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The completion of the classification of the regular near octagons with thick quads

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Abstract Brouwer and Wilbrink [3] showed the nonexistence of regular near octagons whose parameters s, t_2, t_3 and t satisfy $s \geq 2$, $t_2 \geq 2$ and $t_3 \neq t_2(t_2 + 1)$. Later an arithmetical error was discovered in the proof. Because of this error, the existence problem was still open for the near octagons corresponding with certain values of s, t_2 and t_3. In the present paper, we will also show the nonexistence of these remaining regular near octagons.

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1. Introduction

A *near polygon* ([9]) is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point on L nearest to x. Here distances are measured in the *point graph* or *collinearity graph* of $\mathcal{S}$. If d is the diameter of $\mathcal{S}$, then the near polygon is called a *near* $2d$*-gon*. The unique near 0-gon consists of one point (no lines). The near 2-gons are precisely the lines. Near quadrangles are usually called *generalized quadrangles* (GQ's, [7]). We call a generalized quadrangle *thick* if every line is incident with at least three points and if every point is incident with at least three lines.

If A and B are two nonempty sets of points, then $\mathrm{d}(A, B)$ denotes the minimal distance between a point of A and a point of B. If A is a singleton $\{x\}$, then we will

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also write $\mathrm{d}(x, B)$ instead of $\mathrm{d}(\{x\}, B)$. If A is a nonempty set of points and if $i \in \mathbb{N}$, then we denote by $\Gamma_i(A)$ the set of points y for which $\mathrm{d}(y, A) = i$. If A is a singleton $\{x\}$, then we also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A near $2n$-gon, $n \geq 1$, is said to have *order* (s, t) if every line is incident with precisely $s + 1$ points and if every point is incident with precisely $t + 1$ lines. A near $2n$-gon, $n \geq 1$, is called *regular* if its point graph is a so-called *distance-regular graph* ([2]), or equivalently, if it has an order (s, t) and if there exists constants t_i, $i \in \{0, \ldots, n\}$, such that for any two points x and y at distance i from each other, there are precisely $t_i + 1$ lines through y containing a point at distance $i - 1$ from x. Obviously, $t_0 = -1$, $t_1 = 0$ and $t_n = t$.

A sub near polygon $\mathcal{S}'$ of a near polygon $\mathcal{S}$ is called *geodetically closed* if it satisfies the following properties:

(i) the points of $\mathcal{S}'$ determine a subspace of $\mathcal{S}$;
(ii) every point of on a shortest path (in $\mathcal{S}$) between two points of $\mathcal{S}'$ is again a point of $\mathcal{S}'$.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. By [3, Theorem 4], every two points of a dense near polygon at distance δ from each other are contained in a unique geodetically closed sub near 2δ-gon. These sub near polygons are called *quads* if $\delta = 2$ and *hexes* if $\delta = 3$.

In [3, Theorem 7], Brouwer and Wilbrink showed the nonexistence of regular near octagons whose parameters s, t_2, t_3 and t satisfy the following conditions: $s \geq 2, t_2 \geq 2$ and $t_3 \neq t_2(t_2 + 1)$. Later an arithmetical error was discovered in the proof (lines 5 and 6 of page 172), causing a gap in the proof. The authors of [2] ask in their book to fill this gap (see page 206) and in the present paper we will do that.

The regular near octagons which still need to be ruled out all have parameters of the following form: $t_2 = q$, $s = q^2$ and $q^4 + q^3 + 2q \leq t_3 \leq q^4 + q^3 + q^2 + 2q + 18$. The lower bound for t_3 arises from the divisibility condition $t_2 \mid t_3$ and the inequality $t_3 + 1 > (t_2 + 1)(st_2 + 1)$ which are know to hold for any nonclassical dense regular near hexagon which is not isomorphic to the M_{24} near hexagon. We will rule out the remaining regular near octagons by improving this lower bound for $t_3 + 1$ and subsequently dealing with the remaining cases. As a consequence, we have

Theorem 1 ([3, Theorem 7] + Section 4). *There exist no regular near octagons whose parameters s, t_2, t_3 and t satisfy $s \geq 2$, $t_2 \geq 2$ and $t_3 \neq t_2(t_2 + 1)$.*

2. Some properties of generalized quadrangles

As we mentioned before the generalized quadrangles are precisely the near quadrangles. Any generalized quadrangle which is not degenerate, not a grid and not a dual grid must have a certain order (s, t_2). The aim of this section is to collect some known and easy properties of generalized quadrangles.

