a\rangle$ acts transitively (by left multiplication) on the set of left cosets of M that contain elements of the form $PQy_1PQy_2\cdots PQy_m$, with

$1 \neq y_i \in \langle p, q, r\rangle$ for $1 \leq i \leq m$ (so that none of these elements can act trivially on Γ_3), by considering the effect of powers of the element $(haha)^3$ on cosets of the form $PQpqrPQpqr \cdots PQpqrPQpPQyM$, with $1 \neq y \in \langle p, q, r\rangle$. Such a proof is considerably easier than either of the other two.

The result we have achieved is the following:

THEOREM. *Suppose K and L are two arc-transitive subgroups of the automorphism group of the cubic tree, each with finite vertex stabilizer, and intersecting in an arc-regular subgroup M (which is necessarily isomorphic to the modular group). If neither K nor L acts transitively on paths of length 5, then the subgroup generated by K and L is isomorphic to their free product $K \mathop{*}_M L$ with amalgamated subgroup M, whereas in all other cases $K \mathop{*}_M L$ contains nontrivial elements which act trivially on the cubic tree.*

5. Related Questions

Problem 6 at the end of [4] may also be restated as follows: if K and L are distinct copies of the group G_4^1 intersecting in an arc-regular subgroup M (isomorphic to the modular group), as in Problem 5, and if K^+ and L^+ are the subgroups of K and L, respectively, generated by the stabilizers of vertices of Γ_3, then is $\langle K^+, L^+\rangle$ an infinite simple group?

Once again, let h, a, p, q, and r be generators for K that satisfy the relations for G_4^1 as given in Section 2, with $\langle h, a\rangle = M$, and choose P in the normalizer N of M in $\mathrm{Aut}(\Gamma_3)$ so that h, a, and P satisfy the relations for G_2^1; then $L = P^{-1}KP$, generated by h, a, PpP, PqP, and PrP. Now, since each of h, p, q, and r and their conjugates in K fixes at least one vertex, we have $K^+ = \langle h, aha, p, q, r\rangle$ of index 2 in K, and by the same argument we also have $L^+ = \langle h, aha, PpP, PqP, PrP\rangle$; consequently, $\langle K^+, L^+\rangle = \langle h, aha, p, q, r, PpP, PqP, PrP\rangle$, which has index 2 in $\langle K, L\rangle$, with transversal $\{1, a\}$. In particular, because the latter is a Schreier transversal, it is not difficult to obtain a presentation for the group $\langle K^+, L^+\rangle$ in terms of these generators by using the Reidemeister–Schreier process—the relations are all obvious consequences (under conjugation by a or P or both) of those for K.

The aim of [2] was to prove this group to be simple, thereby providing a new example of a finitely presented infinite simple group (and, incidentally, also an example of a finite amalgam that cannot be embedded in any finite group). Unfortunately, a subtle error was made in [2], and in the sequel [3] it was shown instead that the group is residually finite! (Also in [3] it was shown that this group has $PSL_2(17)$ as a finite quotient, but, in fact, it has an even smaller nontrivial quotient, namely $PSL_2(7)$, because the linear fractional transformations

$$h^*: z \mapsto (z-1)/z, \qquad a^*: z \mapsto (2z+1)/(4z-2),$$
$$p^*: z \mapsto (2z+1)/(2z-2), \quad q^*: z \mapsto (3z-3)/(z-3),$$
$$r^*: z \mapsto (z-4)/(4z-1), \qquad P^*: z \mapsto 1-z$$

of the projective line over $GF(7)$ satisfy all those relations required of h, a, p, q, r, and P in the group generated by K and N above and themselves generate $PGL_2(7)$, with h^*, p^*, q^*, and r^* and their conjugates by a^* and P^* giving $PSL_2(7)$, the unique subgroup of index 2.)

The answer for Problem 6 in [4] is a definite "no." On the other hand, it may still make sense to ask a similar question about another group occurring in this context, as follows: Suppose that K and L are copies of the groups G_4^1 and G_3 in $\mathrm{Aut}(\Gamma_3)$ that act regularly on paths of lengths 4 and 3, respectively, and intersecting in the arc-regular subgroup $M(\cong G_1)$. Let h, a, p, q, and r be generators for K that satisfy the relations for G_4^1, with $\langle h, a\rangle = M$, and choose P and Q in L so that h, a, P, and Q satisfy the relations for G_3. As in the case of G_4^1 and G_2^1, we know that $\langle K, L\rangle$ is isomorphic to the free product of K and L with M amalgamated, so $\langle K, L\rangle$ has a presentation in terms of the generators h, a, p, q, r, P, Q, with relations $h^3 = a^2 = p^2 = q^2 = r^2 = P^2 = Q^2 = 1$, $pq = qp$, $pr = rp$, $rq = pqr$, $PQ = QP$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $Ph = hP$, $QhQ = h^{-1}$, $ap = pa$, $aq = ra$, and $aP = Qa$. Now this group has a normal subgroup S of index 8 generated by h, aha, p, PpP, QpQ, and $PQpPQ$, with dihedral complement $\langle a, P, Q\rangle$. (Note that S contains q and r and their conjugates as well, for $h^{-1}ph = q$, $(aha)^{-1}p(aha) = r$, $h^{-1}PpPh = PqP$, and so on.) By letting A, B, C, D, E and F, respectively, be the given generators for S and using the Schreier transversal $\{1, a, P, Q, PQ, aP, aQ, aPQ\}$ as (Schreier) transversal, the following presentation for S is obtainable from the Reidemeister–Schreier process:

$$\begin{aligned}\langle A, B, C, D, E, F \mid A^3 &= B^3 = C^2 = D^2 = E^2 = F^2 \\ &= (AC)^3 = (AD)^3 = (AE)^3 = (AF)^3 \\ &= (BC)^3 = (BD)^3 = (BE)^3 = (BF)^3 \\ &= (ABA^{-1}C)^2 = (ABA^{-1}D)^2 \\ &= (A^{-1}BAE)^2 = (A^{-1}BAF)^2 \\ &= (BAB^{-1}C)^2 = (B^{-1}ABD)^2 \\ &= (BAB^{-1}E)^2 = (B^{-1}ABF)^2 = 1\rangle.\end{aligned}$$

The question is this: can S be a simple group? Or, more generally, does S have a nontrivial finite quotient? It would certainly be nice if the answers were "yes" and "no" to these questions, particularly since the group S would then provide a concrete example of a finite amalgam not embeddable in any finite group. On the other hand, any conjecture to this effect on the author's part would be based on the flimsiest of evidence—only that he has not been able to find (in S) any proper subgroup of finite index! The questions remain open.

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