Journal of Algebraic Combinatorics, 22, 39–63, 2005

Tight Gaussian 4-Designs

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Received August 5, 2003; Revised October 20, 2004; Accepted November 5, 2004

Abstract. A Gaussian t-design is defined as a finite set X in the Euclidean space $\mathbb{R}^n$ satisfying the condition: $\frac{1}{V(\mathbb{R}^n)}\int_{\mathbb{R}^n} f(x)e^{-\alpha^2||x||^2}dx = \sum_{u\in X}\omega(u)f(u)$ for any polynomial $f(x)$ in n variables of degree at most t, here α is a constant real number and ω is a positive weight function on X. It is easy to see that if X is a Gaussian $2e$-design in $\mathbb{R}^n$, then $|X| \geq \binom{n+e}{e}$. We call X a tight Gaussian $2e$-design in $\mathbb{R}^n$ if $|X| = \binom{n+e}{e}$ holds. In this paper we study tight Gaussian $2e$-designs in $\mathbb{R}^n$. In particular, we classify tight Gaussian 4-designs in $\mathbb{R}^n$ with constant weight $\omega = \frac{1}{|X|}$ or with weight $\omega(u) = \frac{e^{-\alpha^2||u||^2}}{\sum_{x\in X} e^{-\alpha^2||x||^2}}$. Moreover we classify tight Gaussian 4-designs in $\mathbb{R}^n$ on 2 concentric spheres (with arbitrary weight functions).

Keywords: Gaussian design, tight design, spherical design, 2-distance set, Euclidean design, addition formula, quadrature formula

1. Main theorems

Definition 1.1 Let $X \subset \mathbb{R}^n$ be a finite set. We say X is a Gaussian t-design if the following condition holds for any polynomial $f(x)$ in n variables $x_1, x_2, \ldots, x_n$ of degree at most t:

$$\frac{1}{V(\mathbb{R}^n)}\int_{\mathbb{R}^n} f(x)e^{-\alpha^2||x||^2}dx = \sum_{x\in X}\omega(x)f(x),$$

where α is a positive real number, $V(\mathbb{R}^n) = \int_{\mathbb{R}^n} e^{-\alpha^2||x||^2}dx$, and ω is a weight function on X satisfying $\omega(x) > 0$ for any $x \in X$ and $\sum_{x\in X}\omega(x) = 1$.

The theorem by Seymour-Zaslavsky [21] assures us that there always exist Gaussian t-designs in $\mathbb{R}^n$ with sufficiently large cardinalities $|X|$. We also have the following theorem which is well known.

Theorem 1.2 *If X is a Gaussian $2e$-design, then $|X| \geq \binom{n+e}{e}$.*

Remark Since Gaussian $2e$-design is a Euclidean $2e$-design as is mentioned in Proposition 2.3 in this paper, better lower bounds for the cardinalities $|X|$ of Gaussian $2e$-designs are sometimes known in some special cases, e.g., if e is odd, $0 \in X$ and $|\{||x|| \mid x \in X\}| \geq \frac{e+3}{2}$, then $|X| \geq \binom{n+e}{e} + 1$ as is proved in [10]. However, we think $\binom{n+e}{e}$ is the most natural and general bound since this is the dimension of the space consisting of all the polynomials of degree at most e on $\mathbb{R}^n$.

Gaussian $2e$-design X is called *tight* if $|X| = \binom{n+e}{e}$ holds. The purpose of this paper is to prove the following two main theorems.

Theorem 1.3 *Let X be a tight Gaussian $2e$-design. Let $\{||x|| \mid x \in X\} = \{r_1, r_2, \ldots, r_p\}$ ($r_i \neq r_j$ for $i \neq j$) and $X_i = \{x \in X \mid ||x|| = r_i\}$. Then the following assertions hold*:

(1) $p \geq [\frac{e}{2}] + 1$.

(2) *$\omega(x)$ is constant on each X_i.*

(3) *Each X_i is an at most e-distance set.*

Theorem 1.4 *Let X be a Gaussian tight 4-design. Then the following assertions hold*:

(1) *If $0 \in X$, then X is a Gaussian tight 4-design if and only if $X - \{0\}$ is a spherical tight 4-design on the sphere of radius $\sqrt{\frac{n+2}{2\alpha^2}}$ and the weight ω is uniquely determined as follows*:

$$\omega(u) = \begin{cases} \dfrac{2}{n+2} & \text{for } u = 0 \\ \dfrac{2}{(n+3)(n+2)} & \text{for } \|u\| = \sqrt{\dfrac{n+2}{2\alpha^2}}. \end{cases}$$

(2) *If $p = 2$ and $0 \notin X$, then $n = 2$ and X equals the 6 points set*

$$\left\{ r_1\left(\cos\frac{2l\pi}{3}, \sin\frac{2l\pi}{3}\right), -r_2\left(\cos\frac{2l\pi}{3}, \sin\frac{2l\pi}{3}\right) \middle| l = 0, 1, 2 \right\}$$

up to orthogonal transformation of $\mathbb{R}^2$, where $r_1 = \frac{\sqrt{5}+1}{\alpha\sqrt{2}}$ and $r_2 = \frac{\sqrt{5}-1}{\alpha\sqrt{2}}$. The weight function is given by

$$\omega(u) = \begin{cases} \omega_1 = \dfrac{1}{6} - \dfrac{\sqrt{5}}{15} & \text{for } u \in X_1 \\ \omega_2 = \dfrac{1}{6} + \dfrac{\sqrt{5}}{15} & \text{for } u \in X_2. \end{cases}$$

(Note that $\frac{\omega_1}{\omega_2} = (\frac{r_2}{r_1})^3$ holds.)

(3) *There is no Gaussian tight 4-design with weight $\omega(u) = \frac{e^{-\alpha^2||u||^2}}{\sum_{x\in X} e^{-\alpha^2||x||^2}}$.*

(4) *There is no Gaussian tight 4-design with constant weight $\omega = \frac{1}{|X|}$.*

Remark It is known that the set $X = X_1 \cup X_2 \subset \mathbb{R}^2$ defined below is a tight Euclidean 4-design (cf. [3]).

$$X_1 = \left\{ r_1\left(\cos\frac{2l\pi}{3}, \sin\frac{2l\pi}{3}\right) \middle| \, l = 0,\ 1,\ 2 \right\},$$
$$X_2 = \left\{ -r_2\left(\cos\frac{2l\pi}{3}, \sin\frac{2l\pi}{3}\right) \middle| \, l = 0,\ 1,\ 2 \right\},$$

where $r_1,\ r_2$ are arbitral positive real numbers and the weight function ω is defined by $\omega(u) = \omega_i$ for $u \in X_i,\ i = 1,\ 2$, with positive real numbers ω_1 and ω_2 satisfying $\frac{\omega_1}{\omega_2} = (\frac{r_2}{r_1})^3$. If $r_1 = r_2$, then X is a regular hexagon, which is a tight spherical 5-design.

Theorems 1.3 and 1.4 will be proved in Sections 2 and 3 respectively. Section 4 will contain some concluding remarks.

2. Preliminaries on Gaussian designs

First we introduce some notation. Let X be a finite set in $\mathbb{R}^n$. Let $\{||x|| \mid x \in X\} = \{r_1,\ r_2,\ \ldots,\ r_p\}$ $(r_i \neq r_j$ if $i \neq j)$. Let $S_i = \{x \in \mathbb{R}^n \mid ||x|| = r_i\}$. Even if $r_i = 0$, we count $S_i = \{0\}$ as a sphere and we say that X is supported by p concentric spheres centered at the origin. Let $X_i = X \cap S_i,\ 1 \leq i \leq p$. Let ω be a positive weight function defined on X. We define $\omega(X_i) = \sum_{x \in X_i} \omega(x)$. If $r_i \neq 0$, then let σ_i be the Haar measure defined on each sphere S_i induced by the ordinary measure of $\mathbb{R}^n$. We denote $|S_i|$ the area of S_i, i.e., $|S_i| = \int_{S_i} d\sigma_i(x)$. If $r_i = 0$, then we define $\int_{S_i} f(x)d\sigma_i(x) = f(0)$. Hence $|S_i| = \int_{S_i} d\sigma_i(x) = 1$ for this case.