Lemma 1 ([7, 1.2.2 and 1.2.3]). *If Q is a generalized quadrangle of order (s, t_2), then*

- *$s + t_2 \mid st_2(s+1)(t_2+1)$;*
- *$s \leq t_2^2$ if $t_2 \neq 1$, or dually, $t_2 \leq s^2$ if $s \neq 1$ (Higman's inequality).*

Lemma 2 ([7, 2.2.1]). *If Q is a generalized quadrangle of order (s, t_2) and if Q' is a proper subquadrangle of order (s, t_2') of Q, then $t_2' \leq \frac{t_2}{s}$. Equality holds if and only if every line of Q meets Q'.*

Lemma 3 ([7, 2.3.1]). *Let Q be a generalized quadrangle of order (s, t_2), $s \neq 1$, and let X be a nonempty set of points of Q. If every line of Q which has at least two points in common with X is completely contained in X, then X is one of the following sets:*

(a) a set of mutually noncollinear points;
(b) the set of points on a pencil of lines (i.e. a set of lines through a distinguished point);
(c) the set of points of a subquadrangle of order (s, t_2').

3. Restrictions on the parameters of regular near hexagons

Let $\mathcal{S}$ be a regular near hexagon with parameters s, t_2 and t_3 and let A denote the collinearity matrix of $\mathcal{S}$. There exist well-known techniques for calculating the eigenvalues and corresponding multiplicities of A, see e.g. [2] or Section 7 of [8]. The graph A has four distinct eigenvalues λ_i, $i \in \{0, 1, 2, 3\}$, with $-(t_3+1) = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 = s(t_3+1)$. Here λ_1 and λ_2 are the roots of the quadratic polynomial

$$X^2 - (s-1)(t_2+2)X + (s^2-s+1)t_2 - st_3 + (s-1)^2.$$

The multiplicity f_3 of the eigenvalue $s(t_3+1)$ is equal to 1. The multiplicity f_0 of the eigenvalue $-(t_3+1)$ is equal to

$$s^3 \frac{(t_2+1) + st_3(t_2+1) + s^2t_3(t_3-t_2)}{s^2(t_2+1) + st_3(t_2+1) + t_3(t_3-t_2)},$$

and the multiplicity f_{3-i} of the eigenvalue λ_{3-i}, $i \in \{1, 2\}$, is equal to

$$\frac{\lambda_i(m-1) + s(t_3+1) - (\lambda_i + t_3 + 1)f_0}{\lambda_i - \lambda_{3-i}},$$

where

$$m = 1 + s(t_3+1) + \frac{s^2(t_3+1)t_3}{t_2+1} + \frac{s^3t_3(t_3-t_2)}{t_2+1}.$$

The fact that all these multiplicities are integers gives already severe restrictions on the parameters. Another restriction is the so-called *Mathon bound* ([3, 5, 6]) which holds if $s \neq 1$:

$$t_3 \leq s^3 + t_2(s^2-s+1).$$

Another inequality which holds if s is different from 1 is the following:

$$t_3^2 - (s^2 t_2 + s^2 + t_2)t_3 + s^4(t_2 + 1) \geq 0.$$

This inequality follows from one of the Krein conditions, see Section (i) of [3] or Remark 2.4 of [6].

Suppose now that $\mathcal{S}$ is dense, so suppose that $s \geq 2$ and $t_2 \geq 1$. Then every two points at distance 2 are contained in a unique quad of order (s, t_2). So, Lemma 1 provides additional parameter restrictions. The number of quads through a point, respectively line, is equal to $\frac{t_3(t_3+1)}{t_2(t_2+1)}$, respectively $\frac{t_3}{t_2}$. Hence, we also obtain the following divisibility conditions:

