J Algebr Comb (2007) 26:83–105
DOI 10.1007/s10801-006-0049-9

Quantum automorphism groups of vertex-transitive graphs of order ≤ 11

Teodor Banica · Julien Bichon

Received: 1 February 2006 / Accepted: 20 November 2006 /
Published online: 10 January 2007

Abstract We study quantum automorphism groups of vertex-transitive graphs having less than 11 vertices. With one possible exception, these can be obtained from cyclic groups $\mathbb{Z}_n$, symmetric groups S_n and quantum symmetric groups $\mathcal{Q}_n$, by using various product operations. The exceptional case is that of the Petersen graph, and we present some questions about it.

Keywords Quantum permutation group · Transitive graph

Introduction

A remarkable fact, discovered by Wang in [18], is that the symmetric group S_n has a quantum analogue $\mathcal{Q}_n$. For $n \geq 4$ this quantum group is bigger than S_n, and fits into Woronowicz's formalism in [19].

The quantum group $\mathcal{Q}_n$ is best understood via its representation theory: with suitable definitions, it appears as the Tannakian realisation of the Temperley-Lieb algebra ([3]). This elucidates a number of questions regarding the Cayley graph, fusion rules, amenability, etc. More generally, this puts $\mathcal{Q}_n$ into the framework of free quantum groups of Van Daele and Wang ([15]), where a whole machinery, inspired by work of Gromov, Jones, Voiculescu, Wassermann, Weingarten is now available.

T. Banica
Departement of Mathematics, Universite Paul Sabatier, 118 route de Narbonne,
31062 Toulouse, France
e-mail: banica@picard.ups-tlse.fr

J. Bichon (✉)
Laboratoire de Mathematiques Appliquees, Universite de Pau et des Pays de l'Adour, IPRA, Avenue de l'Universite, 64000 Pau, France
e-mail: Julien.Bichon@math.univ-bpclermont.fr

J. Bichon
Present Address: Laboratoire de Mathematiques, Universite Blaise Pascal, Campus des Cezeaux,
63177 Aubiere Cedex, France

The study of $\mathcal{Q}_n$, and of free quantum groups in general, focuses now on more technical aspects: matrix models ([5, 6]), ergodic actions ([9, 13]), harmonic analysis ([14, 16]).

The other thing to do is to study subgroups of $\mathcal{Q}_n$. This was started independently by the authors in [2, 3] and [7, 8], and continued in the joint paper [4]. The notion that emerges from this work is that of quantum automorphism group of a vertex-transitive graph.

In this paper we describe quantum automorphism groups of vertex-transitive graphs having $n \leq 11$ vertices, with one graph omitted. This enhances previous classification work from [2–4], where we have $n \leq 9$, also with one graph omitted.

Needless to say, in this classification project the value of n is there only to show how far our techniques go.

The four main features of the present work are:

(1) Product operations. We have general decomposition results for Cartesian and lexicographic products. These are motivated by the graphs $\mathtt{Pr}(C_5)$, $\mathtt{Pr}(K_5)$ and $C_{10}(4)$, which appear at $n = 10$.
(2) The discrete torus. Here $n = 9$. We prove that its quantum group is equal to its classical group, namely $S_3 \wr \mathbb{Z}_2$. This answers a question left open in [3, 4], and provides the first example of a graph having a usual wreath product as quantum symmetry group.
(3) Circulant graphs. It is known from [2] that the n-cycle with $n \neq 4$ has quantum symmetry group D_n. This is extended in [3] to a bigger class of circulant graphs. Here we further enlarge the list of such graphs, with an ad-hoc proof for $C_{10}(2)$, which appears at $n = 10$.
(4) The Petersen graph. This appears at $n = 10$, and the corresponding quantum group seems to be different from the known ones. Our other techniques do not apply here: it cannot be written as a graph product, and is not a circulant graph. Neither could we carry a direct analysis as in the torus case because of the complexity of some computations. However we prove that the corresponding quantum group is not isomorphic to $\mathcal{Q}_5$. In other words, we might have here a "new" quantum group. However, we don't have a proof, and the question is left open.

As a conclusion, we have two questions:

(I) First is to decide whether the Petersen graph produces or not a new quantum group. If it does, this would probably change a bit the landscape: in the big table at the end, based on work since Wang's paper [18], all quantum groups can be obtained from $\mathbb{Z}_n$, S_n, $\mathcal{Q}_n$.
(II) A good question is to try to characterize graphs having no quantum symmetry. This paper provides many new examples, and we have found some more by working on the subject, but so far we were unable to find a conceptual result here.

The paper is organized as follows. Sections 1, 2 are quite detailed preliminary sections, the whole paper, or at least the ideas involved in it, being intended to be accessible to non-specialists. Sections 3, 4, 5, 6 deal with different kinds of graphs, once again in a quite self-contained way. In Section 7 we present the classification

result, in the form of a big, independent table. In the last section we present a technical result about the quantum group of the Petersen graph.

1 Quantum permutation groups

In this paper we use the following simplified version of Woronowicz's compact quantum groups [19], which is the only one we need when dealing with quantum symmetries of classical finite spaces.

Definition 1.1. A Hopf $\mathbb{C}^*$-algebra is a $\mathbb{C}^*$-algebra A with unit, endowed with morphisms

$$\begin{aligned}\Delta&: A \to A \otimes A\\ \varepsilon&: A \to \mathbb{C}\\ S&: A \to A^{op}\end{aligned}$$

satisfying the usual axioms for a comultiplication, counit and antipode, along with the extra condition $S^2 = id$.