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all the polynomials in n variables. Let $\mathrm{Harm}(\mathbb{R}^n)$ be be the set of all the harmonic polynomials in $\mathcal{P}(\mathbb{R}^n)$. Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ consisting of all the homogeneous polynomials of degree l. Let $\mathrm{Harm}_l(\mathbb{R}^n) = \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n)$. We assume that the reader is familiar with the basic concepts related to spherical t-designs, see, e.g. [2, 9].

In [19] A. Neumaier and J. J. Seidel defined Euclidean designs as follows.

Definition 2.1 A finite set X in $\mathbb{R}^n$ is called a Euclidean t-design if

$$\sum_{i=1}^{p} \frac{\omega(X_i)}{|S_i|} \int_{S_i} f(x)d\sigma_i(x) = \sum_{x \in X} \omega(x) f(x)$$

holds for any polynomial $f(x)$ in n variables of degree at most t.

In [19], Neumaier and Seidel also showed the following theorem.

Theorem 2.2 *X is a Euclidean t-design if and only if*

$$\sum_{x\in X}\omega(x)f(x)=0$$

holds for any polynomial $f(x)\in \|x\|^{2j}Harm_l(\mathbb{R}^n)$ *where* $j,\ l$ *are integers satisfying* $1\leq l\leq t$ *and* $0\leq j\leq [\frac{t-l}{2}]$.

We can easily prove the following proposition.

Proposition 2.3 *A Gaussian t-design is a Euclidean t-design.*

Proof: Let σ be the ordinary Haar measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. Let X be a Gaussian t-design with a weight function ω. Let l and j be nonnegative integers satisfying $1\leq l$ and $l+2j\leq t$. Let $\varphi\in \mathrm{Harm}_l(\mathbb{R}^n)$. Then, since $l\geq 1$, we have

$$\begin{aligned}\sum_{x\in X}\omega(x)||x||^{2j}\varphi(x) &= \frac{1}{V(\mathbb{R}^n)}\int_{\mathbb{R}^n}||x||^{2j}\varphi(x)e^{-\alpha^2||x||^2}dx\\ &= \frac{1}{V(\mathbb{R}^n)}\int_0^\infty r^{n-1+2j+l}e^{-\alpha^2r^2}dr\int_{S^{n-1}}\varphi(\xi)d\sigma(\xi)=0.\end{aligned}$$

Hence we have

$$\sum_{x\in X}\omega(x)f(x)=0$$

for any polynomials in $||x||^{2j}\mathrm{Harm}_l(\mathbb{R}^n)$ satisfying $0\leq j\leq[\frac{t-l}{2}]$ and $1\leq l\leq t$. This means X is a Euclidean t-design with a weight function $\omega(x)$. □

Let $\varphi_{l,i}(x)$, $i=1,\dots,N_l$ be a basis of $\mathrm{Harm}_l(\mathbb{R}^n)$ satisfying the following condition.

$$\frac{1}{|S^{n-1}|}\int_{\xi\in S^{n-1}}\varphi_{l_1,i_1}(\xi)\varphi_{l_2,i_2}(\xi)d\sigma(\xi)=\delta_{l_1,l_2}\delta_{i_1,i_2},$$

where $N_l=\dim(\mathrm{Harm}_l(\mathbb{R}^n))$. It is well known that

$$\sum_{i=1}^{N_l}\varphi_{l,i}(\xi)\varphi_{l,i}(\eta)=Q_l((\xi,\ \eta))$$

holds for any $\xi,\ \eta\in S^{n-1}$, where Q_l is the Gegenbauer polynomial of degree l and $(\xi,\ \eta)$ is the ordinary inner product of vectors in $\mathbb{R}^n$ (see e.g. [9, 15].). The above equation is known as the addition formula. The addition formula implies $Q_l(1)=N_l=\dim(\mathrm{Harm}_l(\mathbb{R}^n))$.

For each l we consider the vector space of polynomials in one variable r equipped with the following inner product $<, >_l$. For polynomials $g(r),\ h(r)$ we defined

$$\langle g, h\rangle_l = \frac{1}{\int_0^\infty r^{n-1}e^{-\alpha^2r^2}dr}\int_0^\infty e^{-\alpha^2r^2}g(r)h(r)r^{n-1+2l}dr.$$

Since

$$\{1,\ r^2,\ r^4,\ \ldots, r^{2i},\ \ldots\}$$

is a linearly independent set in the vector space of polynomials in one variable r, applying the Schmidt's orthonormalization method, we can construct polynomials $g_{l,j}(R),\ j = 0, 1, 2, \ldots$ satisfying the following condition:

$g_{l,j}(R)$ is a polynomial in one variable R of degree j and

$$\frac{1}{\int_0^\infty r^{n-1}e^{-\alpha^2r^2}dr}\int_0^\infty e^{-\alpha^2r^2}g_{l,j_1}(r^2)g_{l,j_2}(r^2)r^{n-1+2l}dr = \delta_{j_1,j_2}$$

holds.

Since $g_{l,j}(R)$ is a polynomial of degree j, $g_{l,j}(||x||^2)$ is a polynomial in n variables of degree $2j$.

For each integer $0 \le l \le e$, let $\mathcal{H}_l = \{g_{l,j}(||x||^2)\varphi_{l,i}(x)|\ j \le [\frac{e-l}{2}],\ 1 \le i \le N_l\}$ and $\mathcal{H} = \cup_{l=0}^{e}\mathcal{H}_l$. Then we can easily see that $\mathcal{H}$ is a basis of the vector space $\mathcal{P}_e(\mathbb{R}^n)$ consisting of all the polynomials in n variables of degree at most e (see [10], cf. [6] for a more general result).

Theorem 2.4 *Let X be a Gaussian $2e$-design and $\mathcal{H}$ be the basis of $\mathcal{P}_e(\mathbb{R}^n)$ defined as above. Let M be the matrix which is indexed by the set $X \times \mathcal{H}$, whose $(u, g_{l,j}\varphi_{l,i})$-entry is defined by*

$$\sqrt{\omega(u)}g_{l,j}(||u||^2)\varphi_{l,i}(u).$$

Then we have

$${}^tM\ M = I.$$

Proof: The $(g_{l_1,j_1}\varphi_{l_1,i_1}, g_{l_2,j_2}\varphi_{l_2,i_2})$-entry of ${}^tM\ M$ is given by

$$\sum_{u\in X}\omega(u)g_{l_1,j_1}(||u||^2)\varphi_{l_1,i_1}(u)g_{l_2,j_2}(||u||^2)\varphi_{l_2,i_2}(u)$$
$$= \frac{1}{V(\mathbb{R}^n)}\int_{\mathbb{R}^n}e^{-\alpha^2||x||^2}g_{l_1,j_1}(||x||^2)g_{l_2,j_2}(||x||^2)\varphi_{l_1,i_1}(x)\varphi_{l_2,i_2}(x)\,dx$$

$$= \frac{1}{V(\mathbb{R}^n)} \int_0^\infty e^{-\alpha^2 r^2} g_{l_1,j_1}(r^2) g_{l_2,j_2}(r^2) r^{n-1+l_1+l_2} dr \int_{S^{n-1}} \varphi_{l_1,i_1}(\xi)\varphi_{l_2,i_2}(\xi) d\sigma(\xi)$$
$$= \delta_{l_1,l_2}\delta_{i_1,i_2} \frac{1}{\int_0^\infty r^{n-1} e^{-\alpha^2 r^2} dr} \int_0^\infty e^{-\alpha^2 r^2} g_{l_1,j_1}(r^2) g_{l_1,j_2}(r^2) r^{n-1+2l_1} dr$$
$$= \delta_{l_1,l_2}\delta_{i_1,i_2}\delta_{j_1,j_2}$$

□

The following corollary is well known and proved by a basis-free argument. However, since it is also immediately obtained from Theorem 2.4, we state here.

Corollary 2.5 (= Theorem 1.2) *If X is a Gaussian $2e$-design, then the following hold*:

$$|X| \geq \dim(\mathcal{P}_e(\mathbb{R}^n)) = \binom{n+e}{e}.$$

Proof: Since the rank of ${}^t M M$ is $\binom{n+e}{e}$, we have the Corollary. □

We state Theorem 1.3 here again.