$$t_2 \mid t_3;$$

$$t_2(t_2 + 1) \mid t_3(t_3 + 1).$$

Now, let Q denote an arbitrary quad of $\mathcal{S}$. If x is a point of $\Gamma_1(Q)$, then a unique line through x meets Q and $t_2(t_2 + 1)$ lines through x are completely contained in $\Gamma_1(Q)$. As a consequence, $t_3 \geq t_2(t_2 + 1)$. Moreover, $t_3 = t_2(t_2 + 1)$ if and only if $\Gamma_2(R) = \emptyset$ for every quad R, or equivalently, if and only if $\mathcal{S}$ is a so-called classical near hexagon (i.e. a dual polar space of rank 3 ([4])). If $t_3 \neq t_2(t_2 + 1)$, then there exists a point $x \in \Gamma_2(Q)$. The set $\Gamma_2(x) \cap Q$ is an ovoid of Q and the set of $st_2 + 1$ quads through x which meet Q determine $(t_2 + 1)(st_2 + 1)$ distinct lines through x (see Lemma 25 of [3]). So,

$$(t_3 + 1) \geq (t_2 + 1)(st_2 + 1).$$

If $t_3 + 1 = (t_2 + 1)(st_2 + 1)$, then Theorem 5 of [3] shows that $s = 2$, $t_2 = 2$ and $t = 14$. By [1], there is a unique near hexagon with these parameters, namely the near hexagon which is obtained in the following way from the unique Steiner system $S(5, 8, 24)$ (= the threefold extension of $PG(2, 4)$):

- the points of the near hexagon are the blocks of $S(5, 8, 24)$;
- the lines are the triples of mutually disjoint blocks;
- incidence is containment.

We will refer to this near hexagon as the M_{24} near hexagon since its automorphism group is isomorphic to the Mathieu group M_{24}.

Theorem 2. *Let $\mathcal{S}$ be a nonclassical regular near hexagon with parameters $s \geq 2$, $t_2 \geq 1$ and t_3 which is not isomorphic to the M_{24} near hexagon. Let $t_2' \in \mathbb{N}$ such that no quad of $\mathcal{S}$ has subquadrangles of order (s, α) with $t_2' < \alpha \leq \frac{t_2-1}{s}$. Then $t_3 + 1 = (st_2 + 1)(t_2 + 1) + \eta \cdot t_2$ with $\eta \geq \min\left\{t_2, \frac{(t_2+1)(st_2+1)}{st_2'+1} - (st_2 + s + 1)\right\}$.*

Proof: Since $t_3 + 1 \geq (t_2 + 1)(st_2 + 1)$ and $t_2 \mid t_3$, there exists an $\eta \in \mathbb{N}$ such that $t_3 + 1 = (st_2 + 1)(t_2 + 1) + \eta \cdot t_2$. Let R denote an arbitrary quad of $\mathcal{S}$. Since $\mathcal{S}$ is not classical, $t_3 \neq t_2(t_2 + 1)$ and there exists a point $x \in \Gamma_2(R)$. Since $\mathcal{S}$ is not isomorphic to the M_{24} near hexagon, $t_3 + 1 > (t_2 + 1)(st_2 + 1)$ and hence there exists a line L

through x which is completely contained in $\Gamma_2(R)$. Let V denote the set of lines of $\mathcal{S}$ which meet L and $\Gamma_1(R)$. For every quad Q through L, let V_Q denote the set of lines of V which are contained in Q. Since $|V| = (s+1)(st_2+1)(t_2+1)$ and since there are $\frac{t_3}{t_2}$ quads through L, there exists a quad Q^* through L for which $|V_{Q^*}| \geq \frac{(s+1)t_2(st_2+1)(t_2+1)}{t_3}$.

Put $X := R \cup \Gamma_1(R)$. By Lemma 8 of [3], every line which has at least two points in common with X is completely contained in X. Hence, if Q is a quad through L, then either $Q \cap X$ is empty or satisfies the conditions of Lemma 3. We distinguish the following possibilities:

(i) $Q \cap X$ is a (possibly empty) set of mutually noncollinear points. In this case, $|V_Q| = |Q \cap X| \leq st_2$.
(ii) $Q \cap X$ is the set of points on $k \geq 1$ lines through a distinguished point. Then $|V_Q| = 1 + sk \leq 1 + st_2$.
(iii) $Q \cap X$ determines a subquadrangle of order (s, α). Since Q contains a line which is disjoint from the subquadrangle, $\alpha \leq \frac{t_2-1}{s}$ by Lemma 2. By our assumptions, $\alpha \leq t_2'$. Hence, $|V_Q| = (s+1)(s\alpha+1) \leq (s+1)(st_2'+1)$.