The more traditional terminology for such an object is that of a "universal Hopf $\mathbb{C}^*$-algebra of Kac type". The universality condition refers to the fact that the counit and antipode are assumed to be defined on the whole $\mathbb{C}^*$-algebra A (in full generality, these are only defined on a dense Hopf $*$-subalgebra) and the Kac condition refers to the condition $S^2 = id$.

We warn the reader that the Hopf $\mathbb{C}^*$-algebras we consider here are not Hopf algebras in the usual sense (the tensor product in the definition is a $\mathbb{C}^*$-tensor product). However, they possess canonically defined dense Hopf $*$-subalgebras, from which they can be reconstructed using the universal C^*-completion procedure. See the survey paper [12].

The first example is with a compact group G. We can consider the algebra of continuous functions $A = \mathbb{C}(G)$, with operations

$$\begin{aligned}\Delta(f) &= (g,h) \to f(gh)\\ \varepsilon(f) &= f(1)\\ S(f) &= g \to f(g^{-1})\end{aligned}$$

where we use the canonical identification $A \otimes A = \mathbb{C}(G \times G)$.

The second example is with a discrete group Γ. We have here the algebra $A = \mathbb{C}^*(\Gamma)$, obtained from the usual group algebra $\mathbb{C}[\Gamma]$ by the universal $\mathbb{C}^*$-completion procedure, with operations

$$\begin{aligned}\Delta(g) &= g \otimes g\\ \varepsilon(g) &= 1\\ S(g) &= g^{-1}\end{aligned}$$

where we use the canonical embedding $\Gamma \subset A$.

In general, associated to an arbitrary Hopf $\mathbb{C}^*$-algebra A are a compact quantum group G and a discrete quantum group Γ, according to the following heuristic formulae:

$$A = \mathbb{C}(G) = \mathbb{C}^*(\Gamma)$$
$$G = \widehat{\Gamma}$$
$$\Gamma = \widehat{G}$$

These formulae are made into precise statements in the first section of Woronowicz' seminal paper [19]. They are pieces of Pontryagin duality for locally compact quantum groups, whose latest version is given in [11].

The compact quantum group morphisms are defined in the usual manner: if $A = \mathbb{C}(G)$ and $B = \mathbb{C}(H)$ are Hopf $\mathbb{C}^*$-algebras, a quantum group morphism $H \to G$ arises from a Hopf $\mathbb{C}^*$-algebra morphism $\mathbb{C}(G) \to \mathbb{C}(H)$, and we say that H is a quantum subgroup of G if the corresponding morphism $\mathbb{C}(G) \to \mathbb{C}(H)$ is surjective. We refer to [17] for more details on the compact quantum group language.

A square matrix $u = (u_{ij}) \in M_n(A)$ is said to be multiplicative if

$$\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj} \quad \text{and} \quad \varepsilon(u_{ij}) = \delta_{ij}$$

Multiplicative matrices correspond to corepresentations of the Hopf $\mathbb{C}^*$-algebra A, that is, to representations of the compact quantum group G with $A = \mathbb{C}(G)$. Such a multiplicative matrix u will also be interpreted as a linear map $\mathbb{C}^n \longrightarrow \mathbb{C}^n \otimes A$.

In this paper we are essentially interested in the following special type of multiplicative matrices.

Definition 1.2. A magic unitary matrix is a square matrix, all of whose entries are projections and all of whose rows and columns are partitions of unity.

Here we say that a finite family of projections is a partition of unity if these projections are pairwise orthogonal and if their sum equals 1.

As a first example, consider a finite group G acting on a finite set X. The characteristic functions

$$p_{ij} = \chi\{\sigma \in G \mid \sigma(j) = i\}$$

form a magic unitary matrix, because the corresponding sets form partitions of G, when i or j varies. We have the following formulae for $\mathbb{C}(G)$:

$$\Delta(p_{ij}) = \sum p_{ik} \otimes p_{kj}$$
$$\varepsilon(p_{ij}) = \delta_{ij}$$
$$S(p_{ij}) = p_{ji}$$

and therefore $p = (p_{ij})$ is a multiplicative matrix.

In the particular case of the symmetric group S_n acting on $\{1, \ldots, n\}$, the Stone-Weierstrass theorem shows that entries of p generate $\mathbb{C}(S_n)$. This suggests the following construction, due to Wang ([18]).

Definition 1.3. The $\mathbb{C}^*$-algebra $A_s(n)$ is the universal $\mathbb{C}^*$-algebra generated by n^2 elements u_{ij}, with relations making u into a magic unitary matrix, and with morphisms

$$\begin{aligned}\Delta(u_{ij}) &= \sum u_{ik} \otimes u_{kj}\\ \varepsilon(u_{ij}) &= \delta_{ij}\\ S(u_{ij}) &= u_{ji}\end{aligned}$$

as comultiplication, counit and antipode, making it into a Hopf $\mathbb{C}^*$-algebra.

This Hopf $\mathbb{C}^*$-algebra was discovered by Wang [18]. The corresponding compact quantum group is denoted $\mathcal{Q}_n$ and we call it the quantum permutation group or quantum symmetric group. This is motivated by the fact that the algebra $A_s(n)$ is the biggest Hopf $\mathbb{C}^*$-algebra coacting on the algebra $\mathbb{C}^n$, which is to say that the quantum group $\mathcal{Q}_n$ is the biggest one acting on $\{1, \ldots, n\}$. The coaction u: $\mathbb{C}^n \longrightarrow \mathbb{C}^n \otimes A_s(n)$ is defined on Dirac masses by

$$u(\delta_i) = \sum \delta_j \otimes u_{ji}$$

and verification of axioms of coactions, as well as proof of universality, is by direct computation. See [18].