Theorem 1.3 *Let X be a tight Gaussian design. Let p be the number of the concentric spheres which support X. Then the following assertions hold*:
(1) $[\frac{e}{2}] + 1 \leq p$ *holds.*
(2) $\omega(x)$ *is constant on each X_i, for $i = 1, \ldots, p$.*
(3) *Each X_i is an at most e-distance set for $i = 1, \ldots, p$.*

Proof:

(1) Since $|X| = \binom{n+e}{e}$, the matrix M is a nonsingular square matrix. Hence $M\,{}^t M = I$ holds. To have nonsingular matrix M, we should have the property that the set of the polynomials $\{g_{e,j}(||x||^2) \mid j = 0, \ldots, [\frac{e}{2}]\}$ is linearly independent on X. This implies $p \geq [\frac{e}{2}] + 1$.
(2) For a vector $u \neq 0$ in X, the (u, u)-entry of $M\,{}^t M$ is given by

$$\omega(u) \sum_{l+2j \leq e} g_{l,j}(||u||^2)^2 \sum_{i=1}^{N_l} \varphi_{l,i}(u)^2 = \omega(u) \sum_{l+2j \leq e} ||u||^{2l} g_{l,j}(||u||^2)^2 Q_l(1). \quad (2.1)$$

Let $u \in X_i$ and $R_i = r_i^2$. Since $M\, {}^tM = I$ the Eq. (2.1) implies

$$\omega(u) \sum_{l+2j\le e} {R_i}^l g_{l,j}(R_i)^2 Q_l(1) = 1. \tag{2.2}$$

Hence $\omega(u)$ only depends on the norm r_i of the vector u.

(3) For $u, v \neq 0$, the (u, v)-entry with $u \neq v$ is given by

$$\begin{aligned}&\sqrt{\omega(u)\omega(v)} \sum_{l+2j\le e} g_{l,j}(||u||^2) g_{l,j}(||v||^2) \sum_{i=1}^{N_l} \varphi_{l,i}(u)\varphi_{l,i}(v)\\ &= \sqrt{\omega(u)\omega(v)} \sum_{l+2j\le e} ||u||^l ||v||^l g_{l,j}(||u||^2) g_{l,j}(||v||^2) Q_l\left(\left(\frac{u}{||u||}, \frac{v}{||v||}\right)\right).\end{aligned} \tag{2.3}$$

Suppose that $u, v \in X_i$ and $||u||^2 = ||v||^2 = r_i^2 \neq 0$. Let $R_i = r_i^2$. Then the equation (2.3) implies

$$\sum_{l+2j\le e} {R_i}^l g_{l,j}(R_i)^2 Q_l\left(\frac{(u, v)}{R_i}\right) = 0. \tag{2.4}$$

Here $Q_l(y)$ is a polynomial in y of degree l. Hence for each fixed value R_i, the left hand side of the equation (2.4) is a polynomial in (u, v) of degree at most e. This implies that each X_i is an at most e-distance set.

□

3. Proof of Theorem 1.4

In this section we consider the Gaussian tight 4-designs, i.e., the case $e = 2$. Since

$$\frac{d(r^l e^{-\alpha^2 r^2})}{dr} = -2\alpha^2 r^{l+1} e^{-\alpha^2 r^2} + l r^{l-1} e^{-\alpha^2 r^2}$$

for all $l > 0$, we have

$$\int_0^\infty r^{l+1} e^{-\alpha^2 r^2} dr = \frac{l}{2\alpha^2} \int_0^\infty r^{l-1} e^{-\alpha^2 r^2} dr. \tag{3.1}$$

First we give explicitly the polynomials $g_{l,j}(R)$ of degree j, $0 \le j \le [\frac{2-l}{2}]$, satisfying

$$\frac{1}{\int_0^\infty r^{n-1} e^{-\alpha^2 r^2}} \int_0^\infty g_{l,j_1}(r^2) g_{l,j_2}(r^2) r^{n-1} e^{-\alpha^2 r^2} dr = \delta_{j_1, j_2}.$$

If $l = 0$, then $j = 0, 1$. Since $g_{0,0} = g_{0,0}(R)$ is a constant we have $g_{0,0}^2 = 1$. Let $g_{0,1}(R) = aR + b$. Then

$$\int_0^\infty (ar^2 + b)r^{n-1}e^{-\alpha^2 r^2} dr = 0$$

implies $b = -\frac{na}{2\alpha^2}$, and

$$\frac{1}{\int_0^\infty r^{n-1}e^{-\alpha^2 r^2} dr} \int_0^\infty (ar^2 + b)^2 r^{n-1} e^{-\alpha^2 r^2} dr = 1$$

implies

$$a^2 = \frac{\int_0^\infty r^{n-1}e^{-\alpha^2 r^2} dr}{\int_0^\infty (r^2 - \frac{n}{2\alpha^2})^2 r^{n-1} e^{-\alpha^2 r^2} dr}.$$

Since the Eq. (3.1) implies

$$\begin{aligned}
&\int_0^\infty \left(r^2 - \frac{n}{2\alpha^2}\right)^2 r^{n-1} e^{-\alpha^2 r^2} dr \\
&\quad = \int_0^\infty r^{n+3} e^{-\alpha^2 r^2} dr - \frac{n}{\alpha^2} \int_0^\infty r^{n+1} e^{-\alpha^2 r^2} dr + \frac{n^2}{4\alpha^4} \int_0^\infty r^{n-1} e^{-\alpha^2 r^2} dr \\
&\quad = \left(\frac{(n+2)n}{4\alpha^4} - \frac{n^2}{2\alpha^4} + \frac{n^2}{4\alpha^4}\right) \int_0^\infty r^{n-1} e^{-\alpha^2 r^2} dr = \frac{n}{2\alpha^4} \int_0^\infty r^{n-1} e^{-\alpha^2 r^2} dr,
\end{aligned}$$

we have $a^2 = \frac{2\alpha^4}{n}$. Hence we have

$$g_{0,1}(R)^2 = \frac{2\alpha^4}{n}\left(R - \frac{n}{2\alpha^2}\right)^2. \tag{3.2}$$

If $l = 1$, then $j = 0$ and $g_{1,0} = g_{1,0}(R)$ is a constant. Hence

$$\frac{1}{\int_0^\infty r^{n-1}e^{-\alpha^2 r^2} dr} \int_0^\infty {g_{1,0}}^2 r^{n+1} e^{-\alpha^2 r^2} dr = 1$$

implies

$${g_{1,0}}^2 = \frac{\int_0^\infty r^{n-1}e^{-\alpha^2 r^2} dr}{\int_0^\infty r^{n+1}e^{-\alpha^2 r^2} dr} = \frac{2\alpha^2}{n}. \tag{3.3}$$

If $l = 2$, then $j = 0$ and $g_{2,0} = g_{2,0}(R)$ is a constant. Hence

$$\frac{1}{\int_0^\infty r^{n-1}e^{-\alpha^2 r^2}dr}\int_0^\infty {g_{2,0}}^2 r^{n+3}e^{-\alpha^2 r^2}dr = 1$$

implies

$${g_{2,0}}^2 = \frac{4\alpha^4}{(n+2)n}. \tag{3.4}$$

Substitute the values $g_{l,j}(||u||^2)$ in the Eq. (2.2) we obtain

$$Q_0(1) + \frac{2\alpha^4}{n}\left(R - \frac{n}{2\alpha^2}\right)^2 Q_0(1) + R\frac{2\alpha^2}{n}Q_1(1) + R^2\frac{4\alpha^4}{(n+2)n}Q_2(1) = \frac{1}{\omega(u)},$$

where $R = ||u||^2$. Since $Q_0 \equiv 1,\ Q_1(y) = ny$, and $Q_2(y) = \frac{n+2}{2}(ny^2 - 1)$, we obtain

$$2\alpha^4 R^2 + \frac{n}{2} + 1 = \frac{1}{\omega(u)}. \tag{3.5}$$

Also the Eq. (2.4) implies

$$1 + \frac{2\alpha^4}{n}\left(R - \frac{n}{2\alpha^2}\right)^2 + 2\alpha^2(u,v) + \frac{2\alpha^4}{n}(n(u,v)^2 - R^2) = 0 \tag{3.6}$$

for $u, v \in X$ with $||u||^2 = ||v||^2 = R,\ u \neq v$. Let $||u - v||^2 = A$. Then we have $(u, v) = R - \frac{A}{2}$. Then the Eq. (3.6) yields

$$\frac{1}{2}\alpha^4 A^2 - \alpha^2(2R\alpha^2 + 1)A + 2R^2\alpha^4 + \frac{n}{2} + 1 = 0. \tag{3.7}$$