Taking $Q = Q^*$, we see that $\frac{(s+1)t_2(st_2+1)(t_2+1)}{t_3} \leq st_2 + 1$ or $\frac{(s+1)t_2(st_2+1)(t_2+1)}{t_3} \leq (s+1)(st_2'+1)$. The first inequality is equivalent with $\eta \geq t_2$ and the latter with $\eta \geq \frac{(t_2+1)(st_2+1)}{st_2'+1} - (st_2+s+1)$. The theorem now immediately follows. □

Corollary 1. *Let $\mathcal{S}$ be a nonclassical regular near hexagon with parameters $s \geq 2$, $t_2 \geq 1$ and t_3 which is not isomorphic to the M_{24} near hexagon. Suppose no quad of $\mathcal{S}$ has subquadrangles of order (s, α) with $\alpha \leq \frac{t_2-1}{s}$. Then $t_3 \geq (s+1)t_2(t_2+1)$. In particular, this inequality holds if $t_2 \leq s$.*

Proof: By the proof of Theorem 2, $\eta \geq t_2$ and hence $t_3 \geq (s+1)t_2(t_2+1)$. □

Remark. In Theorem 2 we can take for t_2' the biggest integer smaller than or equal to $\frac{t_2-1}{s}$ for which the divisibility condition $s + t_2' \mid s(s+1)t_2'(t_2'+1)$ is satisfied. If $t_2' \neq \frac{t_2-1}{s}$ (which is certainly the case if $\frac{t_2-1}{s} \notin \mathbb{N}$), then we get an improvement of the lower bound $(t_2+1)(st_2+1)$ for t_3.

4. The nonexistence of the regular near octagons

4.1. Description of the gap

Suppose that $\mathcal{S}$ is a regular near octagon with parameters s, t_2, t_3 and t, with $s \geq 2$, $t_2 \geq 2$ and $t_3 \neq t_2(t_2+1)$. The case $s \neq t_2^2$ has been ruled out in [3]. So, suppose that there exists a $q \geq 2$ such that $t_2 = q$ and $s = q^2$. For $q = 2$, we have $s = 4$, $t_2 = 2$, $t_3 \geq (t_2+1)(st_2+1) = 27$ and $t_3 \leq 90$ by Mathon's bound. Each value of $t_3 \in \{27, \ldots, 90\}$ violates however at least one of the following conditions: (i) $t_2 \mid t_3$, (ii) $t_2(t_2+1) \mid t_3(t_3+1)$, (iii) all multiplicities f_i are integral. So, we may suppose that $q \geq 3$. Put

$$f_q(x) := x^2 - (q^5 + q^4 + q)x + (q^9 + q^8).$$

For $q \geq 3$, this polynomial has two roots $r_1(q)$ and $r_2(q)$ with $0 < r_1(q) < r_2(q)$. By Section 3, we have $f_q(t_3) \geq 0$ and hence either $t_3 \leq r_1(q)$ or $t_3 \geq r_2(q)$. The case $t_3 \geq r_2(q)$ has been ruled out in [3]. The case $t_3 \leq r_1(q)$ has not yet been ruled out because of an arithmetical error. Since $f_q(q^4 + q^3 + q^2 + 2q + 19) = -16q^5 + 23q^4 + 41q^3 + 40q^2 + 57q + 361 < 0$ for every $q \geq 3$, $t_3 \leq q^4 + q^3 + q^2 + 2q + 18$. Because $t_3 > (t_2 + 1)(st_2 + 1)$ and since t_2 is a divisor of t_3, $q^4 + q^3 + 2q \leq t_3$. As a consequence,

$$q^4 + q^3 + 2q \leq t_3 \leq q^4 + q^3 + q^2 + 2q + 18.$$

This is precisely the description of the gap as given on page 206 of [2].