We have a surjective morphism of Hopf $\mathbb{C}^*$-algebras

$$A_s(n) \to \mathbb{C}(S_n)$$

mapping u_{ij} to p_{ij} for any i, j. This morphism expresses the fact that the compact quantum group corresponding to $A_s(n)$ contains S_n.

This map is an isomorphism for $n = 2, 3$, as known from [3, 18], and explained in Section 3 below. At $n = 4$ we have Wang's matrix

$$u = \begin{pmatrix} p & 1-p & 0 & 0\\ 1-p & p & 0 & 0\\ 0 & 0 & q & 1-q\\ 0 & 0 & 1-q & q \end{pmatrix}$$

with p, q free projections, which shows that there exists an epimorphism $A_s(4) \to \mathbb{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ and hence $A_s(4)$ is not commutative and is infinite dimensional. The same remains true for any $n \geq 4$.

2 Quantum automorphism groups of graphs

Consider a finite graph X. In this paper this means that we have a finite set of vertices, and certain pairs of distinct vertices are connected by unoriented edges.

It is convenient to assume that the vertex set is $\{1, \ldots, n\}$.

Definition 2.1. The adjacency matrix of X is the matrix

$$d \in M_n(0, 1)$$

given by $d_{ij} = 1$ if i, j are connected by an edge, and $d_{ij} = 0$ if not.

The adjacency matrix is symmetric, and has 0 on the diagonal. In fact, graphs having vertex set $\{1, \ldots, n\}$ are in one-to-one correspondence with $n \times n$ symmetric 0–1 matrices having 0 on the diagonal.

The quantum automorphism group of X is obtained as an appropriate subgroup of the quantum permutation group of $\{1, \ldots, n\}$. At level of Hopf $\mathbb{C}^*$-algebras, this means taking an appropriate quotient of $A_s(n)$.

Definition 2.2. Associated to a finite graph X is the $\mathbb{C}^*$-algebra

$$A(X) = A_s(n)/\langle du = ud \rangle$$

where n is the number of vertices, and d is the adjacency matrix.

Since a permutation of the set X is a graph automorphism if and only if the corresponding permutation matrix commutes with the adjacency matrix, it is reasonable to say that the quantum group corresponding to $A(X)$ is the quantum automorphism group of X. In this way we have a commutative diagram of Hopf $\mathbb{C}^*$-algebras

$$\begin{array}{ccc} A_s(n) & \to & A(X) \\ \downarrow & & \downarrow \\ \mathbb{C}(S_n) & \to & \mathbb{C}(G) \end{array}$$

where $G = G(X)$ is the usual automorphism group of X, with the kernel of the right arrow being the commutator ideal of $A(X)$. Moreover, for a graph without edges we get indeed $A_s(n)$, and we have the formula

$$A(X) = A(X^c)$$

where X^c is the complement of X. See [3, 4] for details.

The defining equations $ud = du$ of $A(X)$ means that d, considered as a linear map $\mathbb{C}^n \to \mathbb{C}^n$, is a morphism in the category of corepresentations of $A(X)$, i.e. a morphism in the category of representations of the quantum group dual to $A(X)$. General properties of the representation category of a compact quantum group (see e.g. [19]) now ensure that the spectral projections occurring in the spectral decomposition of d are corepresentations morphisms, and hence the corresponding eigensubspaces are subcorepresentations. This key fact will be used freely in the paper.

The following notion will play a central role in this paper.

Definition 2.3. We say that X has no quantum symmetry if

$$A(X) = \mathbb{C}(G)$$

where $G = G(X)$ is the usual automorphism group of X.

This is the same as saying that $A(X)$ is commutative, because by the above considerations, $\mathbb{C}(G)$ is its biggest commutative quotient.

We are particularly interested in the case of graphs X having the property that G acts transitively on the set of vertices. These graphs were called homogeneous in previous work [3, 4], but we use here the following more traditional terminology.

Definition 2.4. The graph X is called vertex-transitive if for any two vertices i, j there is $\sigma \in G(X)$ such that $\sigma(i) = j$.

Each section of the paper ends with a small table, gathering information about vertex-transitive graphs having ≤ 11 vertices. These small tables are to be put together in a single big table, at the end.

What we know so far is that we have

$$A(K_n) = A_s(n)$$

where K_n is the complete graph having n vertices. Moreover, we already mentioned that for $n = 2, 3$ the arrow

$$A_s(n) \to \mathbb{C}(S_n)$$

is an isomorphism, and for $n \geq 4$ it is not.

This information is summarized in the following table.

Order	Graph	Classical group	Quantum group
2	K_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	K_3	S_3	S_3
$n \geq 4$	K_n	S_n	$\mathcal{Q}_n$

Here in the right column $\mathcal{Q}_n$ with $n \geq 4$ is the compact quantum group associated to $A_s(n)$.

3 Circulant graphs

A graph with n vertices is called circulant if its automorphism group contains a cycle of length n (and hence in particular a copy of the cyclic group $\mathbb{Z}_n$). We are particularly interested in connected circulant graphs, which are the cycles with chords.

Definition 3.1. The graph $C_n(k_1, \ldots, k_r)$, where

$$1 < k_1 < \ldots < k_r \leq [n/2]$$

are integers, is obtained by drawing the n-cycle C_n, then connecting all pairs of vertices at distance k_i, for any i.

As basic examples, we have the n-cycle C_n, corresponding to the value $r = 0$, and the $2n$-cycle with diagonals, $C_n^+ = C_{2n}(n)$.

Observe that K_n is a cycle with chords as well.

The adjacency matrix of a cycle with chords, denoted as usual d, is a circulant matrix. We use the following basic fact.