Proof of Theorem 1.4 (1): Assume $0 \in X$. Then $|X - \{0\}| < \binom{n+2}{2}$. By Proposition 2.3, X is a Euclidean 4-design. Hence $X - \{0\}$ is also a Euclidean 4-design. It is known that if the number of the spheres which support a Euclidean 4-design in $\mathbb{R}^n$ is more than 1, then its cardinality must be bounded below by $\binom{n+2}{2}$. Since $|X - \{0\}| < \binom{n+2}{2}$, $X - \{0\}$ must be contained in a sphere centered origin. Hence $X - \{0\}$ is a tight spherical 4-design. We only need to verify the equation given in the definition of Gaussian design for polynomials $||x||^{2j},\ j = 1,\ 2$, that is

$$\frac{1}{V(\mathbb{R}^n)}\int_{\mathbb{R}^n}||x||^{2j}e^{-\alpha^2||x||^2}dx = \left(\frac{(n+2)(n+1)}{2} - 1\right)\omega(u)||u||^{2j}.$$

Let $u \in X - \{0\}$ and $||u||^2 = R$. If $j = 1$, then

$$\frac{\int_0^\infty e^{-\alpha^2 r^2} r^{n+1} dr}{\int_0^\infty e^{-\alpha^2 r^2} r^{n-1} dr} = \omega(u)\left(\binom{n+2}{2} - 1\right) R.$$

Hence we have

$$\frac{n}{2\alpha^2} = \omega(u)\left(\binom{n+2}{2} - 1\right) R.$$

If $j = 2$, then

$$\frac{\int_0^\infty e^{-\alpha^2 r^2} r^{n+3} dr}{\int_0^\infty e^{-\alpha^2 r^2} r^{n-1} dr} = \omega(u)\left(\binom{n+2}{2} - 1\right) R^2.$$

Hence we have

$$\frac{n(n+2)}{4\alpha^4} = \omega(u)\left(\binom{n+2}{2} - 1\right) R^2.$$

This implies

$$\omega(u) = \frac{2}{(n^2 + 5n + 6)}, \quad r = \sqrt{\frac{n+2}{2\alpha^2}}.$$

□

Proof of Theorem 1.4 (2): First we prove the following proposition. □

Proposition 3.1 *Let X be a Gaussian tight 4-design. Assume $p = 2$ and $0 \notin X$. Then the following equation holds*:

$$4(|X_i| - n)\alpha^4 {R_i}^2 - 4|X_i| n R_1 \alpha^2 - n^2 + n^2|X_i| + 2|X_i|n - 2n = 0 \tag{3.8}$$

for $i = 1$ and 2.

Proof: By the assumption of the Proposition 3 we have $X = X_1 \cup X_2$ and $R_1 = {r_1}^2 \neq 0$ and $R_2 = {r_2}^2 \neq 0$. Since the weight function is constant on each X_i, let $\omega(u) = \omega_i$ on $X_i (i = 1, 2)$. Let $N = |X| = \binom{n+2}{2}$. Because the roles of X_1 and X_2 are symmetric it is enough if we prove the Eq. (3.8) holds for $i = 1$. By the definition of Gaussian 4-designs we have

$$|X_1|\omega_1 + (N - |X_1|)\omega_2 = 1, \tag{3.9}$$

and

$$\frac{1}{\int_{\mathbb{R}^n} e^{-\alpha^2||x||^2} dx} \int_{\mathbb{R}^n} ||x||^{2j} e^{-\alpha^2||x||^2} dx = |X_1|\omega_1 {R_1}^j + (N - |X_1|)\omega_2 {R_2}^j$$

for $j = 0,\ 1,\ 2$. If $j = 1$, then we have

$$\frac{n}{2\alpha^2} = \frac{\int_0^\infty r^{n+1} e^{-\alpha^2 r^2} dr}{\int_0^\infty r^{n-1} e^{-\alpha^2 r^2} dr} = |X_1|\omega_1 R_1 + (N - |X_1|)\omega_2 R_2. \tag{3.10}$$

If $j = 2$, then we have

$$\frac{n(n+2)}{4\alpha^4} = \frac{\int_0^\infty r^{n+3} e^{-\alpha^2 r^2} dr}{\int_0^\infty r^{n-1} e^{-\alpha^2 r^2} dr} = |X_1|\omega_1 {R_1}^2 + (N - |X_1|)\omega_2 {R_2}^2. \tag{3.11}$$

Also the Eq. (3.5) implies

$$\omega_1 = \frac{2}{4\alpha^4 R_1^2 + n + 2}. \tag{3.12}$$

By the Eqs. (3.9) and (3.12) we have

$$\omega_2 = \frac{2(1 - w_1|X_1|)}{n^2 + 3n + 2 - 2|X_1|} = \frac{2(-2|X_1| + 4\alpha^4 {R_1}^2 + n + 2)}{\left(4\alpha^4 {R_1}^2 + n + 2\right)\left(n^2 + 3n + 2 - 2|X_1|\right)}. \tag{3.13}$$

The assumption $\omega_2 > 0$ implies $4\alpha^4 {R_1}^2 + n + 2 - 2|X_1| > 0$. The Eqs. (3.10), (3.12) and (3.13) imply

$$R_2 = \frac{n - 2|X_1|\omega_1 R_1 \alpha^2}{2\alpha^2 (N - |X_1|)\omega_2} = \frac{-4|X_1|R_1\alpha^2 + 4n\alpha^4 {R_1}^2 + n^2 + 2n}{2\alpha^2(-2|X_1| + 4\alpha^4 {R_1}^2 + n + 2)}. \tag{3.14}$$

Then the Eqs. (3.11), (3.12), (3.13) and (3.14) imply the following equation:

$$\frac{-n^2 + n^2|X_1| + 2|X_1|n - 4|X_1|R_1\alpha^2 n - 2n - 4n\alpha^4 {R_1}^2 + 4|X_1|\alpha^4 {R_1}^2}{2(-2|X_1| + 4\alpha^4 {R_1}^2 + n + 2)\alpha^4} = 0.$$

Hence we have

$$4(|X_1| - n)\alpha^4 {R_1}^2 - 4|X_1|nR_1\alpha^2 - n^2 + n^2|X_1| + 2|X_1|n - 2n = 0.$$

□

Let $F(x, R)$ be the polynomial defined by

$$F(x, R) = 4(x - n)\alpha^4 R^2 - 4xnR\alpha^2 - n^2 + n^2 x + 2xn - 2n. \tag{3.15}$$

Proposition 3.2 *For $i = 1$ and 2, $|X_i| > n$ holds.*

Proof: Assume one of X_i is of size n. We may assume $|X_1| = n$. Then the Eq. (3.8) implies

$$R_1 = \frac{(n^2+n-2)}{4n\alpha^2}. \tag{3.16}$$

Then the Eqs. (3.7) and (3.16) imply

$$4\alpha^4n^2A^2 + (8n - 12n^2 - 4n^3)\alpha^2 A + n^4 + 6n^3 + 5n^2 - 4n + 4 = 0.$$

However the discriminant of this quadratic equation is $-128\alpha^4n^3 < 0$, so there is no solution for A. Hence $|X_i| \neq n$ for $i = 1,\ 2$.