4.2. Filling of the gap

By Corollary 1, we have $t_3 \geq (s + 1)t_2(t_2 + 1) \geq q^4 + q^3 + q^2 + q$. Since $q \mid t_3$ and $t_3 \leq q^4 + q^3 + q^2 + 2q + 18$, either $t_3 = q^4 + q^3 + q^2 + q$, $t_3 = q^4 + q^3 + q^2 + 2q$ or $q^4 + q^3 + q^2 + 3q \leq t_3 \leq q^4 + q^3 + q^2 + 2q + 18$. (So, $q \leq 18$ in the latter case.) We will kill each of these cases in the following lemmas.

Lemma 4. *The case $t_3 = q^4 + q^3 + q^2 + q$ cannot occur.*

Proof: In this case the multiplicity of the eigenvalue $-(t_3 + 1)$ is equal to

$$\frac{q^{15} + 2q^{14} + 3q^{13} + 3q^{12} + 2q^{11} + 2q^{10} + 2q^9 + 2q^8 + 2q^7 + q^6 + q^4 + q^3}{q^5 + 3q^4 + 5q^3 + 6q^2 + 5q + 2}$$
$$= q^{10} - q^9 + q^8 - q^7 + q^6 + q^5 - 3q^4 + 3q^3 - q^2 - \frac{2q^4 - 2q^2}{q^5 + 3q^4 + 5q^3 + 6q^2 + 5q + 2}.$$

Since $0 < 2q^4 - 2q^2 < q^5 + 3q^4 + 5q^3 + 6q^2 + 5q + 2$, this multiplicity would not be integral, a contradiction. □

Lemma 5. *The case $t_3 = q^4 + q^3 + q^2 + 2q$ cannot occur.*

Proof: In this case the multiplicity of the eigenvalue $-(t_3 + 1)$ is equal to

$$\frac{q^{16} + 2q^{15} + 3q^{14} + 5q^{13} + 4q^{12} + 4q^{11} + 4q^{10} + 2q^9 + 3q^8 + 2q^7 + q^5 + q^4}{q^6 + 3q^5 + 5q^4 + 8q^3 + 8q^2 + 5q + 2},$$

or to

$$q^{10} - q^9 + q^8 - q^7 + 2q^6 - 2q^5 + 3q^4 - 8q^3 + 15q^2 - 19q + 28$$
$$- \frac{55q^5 + 73q^4 + 131q^3 + 159q^2 + 102q + 56}{q^6 + 3q^5 + 5q^4 + 8q^3 + 8q^2 + 5q + 2}.$$

If $q \geq 54$, then $0 < 55q^5 + 73q^4 + 131q^3 + 159q^2 + 102q + 56 < q^6 + 3q^5 + 5q^4 + 8q^3 + 8q^2 + 5q + 2$, contradicting the fact that the multiplicity is integral. So, $3 \leq q \leq 54$. Also for the remaining possibilities of q one can verify (individually) that the multiplicity of the eigenvalue $-(t_3+1)$ is not integral. So, also this case cannot occur. □

Lemma 6. *The case* $q^4+q^3+q^2+3q \le t_3 \le q^4+q^3+q^2+2q+18$ *cannot occur.*

Proof: If $q \ge 6$, then $t_3 \ge q^4+q^3+q^2+2q+6$. On the other hand, since $f_q(q^4+q^3+q^2+2q+6) = -3q^5+10q^4+15q^3+14q^2+18q+36 < 0$ if $q \ge 6$, $t_3 < q^4+q^3+q^2+2q+6$. So, we have a contradiction.

If $q=5$, then $q^4+q^3+q^2+3q \le t_3 \le r_1(q)$ implies that $t_3 = 790$. But for this value of t_3, the divisibility condition $t_2(t_2+1) \mid t_3(t_3+1)$ is not satisfied.

If $q=4$, then $q^4+q^3+q^2+3q \le t_3 \le r_1(q)$ implies that $348 \le t_3 \le 351$. No possible value of t_3 survives the conditions $t_2 \mid t_3$ and $t_2(t_2+1) \mid t_3(t_3+1)$.

If $q=3$ then $q^4+q^3+q^2+3q \le t_3 \le r_1(q)$ implies that $126 \le t_3 \le 141$. From $t_2 \mid t_3$ and $t_2(t_2+1) \mid t_3(t_3+1)$, $t_3 \in \{132, 135\}$. None of the possible values of t_3 gives rise a integral multiplicity for the eigenvalue $-(t_3+1)$. □

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