Proposition 3.1. *We have* $d(\xi^s) = 2f(s)\xi^s$, *where*

$$f(s) = \sum_{i=0}^{r} \cos\left(\frac{2k_i s\pi}{n}\right) \quad (\text{with } k_0 = 1)$$

and ξ *is the vector whose coordinates are* $1, \omega, \ldots, \omega^{n-1}$ *in the canonical basis of* $\mathbb{C}^n$, *with* $\omega = e^{\frac{2i\pi}{n}}$.

This tells us that we have the following eigenspaces for d:

$$\begin{array}{ll} V_0 = \mathbb{C}1 \\ V_1 = \mathbb{C}\xi + \mathbb{C}\xi^{n-1} \\ V_2 = \mathbb{C}\xi^2 + \mathbb{C}\xi^{n-2} \\ \ldots \quad \ldots \\ V_m = \mathbb{C}\xi^m + \mathbb{C}\xi^{n-m} \end{array}$$

where $m = [n/2]$ and all sums are direct, except maybe for the last one, which depends on the parity of n.

The fact that these eigenspaces correspond or not to different eigenvalues depends of course on f.

We use the following result from [3], whose proof is briefly explained, because several versions of it will appear throughout the paper.

Theorem 3.1. *If* $n \neq 4$ *and the associated function*

$$f: \{1, 2, \ldots, [n/2]\} \to \mathbb{R}$$

is injective, then $C_n(k_1, \ldots, k_r)$ *has no quantum symmetry.*

Proof: Since $\mathbb{C}\xi \oplus \mathbb{C}\xi^{n-1}$ is invariant, the coaction can be written as

$$v(\xi) = \xi \otimes a + \xi^{n-1} \otimes b$$

for some a, b. By taking powers and using $n \neq 4$ we get by induction

$$v(\xi^s) = \xi^s \otimes a^s + \xi^{n-s} \otimes b^s$$

for any s, along with the relations $ab = -ba$ and $ab^2 = ba^2 = 0$.

Now from $v(\xi^*) = v(\xi)^*$ we get $b^* = b^{n-1}$, so $(ab)(ab)^* = ab^n a^* = 0$. Thus $ab = ba = 0$, so $A(X)$ is commutative and we are done. □

Corollary 3.1. *The following graphs have no quantum symmetry:*

1. *The cycles C_n with $n \neq 4$.*
2. *The cycles with diagonals C_8^+, C_{10}^+.*
3. *The cycles with chords $C_9(3)$, $C_{11}(2)$, $C_{11}(3)$.*

Proof: (1) follows from the fact that f is decreasing, hence injective. As for (2) and (3), the corresponding 5 functions are given by

$$\begin{aligned}
C_8^+ &: -0.29, 1, -1.7, 0\\
C_{10}^+ &: -0.19, 1.3, -1.3, 0.19, -2\\
C_9(3) &: 0.26, -0.32, 0.5, -1.43\\
C_{11}(2) &: 1.25, -0.23, -1.10, -0.79, -0.11\\
C_{11}(3) &: 0.69, -0.54, 0.27, 0.18. - 1.61
\end{aligned}$$

with 0.01 error, so they are injective, and Theorem 3.1 applies. □

The graphs in Corollary 3.1 have usual symmetry group D_n, where n is the number of vertices. We don't know if $G = D_n$ with $n \neq 4$ implies that the graph has no quantum symmetry. However, we are able to prove this for $n \leq 11$: graphs satisfying $G = D_n$ are those in Corollary 3.1, plus the cycle with chords $C_{10}(2)$, discussed below.

Theorem 3.2. *The graph $C_{10}(2)$ has no quantum symmetry.*

Proof: The function f is given by

$$f(s) = \cos\left(\frac{s\pi}{5}\right) + \cos\left(\frac{2s\pi}{5}\right)$$

and we have $f(1) = -f(3) \simeq 1.11$, $f(2) = f(4) = -0.5$ and $f(5) = 0$. Thus the list of eigenspaces is:

$$\begin{aligned}
V_0 &= \mathbb{C}1\\
V_1 &= \mathbb{C}\xi \oplus \mathbb{C}\xi^9\\
V_2 &= \mathbb{C}\xi^2 \oplus \mathbb{C}\xi^4 \oplus \mathbb{C}\xi^6 \oplus \mathbb{C}\xi^8\\
V_3 &= \mathbb{C}\xi^3 \oplus \mathbb{C}\xi^7\\
V_5 &= \mathbb{C}\xi^5
\end{aligned}$$

Since coactions preserve eigenspaces, we can write

$$v(\xi) = \xi \otimes a + \xi^9 \otimes b$$

for some a, b. Taking the square of this relation gives

$$v(\xi^2) = \xi^2 \otimes a^2 + \xi^8 \otimes b^2 + 1 \otimes (ab + ba)$$

and once again since v preserves eigenspaces, we get $ab = -ba$. Taking now the cube of the above relation gives

$$v(\xi^3) = \xi^3 \otimes a^3 + \xi^7 \otimes b^3 + \xi \otimes ba^2 + \xi^9 \otimes ab^2$$

and once again since v preserves eigenspaces, we get:

$$ab^2 = 0 = ba^2$$

With the relations $ab = -ba$ and $ab^2 = ba^2 = 0$ in hand, we get by induction the formula

$$v(\xi^s) = \xi^s \otimes a^s + \xi^{n-s} \otimes b^s$$

and we can conclude by using adjoints, as in proof of Theorem 3.1. □

For graphs having $n \leq 11$ vertices, results in this section are summarized in the following table.

Order	Graph	Classical group	Quantum group
$n \geq 5$	C_n	D_n	D_n
8	C_8, C_8^+	D_8	D_8
9	$C_9, C_9(3)$	D_9	D_9
10	$C_{10}, C_{10}(2), C_{10}^+$	D_{10}	D_{10}
11	$C_{11}, C_{11}(2), C_{11}(3)$	D_{11}	D_{11}

As already mentioned, we don't know if these computations are particular cases of some general result.