Next assume one of X_i has the cardinality less than n. Then we may assume $|X_1| < n$. The Eq. (3.8) implies

$$R_1 = \frac{-|X_1|n \pm \sqrt{(|X_1|-1)n^3 + (3|X_1|-2)n^2 - 2|X_1|(|X_1|-1)n}}{2\alpha^2(n-|X_1|)}.$$

Since $R_1 > 0$ and $|X_1| < n$ we have

$$R_1 = \frac{-|X_1|n + \sqrt{(|X_1|-1)n^3 + (3|X_1|-2)n^2 - 2|X_1|(|X_1|-1)n}}{2\alpha^2(n-|X_1|)}. \tag{3.17}$$

Then the Eqs. (3.7) and (3.17) imply

$$\begin{aligned}&\frac{1}{2}\alpha^4A^2 + \frac{\alpha^2\big((n+1)|X_1| - n - \sqrt{(|X_1|-1)n^3 + (3|X_1|-2)n^2 - 2|X_1|(|X_1|-1)n}\big)}{n-|X_1|}A\\ &\quad + \frac{|X_1|}{2(n-|X_1|)^2} \times (n(n^2+n-2) + (n^2-n+2)|X_1|\\ &\quad - 2n\sqrt{(|X_1|-1)n^3 + (3|X_1|-2)n^2 - 2|X_1|(|X_1|-1)n}) = 0.\end{aligned}$$

Then the discriminant of the quadratic equation of A given above is

$$-\frac{\alpha^4(n^2+n+|X_1|n - |X_1| - 2\sqrt{(|X_1|-1)n^3 + (3|X_1|-2)n^2 - 2|X_1|(|X_1|-1)n})}{n-|X_1|}.$$

Since $n > |X_1|$ we have

$$\begin{aligned}&(n^2+n+|X_1|n-|X_1|)^2 - 4((|X_1|-1)n^3 + 3n^2|X_1| - 2n^2 - 2|X_1|^2n + 2|X_1|n)\\ &\quad = (n-|X_1|)(n(n^2+6n+9) - |X_1|(n^2+6n-1)) > 0.\end{aligned}$$

Hence the discriminant of the quadratic equation of A is a negative number and there is no real valued solution for A. This is a contradiction. Therefore we have $|X_i| > n$ for $i = 1,\ 2$. □

Now, we may assume that $|X_1| \geq |X_2|$. Then Proposition 3.2 implies

$$\max\left\{n+1, \frac{(n+2)(n+1)}{4}\right\} \leq |X_1| \leq \frac{(n+2)(n+1)}{2} - (n+1) = \frac{n(n+1)}{2}.$$

First we prove Theorem 1.4 (2) for $n = 2$. Let $n = 2$. Since $|X| = 6$ and $|X_i| > 2,\ (i = 1,\ 2)$, we have $|X_1| = |X_2| = 3$. Then Proposition 3.1 implies

$$r_1 = \sqrt{R_1} = \frac{\sqrt{3+\sqrt{5}}}{\alpha} \text{ or } \frac{\sqrt{3-\sqrt{5}}}{\alpha}.$$

Let $R = \frac{3+\varepsilon\sqrt{5}}{\alpha^2}$. Then the Eq. (3.7) implies

$$A = \frac{3(3+\varepsilon\sqrt{5})}{\alpha^2}, \quad \frac{(5+\varepsilon\sqrt{5})}{\alpha^2}.$$

Since the regular triangle on the circle of radius $\frac{\sqrt{3+\varepsilon\sqrt{5}}}{\alpha}$ has edges of length $\frac{\sqrt{3}\sqrt{3+\varepsilon\sqrt{5}}}{\alpha}$, X_i must form a regular triangle for $i = 1,\ 2$. The Eq. (2.3) for $u \in X_1$, $v \in X_2$ implies

$$2\left(\frac{u}{||u||}, \frac{v}{||v||}\right)^2 + \left(\frac{u}{||u||}, \frac{v}{||v||}\right) - 1 = 0$$

Hence we have

$$\left(\frac{u}{||u||}, \frac{v}{||v||}\right) = \frac{1}{2} \quad \text{or} \quad -1.$$

This gives the design given in the Theorem 1.4 (2). (i).

Next we assume $n \geq 3$. Since the maximum cardinality of the 1-distance sets in $\mathbb{R}^n$ is $n+1$ and $|X_1| \geq \frac{(n+2)(n+1)}{4} > n+1$ for $n \geq 3$, X_1 is a 2-distance set. Let $\alpha_1,\ \alpha_2$ be the two distances of X_1 satisfying $\alpha_1 > \alpha_2$. Let $A_1 = {\alpha_1}^2$ and $A_2 = {\alpha_2}^2$. Then A_1 and A_2 are the distinct solution of the Eq. (3.7) for $R = R_1$, where $R_1 = r_1^2$.

Proposition 3.3 *If $n \geq 7$, then the following assertions hold*:
(1) $(\frac{A_2+A_1}{A_2-A_1})^2 = (2k-1)^2$,
(2) $\frac{(1+2\alpha^2 R_1)^2}{4\alpha^2 R_1 - n - 1} = (2k-1)^2$,

with an integer k satisfying $2 \leq k < \sqrt{\frac{n}{2}} + \frac{1}{2}$.

Proof: Since $n \geq 7$, we have $|X_1| \geq \frac{(n+2)(n+1)}{4} > 2n+3$. The theorem of Larman-Rogers-Seidel [18] implies that if $|X_1| > 2n+3$ then

$$\frac{A_2}{A_1} = \frac{k-1}{k} \tag{3.18}$$

with an integer k satisfying $2 \leq k < \sqrt{\frac{n}{2}} + \frac{1}{2}$. The Eq. (3.18) implies

$$\left(\frac{A_2 + A_1}{A_2 - A_1}\right)^2 = (2k-1)^2.$$

Since the (3.7) must have two distinct positive solutions A_1 and A_2 the discriminant of the quadratic Eq. (3.7) of A has to be positive. This implies $4\alpha^2 R_1 - n - 1 > 0$. Solving for A_1 and A_2 with $A_1 > A_2$ explicitly we obtain

$$\left(\frac{A_2 + A_1}{A_2 - A_1}\right)^2 = \frac{(1 + 2\alpha^2 R_1)^2}{4\alpha^2 R_1 - n - 1}.$$

□

Let $G(R)$ be the rational function of R defined by

$$G(R) = \frac{(1 + 2\alpha^2 R)^2}{4\alpha^2 R - n - 1}$$

and let $R(x)$ be a continuous function of x satisfying

$$F(x, R(x)) = 0,$$

where $F(x, R)$ is the polynomial defined by the Eq. (3.15). Then

$$R(x) = \frac{xn + \varepsilon\sqrt{-n^3 + xn^3 + 3n^2x - 2n^2 - 2x^2n + 2xn}}{2\alpha^2(x-n)}, \tag{3.19}$$

where $\varepsilon = 1$ or -1. Then Proposition 3.1 implies that if there exists a Gaussian tight 4-design X satisfying $0 \notin X$ and $p = 2$, then $R_1 = R(|X_1|)$, $F(|X_1|, R(|X_1|)) = 0$ for one of the solution $R(x)$. Moreover if $|X_1| > 2n+3$, then $G(R(|X_1|))$ is a square of an odd integer. We have the following proposition on the property of the function $G(R(x))$.

Proposition 3.4 *Assume $n \geq 10$ and $n < \frac{(n+2)(n+1)}{4} \leq x \leq \frac{n(n+1)}{2}$, then the following conditions hold*:

(1)

$$\frac{dG(R(x))}{dx} < 0,$$

(2)

$$n + 3 < G(R(x)) < n + 6.$$

Proof: Let $R = R(x)$.

$$\frac{dG(R(x))}{dx} = \frac{dG(R)}{dR}\frac{dR}{dx}.$$

$$\frac{dG(R)}{dR} = \frac{d}{dR}\left(\frac{(1+2\alpha^2 R)^2}{4\alpha^2 R - n - 1}\right) = \frac{4\alpha^2(1+2\alpha^2 R)(2\alpha^2 R - n - 2)}{(4\alpha^2 R - n - 1)^2}.$$

Since $R = R(x)$ we have

$$2\alpha^2 R - n - 2 = -\frac{n^2 + 2n - 2x + \varepsilon\sqrt{-n(n^2 - xn^2 + 2n - 3nx - 2x + 2x^2)}}{x - n}.$$

Since $n < \frac{(n+2)(n+1)}{4} \le x \le \frac{n(n+1)}{2}$,

$$\begin{aligned}&(n^2 + 2n - 2x)^2 - (\sqrt{-n(n^2 - xn^2 + 2n - 3nx - 2x + 2x^2)})^2\\ &\quad = (2+n)(x-n)(2x - n^2 - 3n) < 0\end{aligned}$$

holds. Hence if $\varepsilon = +1$, then $2\alpha^2 R - n - 2 < 0$ and if $\varepsilon = -1$, then $2\alpha^2 R - n - 2 > 0$. This implies

$$\varepsilon\frac{dG(R)}{dR} < 0$$

for any $R = R(x)$. On the other hand

$$\frac{dR}{dx} = \frac{n(\varepsilon(n^3 + n^2 + xn^2 - 2n - nx + 2x) - 2n\sqrt{-n(n^2 - xn^2 + 2n - 3nx - 2x + 2x^2)})}{4(x-n)^2\alpha^2\sqrt{-n(n^2 - xn^2 + 2n - 3nx - 2x + 2x^2)}}.$$

Since

$$\begin{aligned}&(n^3 + n^2 + xn^2 - 2n - nx + 2x)^2 - (2n\sqrt{-n(n^2 - xn^2 + 2n - 3nx - 2x + 2x^2)})^2\\ &\quad = (n+2)(n^3 + 4n^2 - 3n + 2)(x-n)^2 > 0,\end{aligned}$$

we have $\varepsilon\frac{dR}{dx} > 0$. Hence we have $\frac{dG(R(x))}{dx} < 0$. This completes the proof for (1).

Next we prove (2). Since $G(R(x))$ is a decreasing function for $\frac{(n+2)(n+1)}{4} \leq x \leq \frac{n(n+1)}{2}$ we only need to show that $n+6 > G(R(\frac{(n+2)(n+1)}{4}))$ and $n+3 < G(R(\frac{n(n+1)}{2}))$. We have

$$\begin{aligned} & n+6-G\left(R\left(\frac{(n+2)(n+1)}{4}\right)\right) \\ & \quad = \frac{1}{(n^2-n+2)(n^3+6n^2+3n-2+2\varepsilon\sqrt{2n(n+2)(n^3+4n^2-3n+2)})} \\ & \quad \times (n^5-n^4-21n^3+41n^2+32n-28 \\ & \quad + 2\varepsilon(n^2-5n+10)\sqrt{2n(n+2)(n^3+4n^2-3n+2)}). \end{aligned} \tag{3.20}$$

If $n \geq 10$, then the numerator of the right hand side of the Eq. (3.20) is positive because

$$\begin{aligned} & (n^5-n^4-21n^3+41n^2+32n-28)^2 \\ & \quad -(2(n^2-5n+10)\sqrt{2n(n+2)(n^3+4n^2-3n+2)})^2 \\ & \quad = (n^6-8n^5-30n^4+188n^3-15n^2-1052n+196)(n^2-n+2)^2 > 0 \end{aligned}$$

for $n \geq 10$. And the denominator of (3.20) is positive because

$$\begin{aligned} & (n^3+6n^2+3n-2)^2-(2\sqrt{2n(n+2)(n^3+4n^2-3n+2)})^2 \\ & \quad = (n^2-n+2)(n^4+5n^3-3n^2-21n+2) > 0 \end{aligned}$$

for $n \geq 2$. Hence we have

$$G(R(x)) \leq G\left(R\left(\frac{(n+2)(n+1)}{4}\right)\right) < n+6$$

for any x satisfying $\frac{(n+2)(n+1)}{4} \leq x \leq \frac{n(n+1)}{2}$. Next we will show the second inequality. We have

$$G\left(R\left(\frac{n(n+1)}{2}\right)\right) - (n+3) = \frac{4(n^2+n+2\varepsilon\sqrt{n^2+n-1})}{(n-1)(n^2+2n+1+4\varepsilon\sqrt{n^2+n-1})}.$$

The numerator of the right hand side is positive because

$$(n^2+n)^2-(2\sqrt{n^2+n-1})^2 = (n+2)^2(n-1)^2 > 0$$

and the denominator of the right is positive because

$$(n^2+2n+1)^2-(4\sqrt{n^2+n-1})^2 = (n-1)(n^3+5n^2-5n-17) > 0$$

for $n \geq 2$. Hence we have $G(R(\frac{n(n+1)}{2})) > (n+3)$. □

Since the function $G(R(x))$ is decreasing monotonously, Proposition 3.4 implies the following proposition.

Proposition 3.5 *Let X be a Gaussian tight 4-design. Assume $p = 2$ and $0 \notin X$ and $|X_1| \geq |X_2|$. With these conditions, if $n \geq 10$, then there exists an integer $k \geq 2$ satisfying*

$$n = (2k-1)^2 - 4, \quad \textit{or} \quad n = (2k-1)^2 - 5,$$

and

$$\left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2 = (2k-1)^2.$$

Next we prove the following proposition.

Proposition 3.6

(1) *If $n = (2k-1)^2 - 5$, then there is no integer x satisfying $\frac{n+2}{4} \leq x \leq \frac{n(n+1)}{2}$ and $G(R(x)) = (2k-1)^2$.*

(2) *If $n = (2k-1)^2 - 4$, then there is no integer x satisfying $\frac{n+2}{4} \leq x \leq \frac{n(n+1)}{2}$ and $G(R(x)) = (2k-1)^2$.*

Proof:

(1) Let $n = (2k-1)^2 - 5$. Then equation $G(R(x)) = n + 5$ implies

$$\begin{aligned}&(6-4n)x^2 - xn^2 + (n^3 - 10n)x + n^4 + 5n^3 + 4n^2 \\ &\quad + 2\varepsilon\sqrt{n(-n^2 + xn^2 - 2n + 3xn + 2x - 2x^2)}(-4x + 4n + n^2) = 0.\end{aligned}$$

Then

$$\begin{aligned}&((6-4n)x^2 - xn^2 + (n^3 - 10n)x + n^4 + 5n^3 + 4n^2)^2 \\ &\qquad - (2\varepsilon\sqrt{n(-n^2 + xn^2 - 2n + 3xn + 2x - 2x^2)}(-4x + 4n + n^2))^2 \\ &\quad = ((16n^2 + 80n + 36)x^2 - (8n^4 + 76n^3 + 220n^2 + 176n)x \\ &\qquad + n^6 + 14n^5 + 73n^4 + 168n^3 + 144n^2)(x - n)^2\end{aligned}$$

implies

$$\begin{aligned}&(16n^2 + 80n + 36)x^2 - (8n^4 + 76n^3 + 220n^2 + 176n)x + n^6 + 14n^5 + 73n^4 \\ &\quad + 168n^3 + 144n^2 = 0. \qquad (3.21)\end{aligned}$$

The discriminant of the quadratic Eq. (3.21) of x is equal to

$$128n^2(n+5)(n+4)^2 = 2\cdot 8^2 n^2(2k-1)^2(n+4)^2.$$

Hence the solution x of the Eq. (3.21) is not an integer.

(2) Let $n=(2k-1)^2-4$. Then $\frac{n(n+1)}{3}=\frac{2}{3}(2k+1)(2k-3)(2k^2-2k-1)$ is an integer. We compute $n+4-G(R(\frac{n(n+1)}{3}))$. Then we have

$$n+4-G\left(R\left(\frac{n(n+1)}{3}\right)\right)=\frac{-4(3\varepsilon\sqrt{n^3+8n^2+4n-12}+2n^2+4n+2)}{(n^2+3n+2+2\varepsilon\sqrt{n^3+8n^2+4n-12})(n-2)}.$$

Since

$$(2n^2+4n+2)^2-\left(3\varepsilon\sqrt{n^3+8n^2+4n-12}\right)^2=(n+4)(4n+7)(n-2)^2>0$$

and

$$\begin{aligned}&(n^2+3n+2)^2-\left(2\varepsilon\sqrt{n^3+8n^2+4n-12}\right)^2\\&\quad=(n-2)(n^3+4n^2-11n-26)>0,\end{aligned}$$

we have

$$n+4-G\left(R\left(\frac{n(n+1)}{3}\right)\right)<0. \tag{3.22}$$

Next we compute $(n+4)-G(R(\frac{n(n+1)}{3}+1))$. Then we have

$$\begin{aligned}&(n+4)-G\left(R\left(\frac{n(n+1)}{3}+1\right)\right)\\&\quad=\frac{8n^4+7n^3+11n^2-69n+45+6\varepsilon n(2n-3)\sqrt{n^3+8n^2+n+3}}{3\left(n^3+3n^2+5n-3+2\varepsilon n\sqrt{n^3+8n^2+n+3}\right)}.\end{aligned}$$

Since

$$\begin{aligned}&(8n^4+7n^3+11n^2-69n+45)^2-\left(6\varepsilon n(2n-3)\sqrt{n^3+8n^2+n+3}\right)^2\\&\quad=(64n^4+224n^3-239n^2-390n+225)(n^2-2n+3)^2>0\end{aligned}$$

and

$$(n^3+3n^2+5n-3)^2-\left(2\varepsilon n\sqrt{n^3+8n^2+n+3}\right)^2$$
$$=(n+1)(n^2-2n+3)(n^3+3n^2-11n+3)>0,$$

we have

$$n+4-G\left(R\left(\frac{n(n+1)}{3}+1\right)\right)>0. \tag{3.23}$$

The Eqs. (3.22) and (3.23) imply

$$G\left(R\left(\frac{n(n+1)}{3}+1\right)\right)<n+4<G\left(R\left(\frac{n(n+1)}{3}\right)\right).$$

Since $\frac{n(n+1)}{3}$ and $\frac{n(n+1)}{3}+1$ are integers and the function $G(R(x))$ decreases monotonously as x increases, there is no integer x satisfying $G(R(x))=n+4$. □

Proposition 3.6 implies Theorem 1.4 (2) for $n\geq 10$. If $n=7,\ 8,\ 9$(consequently $|X_1|>2n+3$) we compute $G(R(|X_1|))$ explicitly for each case and find out $G(R(|X_1|))$ is not a square of any odd integer.