4 Products of graphs

For a finite graph X, it is convenient to use the notation

$$X = (X, \sim)$$

where the X on the right is the set of vertices, and where we write $i \sim j$ when two vertices i, j are connected by an edge.

Definition 4.1. Let X, Y be two finite graphs.

1. The direct product $X \times Y$ has vertex set $X \times Y$, and edges

$$(i, \alpha) \sim (j, \beta) \iff i \sim j,\ \alpha \sim \beta.$$

2. The Cartesian product $X \square Y$ has vertex set $X \times Y$, and edges

$$(i, \alpha) \sim (j, \beta) \iff i = j,\ \alpha \sim \beta \text{ or } i \sim j, \alpha = \beta.$$

The direct product is the usual one in a categorical sense. As for the Cartesian product, this is a natural one from a geometric viewpoint: for instance a product by a segment gives a prism.

Definition 4.2. The prism having basis X is $\mathtt{Pr}(X) = K_2 \square X$.

We have embeddings of usual symmetry groups

$$G(X) \times G(Y) \subset G(X \times Y)$$
$$G(X) \times G(Y) \subset G(X \square Y)$$

which have the following quantum analogues.

Proposition 4.1. *We have surjective morphisms of Hopf $\mathbb{C}^*$-algebras*

$$A(X \times Y) \longrightarrow A(X) \otimes A(Y)$$
$$A(X \square Y) \longrightarrow A(X) \otimes A(Y).$$

Proof: We use the canonical identification

$$\mathbb{C}(X \times Y) = \mathbb{C}(X) \otimes \mathbb{C}(Y)$$

given by $\delta_{(i,\alpha)} = \delta_i \otimes \delta_\alpha$. The adjacency matrices are given by

$$d_{X \times Y} = d_X \otimes d_Y$$
$$d_{X \square Y} = d_X \otimes 1 + 1 \otimes d_Y$$

so if u commutes with d_X and v commutes with d_Y, the matrix

$$u \otimes v = (u_{ij} v_{\alpha\beta})_{(i\alpha, j\beta)}$$

is a magic unitary that commutes with both $d_{X \times Y}$ and $d_{X \square Y}$. This gives morphisms as in the statement, and surjectivity follows by summing over i and β. □

Theorem 4.1. *Let X and Y be finite connected regular graphs. If their spectra $\{\lambda\}$ and $\{\mu\}$ do not contain 0 and satisfy*

$$\{\lambda_i/\lambda_j\} \cap \{\mu_k/\mu_l\} = \{1\}$$

then $A(X \times Y) = A(X) \otimes A(Y)$. Also, if their spectra satisfy

$$\{\lambda_i - \lambda_j\} \cap \{\mu_k - \mu_l\} = \{0\}$$

then $A(X \square Y) = A(X) \otimes A(Y)$.

Proof: We follow [3], where the first statement is proved. Let λ_1 be the valence of X. Since X is regular we have $\lambda_1 \in \mathrm{Sp}(X)$, with 1 as eigenvector, and since X is connected λ_1 has multiplicity one. Hence if P_1 is the orthogonal projection onto $\mathbb{C}1$, the spectral decomposition of d_X is of the following form:

$$d_X = \lambda_1 P_1 + \sum_{i \neq 1} \lambda_i P_i$$

We have a similar formula for d_Y:

$$d_Y = \mu_1 Q_1 + \sum_{j \neq 1} \mu_j Q_j$$

This gives the following formulae for products:

$$d_{X \times Y} = \sum_{ij} (\lambda_i \mu_j) P_i \otimes Q_j$$

$$d_{X \square Y} = \sum_{i,j} (\lambda_i + \mu_j) P_i \otimes Q_j$$

Here projections form partitions of unity, and the scalar are distinct, so these are spectral decomposition formulae. We can conclude as in [3]. The universal coactions will commute with any of the spectral projections, and hence with both $P_1 \otimes 1$ and $1 \otimes Q_1$. In both cases the universal coaction v is the tensor product of its restrictions to the images of $P_1 \otimes 1$ (i.e. $1 \otimes \mathbb{C}(Y)$) and of $1 \otimes Q_1$ (i.e. $\mathbb{C}(X) \otimes 1$). □

Corollary 4.1.

1. *We have $A(K_m \times K_n) = A(K_m) \otimes A(K_n)$ for $m \neq n$.*
2. *We have $A(\mathtt{Pr}(K_n)) = \mathbb{C}(\mathbb{Z}_2) \otimes A_s(n)$, for any n.*
3. *We have $A(\mathtt{Pr}(C_n)) = \mathbb{C}(D_{2n})$, for n odd.*
4. *We have $A(\mathtt{Pr}(C_4)) = \mathbb{C}(\mathbb{Z}_2) \otimes A_s(4)$.*

Proof: The spectra of graphs involved are $\mathrm{Sp}(K_2) = \{-1, 1\}$ and

$$\mathrm{Sp}(K_n) = \{-1,\ n-1\}$$
$$\mathrm{Sp}(C_n) = \{2\cos(2k\pi/n) \mid k = 1, \ldots, n\}$$

so the first three assertions follow from Theorem 4.1. We have

$$\mathtt{Pr}(C_4) = K_2 \times K_4$$

(this graph is the cube) and the fourth assertion follows from the first one. □

We get the following table, the product operation $\times$ on quantum groups being the one dual to the tensor product of Hopf $\mathbb{C}^*$-algebras.