The remaining cases are listed below. In the following list ε is the sign given in the definition of $R(x)$ (see Eq. (3.19)).

Case $n=6$, then $14\leq|X_1|\leq 21$. If $|X_1|>2n+3=15$, then we find out $G(R(|X_1|))$ is not a square of any odd integer.

If $|X_1|=14$, then $A_1/A_2=1.829374832(\varepsilon=1)$ or $1.774847299(\varepsilon=-1)$
If $|X_1|=15$, then $A_1/A_2=1.855307824(\varepsilon=1)$ or $1.805245000(\varepsilon=-1)$

Case $n=5$, then $11\leq|X_1|\leq 15$. If $|X_1|>2n+3=13$, then we find out $G(R(|X_1|))$ is not a square of any odd integer.

If $|X_1|=11$, then $A_1/A_2=1.903339703(\varepsilon=1)$ or $1.819514523(\varepsilon=-1)$
If $|X_1|=12$, then $A_1/A_2=1.942631710(\varepsilon=1)$ or $1.868010544(\varepsilon=-1)$

If $|X_1|=13$, then $A_1/A_2=1.975053872(\varepsilon=1)$ or $1.908655884(\varepsilon=-1)$

Case $n=4$, then $7<\frac{(n+2)(n+1)}{4}\leq|X_1|\leq\frac{n(n+1)}{2}=10<2n+3=11$.

If $|X_1|=8$, then $A_1/A_2=1.983993349(\varepsilon=1)$ or $1.837942554(\varepsilon=-1)$
If $|X_1|=9$, then $A_1/A_2=2.052139475(\varepsilon=1)$ or $1.928970215(\varepsilon=-1)$
If $|X_1|=10$, then $A_1/A_2=2.104297490(\varepsilon=1)$ or $2.000947207(\varepsilon=-1)$

Case $n = 3$, then $5 = \frac{(n+2)(n+1)}{4} \le |X_1| \le \frac{n(n+1)}{2} = 6 < 2n + 3 = 9$.
If $|X_1| = 5$, then $A_1/A_2 = 2.022725571(\varepsilon = 1)$ or $1.691808568(\varepsilon = -1)$
If $|X_1| = 6$, then $A_1/A_2 = 2.178609474(\varepsilon = 1)$ or $1.929947671(\varepsilon = -1)$

Compare with the list of ratios obtained by the method given by Einhorn-Schoeneberg ([13, 14]) we find that there is no 2-distance set with the ratios given above. The reader is referred to [3] for further explanation of the details of the proof. The authors are indebted to Makoto Tagami for the verification of this claim by using computer.

Proof of Theorem 1.4 (3): Let $\omega(u) = \frac{e^{-\alpha^2||u||^2}}{\sum_{x\in X} e^{-\alpha^2||x||^2}}$. Then the Eq. (3.5) implies

$$e^{\alpha^2 R}\sum_{x\in X} e^{-\alpha^2||x||^2} = 2\alpha^4 R^2 + \frac{n}{2} + 1.$$

Let $Y = \alpha^2 R$ and $C = \frac{1}{\sum_{x\in X} e^{-\alpha^2||x||^2}}$. Then

$$e^Y - C\left(2Y^2 + \frac{n}{2} + 1\right) = 0.$$

Let $F(Y) = e^Y - C(2Y^2 + \frac{n}{2} + 1)$. If $4C \le 1$, then $\frac{\partial^2 F(Y)}{\partial Y^2} = e^Y - 4C \ge 0$ for any $Y \ge 0$. Then $\frac{\partial F(Y)}{\partial Y}|_{Y=0} = 1 > 0$. Hence $F(Y)$ is increasing monotonously and has only one solution for $Y \ge 0$. So we assume $4C > 1$. The second derivative $\frac{\partial F(Y)}{\partial Y}$ takes local minimum at $Y = \ln(4C)$. If $\frac{\partial F(Y)}{\partial Y}|_{Y=\ln(4C)} \ge 0$, i.e., if $\ln(4C) \le 1$, then $\frac{\partial F(Y)}{\partial Y} \ge 0$ for any $Y \ge 0$. Hence again $F(Y)$ is increasing monotonously and has only one solution for $Y \ge 0$. So we assume $\ln(4C) > 1$. Then $\frac{\partial F(Y)}{\partial Y} = 0$ has two solutions $0 < Y_1 < Y_2$ and $F(Y)$ takes the local maximum at $Y = Y_1$ and local minimum at $Y = Y_2$. Then $e^{Y_i} = 4CY_i$ implies

$$F(Y_i) = 4CY_i - C\left(2Y_i^2 + \frac{n}{2} + 1\right) = -C\left(2(Y_i - 1)^2 + \frac{n}{2} - 1\right) < 0$$

for any $n \ge 3$. Therefore $F(Y) = 0$ has only one solution for $Y > 0$. This implies that the number of the spheres which support X having positive radius is one. Hence X contains the origin 0. Let $R = R_1 = {r_1}^2$ and $R_2 = {r_2}^2 = 0$. Applying the equation of the definition of Gaussian 4-design for $f(x) = ||x||^{2j}$, $j = 1,\ 2$, we obtain

$$\frac{1}{V(\mathbb{R}^n)}\int_{\mathbb{R}^n} ||x||^{2j} e^{-\alpha^2||x||^2} dx = \sum_{u\in X}\omega(u)||u||^{2j} = \frac{\left(\binom{n+2}{2} - 1\right)R^j e^{-\alpha^2 R}}{1 + \left(\binom{n+2}{2} - 1\right)e^{-\alpha^2 R}}.$$

If $j = 1$, then

$$\frac{n}{2\alpha^2} = \frac{\int_0^\infty e^{-\alpha^2 r^2} r^{n+1} dr}{\int_0^\infty e^{-\alpha^2 r^2} r^{n-1} dr} = \frac{\left(\binom{n+2}{2} - 1\right)R e^{-\alpha^2 R}}{1 + \left(\binom{n+2}{2} - 1\right)e^{-\alpha^2 R}}.$$

If $j = 2$, then

$$\frac{n(n+2)}{4\alpha^4} = \frac{\int_0^\infty e^{-\alpha^2 r^2} r^{n+3} dr}{\int_0^\infty e^{-\alpha^2 r^2} r^{n-1} dr} = \frac{\left(\binom{n+2}{2} - 1\right) R^2 e^{-\alpha^2 R}}{1 + \left(\binom{n+2}{2} - 1\right) e^{-\alpha^2 R}}.$$

Let $Y = \alpha^2 R$. Then we have

$$\frac{n}{2} = \frac{\left(\binom{n+2}{2} - 1\right) Y e^{-Y}}{1 + \left(\binom{n+2}{2} - 1\right) e^{-Y}}, \quad \frac{n(n+2)}{4} = \frac{\left(\binom{n+2}{2} - 1\right) Y^2 e^{-Y}}{1 + \left(\binom{n+2}{2} - 1\right) e^{-Y}}.$$

The first equation implies

$$e^{-Y} = \frac{2}{-n^2 - 3n + 2Yn + 6Y}.$$

Substitute in the second equation we get,

$$\frac{4(-n-2+2Y)Y}{-n+2Y} = 0.$$

Hence we get $Y = \frac{n}{2} + 1$. Then we have

$$\frac{1}{n+3} = e^{-\frac{n}{2}-1}.$$

There is no integer n satisfying the above equation. This completes the proof of Theorem 1.4 (3). □

Proof of Theorem 1.4 (4): Let $\omega(x) = \frac{1}{|X|}$. Then the Eq. (3.5) implies

$$R^2 = \frac{1}{2\alpha^4}\left(|X| - \frac{n+2}{2}\right).$$

This implies that $p = 2$ and $0 \in X$. Then Theorem 1.4 (1) implies that X is not of constant weight. This completes the proof of Theorem 1.4 (4). □

4. Concluding remarks

(1) In the previous paper [3], we determined tight Euclidean 4-designs (i.e., tight rotatable designs of degree 2) in $\mathbb{R}^n$ with constant weight. (As for the definition of Euclidean t-designs in $\mathbb{R}^n$, see Definition 2.1 as well as [19] and [3].) The method employed in this present paper is similar to that of [3]. Generally the treatment in the present paper is slightly simpler than the one in [3].