Order	Graph	Classical group	Quantum group
8	$\mathtt{Pr}(C_4)$	$S_4 \times \mathbb{Z}_2$	$\mathcal{Q}_4 \times \mathbb{Z}_2$
10	$\mathtt{Pr}(C_5)$	D_{10}	D_{10}
10	$\mathtt{Pr}(K_5)$	$S_5 \times \mathbb{Z}_2$	$\mathcal{Q}_5 \times \mathbb{Z}_2$

5 The torus graph

Theorem 4.1 doesn't apply to the case $X = Y$, and the problem of computing the algebras $A(X \times X)$ and $A(X \square X)$ appears. At level of classical symmetry groups, there is no simple formula describing $G(X \times X)$ and $G(X \square X)$. Thus we have reasons to believe that the above problem doesn't have a simple solution either.

A simpler question is to characterize graphs X such that $X \times X$ or $X \square X$ has no quantum symmetry. We don't have a general result here, but we are able however to deal with the case $X = K_3$.

Definition 5.1. The graph `Torus` is the graph $K_3 \times K_3 = K_3 \square K_3$.

The result below answers a question asked in [3, 4]. It also provides the first example of graph having a classical wreath product as quantum symmetry group.

Theorem 5.1. *The graph* `Torus` *has no quantum symmetry.*

Proof: The spectrum of K_3 is known to be

$$\mathrm{Sp}(K_3) = \{-1, 2\}$$

with corresponding eigenspaces given by

$$F_2 = \mathbb{C}1$$
$$F_{-1} = \mathbb{C}\xi \oplus \mathbb{C}\xi^2$$

where ξ is the vector formed by third roots of unity.

Tensoring the adjacency matrix of K_3 with itself gives

$$\mathrm{Sp}(\mathtt{Torus}) = \{-2, 1, 4\}$$

with corresponding eigenspaces given by

$$\begin{aligned} E_4 &= \mathbb{C}\xi_{00} \\ E_{-2} &= \mathbb{C}\xi_{10} \oplus \mathbb{C}\xi_{01} \oplus \mathbb{C}\xi_{20} \oplus \mathbb{C}\xi_{02} \\ E_1 &= \mathbb{C}\xi_{11} \oplus \mathbb{C}\xi_{12} \oplus \mathbb{C}\xi_{21} \oplus \mathbb{C}\xi_{22} \end{aligned}$$

where we use the notation $\xi_{ij} = \xi^i \otimes \xi^j$.

The universal coaction v preserves eigenspaces, so we have

$$\begin{aligned} v(\xi_{10}) &= \xi_{10} \otimes a + \xi_{01} \otimes b + \xi_{20} \otimes c + \xi_{02} \otimes d \\ v(\xi_{01}) &= \xi_{10} \otimes \alpha + \xi_{01} \otimes \beta + \xi_{20} \otimes \gamma + \xi_{02} \otimes \delta \end{aligned}$$

for some $a, b, c, d, \alpha, \beta, \gamma, \delta \in A$. Taking the square of $v(\xi_{10})$ gives

$$v(\xi_{20}) = \xi_{20} \otimes a^2 + \xi_{02} \otimes b^2 + \xi_{10} \otimes c^2 + \xi_{01} \otimes d^2$$

along with relations coming from eigenspace preservation:

$$ab = -ba,\ ad = -da,\ bc = -cb,\ cd = -dc$$

$$ac + ca = -(bd + db)$$

Now since a, b anticommute, their squares have to commute. On the other hand, by applying v to the equality $\xi_{10}^* = \xi_{20}$, we get the following formulae for adjoints:

$$a^* = a^2, \quad b^* = b^2, \quad c^* = c^2, \quad d^* = d^2$$

The commutation relation $a^2b^2 = b^2a^2$ reads now $a^*b^* = b^*a^*$, and by taking adjoints we get $ba = ab$. Together with $ab = -ba$ this gives:

$$ab = ba = 0$$

The same method applies to ad, bc, cd, and we end up with:

$$ab = ba = 0, \quad ad = da = 0, \quad bc = cb = 0, \quad cd = dc = 0$$

We apply now v to the equality $1 = \xi_{10}\xi_{20}$. We get that 1 is the sum of 16 terms, all of them of the form $\xi_{ij} \otimes P$, where P are products between a, b, c, d and their squares. Due to the above formulae 8 terms vanish, and the 8 remaining ones produce the formula

$$1 = a^3 + b^3 + c^3 + d^3$$

along with relations coming from eigenspace preservation:

$$ac^2 = ca^2 = bd^2 = db^2 = 0$$

Now from $ac^2 = 0$ we get $a^2c^2 = 0$, and by taking adjoints this gives $ca = 0$. The same method applies to ac, bd, db, and we end up with:

$$ac = ca = 0, \quad bd = db = 0$$

In the same way one shows that $\alpha, \beta, \gamma, \delta$ pairwise commute:

$$\alpha\beta = \beta\alpha = \ldots = \gamma\delta = \delta\gamma = 0$$

It remains to show that a, b, c, d commute with $\alpha, \beta, \gamma, \delta$. For, we apply v to the following equality:

$$\xi_{10}\xi_{01} = \xi_{01}\xi_{10}$$

We get an equality between two sums having 16 terms each, and by using again eigenspace preservation we get the following formulae relating the corresponding 32 products $a\alpha$, αa etc.:

$$\begin{aligned}
&a\alpha = 0 = \alpha a, \quad b\beta = 0 = \beta b, \quad c\gamma = 0 = \gamma c, \quad d\delta = 0 = \delta d,\\
&a\gamma + c\alpha + b\delta + d\beta = 0 = \alpha c + \gamma a + \beta d + \delta b,\\
&a\beta + b\alpha = \alpha b + \beta a, \quad b\gamma + c\beta = \beta c + \gamma b,\\
&c\delta + d\gamma = \gamma d + \delta c, \quad a\delta + d\alpha = \alpha d + \delta a
\end{aligned}$$

Multiplying the first equality in the second row on the left by a and on the right by γ gives $a^2\gamma^2 = 0$, and by taking adjoints we get $\gamma a = 0$. The same method applies to the other 7 products involved in the second row, so all 8 products involved in the second row vanish:

$$a\gamma = c\alpha = b\delta = d\beta = \alpha c = \gamma a = \beta d = \delta b = 0$$

We use now the first equality in the third row. Multiplying it on the left by a gives $a^2\beta = a\beta a$, and multiplying it on the right by a gives $a\beta a = \beta a^2$. Thus we get the commutation relation $a^2\beta = \beta a^2$.

On the other hand from $a^3 + b^3 + c^3 + d^3 = 1$ we get $a^4 = a$, so:

$$a\beta = a^4\beta = a^2a^2\beta = \beta a^2a^2 = \beta a$$

One shows in a similar manner that the missing commutation formulae $a\delta = \delta a$ etc. hold as well. Thus A is commutative. □

Order	Graph	Classical group	Quantum group
9	`Torus`	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr \mathbb{Z}_2$

6 Lexicographic products

Let X and Y be two finite graphs. Their lexicographic product is obtained by putting a copy of X at each vertex of Y:

Definition 6.1. The lexicographic product $X \circ Y$ has vertex set $X \times Y$, and edges are given by

$$(i, \alpha) \sim (j, \beta) \Longleftrightarrow \alpha \sim \beta \quad \text{or} \quad \alpha = \beta,\ i \sim j.$$

The terminology comes from a certain similarity with the ordering of usual words, which is transparent when iterating $\circ$.

The simplest example is with $X \circ X_n$, where X_n is the graph having n vertices and no edges: the graph $X \circ X_n$ is the graph consisting of n disjoint copies of X.

Definition 6.2. nX is the disjoint union of n copies of X.

When X is connected, we have an isomorphism

$$G(nX) = G(X) \wr S_n$$

where $\wr$ is a wreath product. In other words, we have:

$$G(X \circ X_n) = G(X) \wr G(X_n)$$

In the general case, we have the following embedding of usual symmetry groups:

$$G(X) \wr G(Y) \subset G(X \circ Y)$$

The quantum analogues of these results use the notion of free wreath product from [4, 8]. In the following definition, a pair (A, u) is what we call a quantum permutation group in [4]: A is a Hopf $\mathbb{C}^*$-algebra and u is a multiplicative magic unitary matrix.

Definition 6.3. The free wreath product of (A, u) and (B, v) is

$$A *_w B = (A^{*n} * B) / < [u_{ij}^{(a)}, v_{ab}] = 0 >$$

where n is the size of v, with magic unitary matrix $w_{ia,jb} = u_{ij}^{(a)} v_{ab}$.

In other words, $A *_w B$ is the universal $\mathbb{C}^*$-algebra generated by n copies of A and a copy of B, with the a-th copy of A commuting with the a-th row of v, for any a. The Hopf $\mathbb{C}^*$-algebra structure on $A *_w B$ is the unique one making w into a multiplicative matrix. With this definition, we have the following result ([4]).

Theorem 6.1. *If X is connected we have $A(nX) = A(X) *_w A_s(n)$.*

Note that the embedding $A(X)^{*n} \hookrightarrow A(X) *_w A_s(n)$ ensures that $A(nX)$ is an infinite-dimensional algebra whenever $n \geq 2$ and $G(X)$ is non trivial.

In the general case, we have the following quantum analogue of the embedding result for $G(X) \wr G(Y)$.

Proposition 6.1. *We have a surjective morphism of Hopf $\mathbb{C}^*$-algebras*

$$A(X \circ Y) \longrightarrow A(X) *_w A(Y).$$

Proof: We use the canonical identification

$$\mathbb{C}(X \times Y) = \mathbb{C}(X) \otimes \mathbb{C}(Y)$$

given by $\delta_{(i,\alpha)} = \delta_i \otimes \delta_\alpha$. The adjacency matrix of $X \circ Y$ is

$$d_{X\circ Y} = d_X \otimes 1 + \mathbb{I} \otimes d_Y$$

where $\mathbb{I}$ is the square matrix filled with 1's.

Let u, v be the magic unitary matrices of $A(X)$, $A(Y)$. The magic unitary matrix of $A(X) *_w A(Y)$ is given by

$$w_{ia,jb} = u^{(a)}_{ij} v_{ab}$$

and from the fact that u commutes with d_X (and $\mathbb{I}$) and v commutes with d_Y, we get that w commutes with $d_{X\circ Y}$. This gives a morphism as in the statement, and surjectivity follows by summing over i and b. □

Theorem 6.2. *Let X, Y be regular graphs, with X connected. If their spectra $\{\lambda_i\}$ and $\{\mu_j\}$ satisfy the condition*

$$\{\lambda_1 - \lambda_i \mid i \neq 1\} \cap \{-n\mu_j\} = \emptyset$$

where n and λ_1 are the order and valence of X, then $A(X \circ Y) = A(X) *_w A(Y)$.

Proof: We denote by P_i, Q_j the spectral projections corresponding to λ_i, μ_j. Since X is connected we have $P_1 = \frac{1}{n}\mathbb{I}$, and we get:

$$\begin{aligned} d_{X\circ Y} &= d_X \otimes 1 + \mathbb{I} \otimes d_Y \\ &= \left(\sum_i \lambda_i P_i\right) \otimes \left(\sum_j Q_j\right) + (nP_1) \otimes \left(\sum_i \mu_j Q_j\right) \\ &= \sum_j (\lambda_1 + n\mu_j)(P_1 \otimes Q_j) + \sum_{i\neq 1} \lambda_i (P_i \otimes 1) \end{aligned}$$

In this formula projections form a partition of unity and scalars are distinct, so this is the spectral decomposition of $d_{X\circ Y}$.