Although we classified tight Gaussian 4-designs and tight Euclidean 4-designs with constant weight, we are still short of complete classification of those tight 4-designs with an arbitrary weight function. The difficulty lies in the fact that generally we cannot bound the number p (the number of concentric spheres on which X lies). As we have shown in Theorem 1.4, we classified tight Gaussian 4-designs with $p = 2$ and an arbitrary weight function. It would be interesting to classify tight Euclidean designs with $p = 2$ and an arbitrary weight function. In a separate paper under preparation, we are dealing with the classification of optimal tight 4-designs on 2 concentric spheres (cf. [8,16,17,19] etc. for the concept of optimal designs and related statistical background). This classification problem will be reduced to the determination of tight Euclidean 4-designs with $p = 2$ and an arbitrary weight function. For that purpose, the method we used in Theorem 1.4 (2) should be helpful.

(2) In this paper and also in the previous paper [3], we have mostly considered tight 4-designs. It would be interesting to study tight $2e$-designs with $e \geq 3$. One of the reasons of difficulty of this generalization is that we utilized the work of Larman-Rogers-Seidel [18] on 2-distance sets in $\mathbb{R}^n$ in a very crucial way. (see also [13, 14].) So it would be very desirable to obtain similar results for s-distance sets in $\mathbb{R}^n$ with $s \geq 3$, in particular, to study the following problem:

Problem Let X be a 3-distance set in $\mathbb{R}^n$ (or S^{n-1}) with $A(X) := \{d(x, y) \mid x,\ y \in X,\ x \neq y\} = \{\alpha,\ \beta,\ \gamma\}$, where $\alpha,\ \beta,\ \gamma$ are 3 distinct positive real numbers. Then what relations exist among $\alpha,\ \beta,\ \gamma$, if $|X|$ is relatively large.

(3) Let us consider the weight function $e^{-||x||^2}$ on $\mathbb{R}^n$. The suggestion to consider (Gaussian) t-design $X \subset \mathbb{R}^n$ satisfying

$$\frac{1}{V(\mathbb{R}^n)} \int_{\mathbb{R}^n} f(x)e^{-||x||^2} dx = \frac{1}{V(X)} \sum_{x \in X} f(x)e^{-||x||^2} \tag{A}$$

for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most t, was proposed in [1], but was not much studied before. The authors thank de la Harpe and Pache (see [11]) for renewing our interest on this study.

(4) Another natural setting of Gaussian t-design is to consider finite set $X \subset \mathbb{R}^n$ satisfying

$$\frac{1}{V(\mathbb{R}^n)} \int_{\mathbb{R}^n} f(x)e^{-||x||^2} dx = \frac{1}{|X|} \sum_{x \in X} f(x) \tag{B}$$

for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most t, has been a topic of approximation theory for a long time. In some literature, it is called Tchebycheff type quadrature formula. We can regard the setting (A) as the Tchebycheff type quadrature formula for the set of functions $\{f_i(x)e^{-||x||^2} | 1 \leq i \leq N\}$ where $\{f_i | 1 \leq i \leq N\}$ is the basis of the space of the polynomials of degree at most $2e$. So we believe the setting (A) and setting (B) are both interesting.

(5) The famous Jacobi-Gauss quadrature means that for each interval $[a,\ b]$ in $\mathbb{R}^1$ and for any weight function $k(x)$ on $[a,\ b]$, there is a set of points $\{x_1,\dots x_{t+1}\}\subset[a,\ b]$ satisfying

$$\frac{1}{\int_a^b k(x)dx}\int_a^b f(x)k(x)\,dx=\frac{1}{|X|}\sum_{i=1}^{e+1}w(x_i)f(x_i) \tag{C}$$

for all polynomials $f(x)$ of degree $t\le 2e+1$, where the $w(x_i)$ are the Christoffel numbers (cf. [12, 22]). This quadrature is considered as a t-design on $[a,\ b]$ with weight functions $w(x)$.

Dunkl-Xu [12] (see also many references listed in the Reference at the end of this book) studied higher dimensional version, i.e., finite set $X\subset\Omega\subset\mathbb{R}^n$ satisfying

$$\frac{1}{\int_\Omega k(x)dx}\int_\Omega f(x)k(x)\,dx=\frac{1}{|X|}\sum_{i=1}^{\binom{n+e}{e}}w(x_i)f(x_i) \tag{D}$$

for all polynomials $f(x)$ of degree $t\le 2e+1$. Since this is an exact quadrature formula for the degree up to $2e+1$, this can be regarded as a stronger version of the quadrature formula studied here (i.e. the degree up to $2e$). Dunkl-Xu [12] discussed examples of $k(x)$ which has the quadrature formula (D) for some domain $\Omega\subseteq\mathbb{R}^n$

(6) On $\mathbb{R}^1$ or on an interval $(a,\ b)$, we consider the following quadrature

$$\frac{1}{\int_a^b k(x)dx}\int_a^b f(x)k(x)dx=\frac{1}{|X|}\sum_{x\in X}f(x) \tag{E}$$

for all polynomials $f(x)$ of degree at most t. Such a quadrature is called a Tchebycheff type quadrature. Suppose $|X|=e+1$. Then it is known that $t\le 2e+1$. There are some examples, i.e., $a=-1,\ b=1,\ k(x)=(1-x^2)^{-\frac{1}{2}}$, for which this quadrature (E) hold for $t=2e+1$. It is an interesting question whether there are such formulas for smaller values of t with $|X|=e+1$. Some other examples with $t=e$ are known (see e.g. [23]). We consider whether there is $k(x)$ (other than the one mentioned above) for which the Tchebycheff type quadrature hold for $t=2e$ and $|X|=e+1$.

It is interesting to consider higher dimensional analogue of this result. In a certain domain $\Omega\subset\mathbb{R}^n$ and for a certain weight function $k(x)$, there are some examples of $X\subset\Omega$ with $|X|=\binom{n+e}{e}$ when the equation

$$\frac{1}{\int_\Omega k(x)dx}\int_\Omega f(x)k(x)dx=\frac{1}{|X|}\sum_{x\in X}f(x) \tag{F}$$

is satisfied for any polynomials $f(x) = f(x_1, \ldots, x_n)$ of degree $t \leq 2e + 1$ (cf. Dunkl-Xu [12]). From our point of view, it would be interesting to consider weight function $k(x) = h(r)$ which depends only on $r = \sqrt{x_1{}^2 + \cdots + x_n{}^2}$ having Tchebycheff quadrature (F) with the size $|X| = \binom{n+e}{e}$ and $t = 2e$. The main theorem in [3] implies the following theorem which may have an independent interest: (see also [2,4,5,7,9].)

Theorem 4.1 *Let n (≥ 3) be not of the form $n = (2l+1)^2 - 3$ and let $t = 2e = 4$. Then there is no weight function $k(x) = h(r)$ satisfying the condition (F) with a finite set X of cardinality $\binom{n+2}{2}$ for any Ω which is invariant under the action of orthogonal group $O(n)$ of $\mathbb{R}^n$ and satisfying $\int_\Omega f(x)k(x)dx < \infty$ for polynomials of degree at most* 4.

It seems interesting to know whether there is a quadrature formula (F) with $|X| = \binom{n+e}{e}$, $t = 2e$, and $k(x) = h(r)$, for larger values of e. Although it is not yet answered, it seems that, in view of Theorem 4.1, it is unlikely that there are such quadratures for larger values of e.

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