Let W be the universal coaction on $X \circ Y$. Then W must commute with all spectral projections, and in particular:

$$[W, P_1 \otimes Q_j] = 0$$

Summing over j gives $[W, P_1 \otimes 1] = 0$, so $1 \otimes \mathbb{C}(Y)$ is invariant under the coaction. The corresponding restriction of W gives a coaction of $A(X \circ Y)$ on $1 \otimes \mathbb{C}(Y)$, say

$$W(1 \otimes e_a) = \sum_b 1 \otimes e_b \otimes y_{ba}$$

where y is a magic unitary. On the other hand we can write

$$W(e_i \otimes 1) = \sum_{jb} e_j \otimes e_b \otimes x^b_{ji}$$

and by multiplying by the previous relation we get:

$$\begin{aligned} W(e_i \otimes e_a) &= \sum_{jb} e_j \otimes e_b \otimes y_{ba} x^b_{ji} \\ &= \sum_{jb} e_j \otimes e_b \otimes x^b_{ji} y_{ba} \end{aligned}$$

This shows that coefficients of W have the following form:

$$W_{jb,ia} = y_{ba} x^b_{ji} = x^b_{ji} y_{ba}$$

Consider now the matrix $x^b = (x^b_{ij})$. Since W is a morphism of algebras, each row of x^b is a partition of unity. Also using the antipode, we have

$$\begin{aligned} S\left(\sum_j x^b_{ji}\right) &= S\left(\sum_{ja} x^b_{ji} y_{ba}\right) \\ &= S\left(\sum_{ja} W_{jb,ia}\right) \\ &= \sum_{ja} W_{ia,jb} \\ &= \sum_{ja} x^a_{ij} y_{ab} \\ &= \sum_a y_{ab} \\ &= 1 \end{aligned}$$

so we conclude that x^b is a magic unitary.

We check now that x^a, y commute with d_X, d_Y. We have

$$(d_{X\circ Y})_{ia,jb} = (d_X)_{ij}\delta_{ab} + (d_Y)_{ab}$$

so the two products between W and $d_{X\circ Y}$ are given by:

$$(Wd_{X\circ Y})_{ia,kc} = \sum_j W_{ia,jc}(d_X)_{jk} + \sum_{jb} W_{ia,jb}(d_Y)_{bc}$$

$$(d_{X\circ Y}W)_{ia,kc} = \sum_j (d_X)_{ij}W_{ja,kc} + \sum_{jb}(d_Y)_{ab}W_{jb,kc}$$

Now since W commutes with $d_{X\circ Y}$, the terms on the right are equal, and by summing over c we get:

$$\sum_j x^a_{ij}(d_X)_{jk} + \sum_{cb} y_{ab}(d_Y)_{bc} = \sum_j (d_X)_{ij}x^a_{jk} + \sum_{cb}(d_Y)_{ab}y_{bc}$$

The graph Y being regular, the second sums in both terms are equal to the valency of Y, so we get $[x^a, d_X] = 0$.

Now once again from the formula coming from commutation of W with $d_{X\circ Y}$, we get $[y, d_Y] = 0$.

Summing up, the coefficients of W are of the form

$$W_{jb,ia} = x^b_{ji}y_{ba}$$

where x^b are magic unitaries commuting with d_X, and y is a magic unitary commuting with d_Y. This gives a morphism

$$A(X) *_w A(Y) \longrightarrow A(X \circ Y)$$

mapping $u^{(b)}_{ji} \to x^b_{ji}$ and $v_{ba} \to y_{ba}$, which is inverse to the morphism in the previous proposition. □

Corollary 6.1. *We have* $A(C_{10}(4)) = \mathbb{C}(\mathbb{Z}_2) *_w \mathbb{C}(D_5)$.

Proof: We have isomorphisms

$$C_{10}(4) = C_{10}(4,5)^c = K_2 \circ C_5$$

and Theorem 6.2 applies to the product on the right. □

Together with Theorem 6.1, this corollary gives the following table, where $\wr_*$ is defined by $\mathbb{C}(G \wr_* H) = \mathbb{C}(G) *_w \mathbb{C}(H)$.

Order	Graph	Classical group	Quantum group
4	$2K_2$	$\mathbb{Z}_2 \wr \mathbb{Z}_2$	$\mathbb{Z}_2 \wr_* \mathbb{Z}_2$
6	$2K_3$	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr_* \mathbb{Z}_2$
6	$3K_2$	$\mathbb{Z}_2 \wr S_3$	$\mathbb{Z}_2 \wr_* S_3$
8	$2K_4$	$S_4 \wr \mathbb{Z}_2$	$\mathcal{Q}_4 \wr_* \mathbb{Z}_2$
8	$2C_4$	$(\mathbb{Z}_2 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$	$(\mathbb{Z}_2 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$
8	$4K_2$	$\mathbb{Z}_2 \wr S_4$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_4$
9	$3K_3$	$S_3 \wr S_3$	$S_3 \wr_* S_3$
10	$2C_5$	$D_5 \wr \mathbb{Z}_2$	$D_5 \wr_* \mathbb{Z}_2$
10	$2K_5$	$S_5 \wr \mathbb{Z}_2$	$\mathcal{Q}_5 \wr_* \mathbb{Z}_2$
10	$5K_2$	$\mathbb{Z}_2 \wr S_5$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_5$
10	$C_{10}(4)$	$\mathbb{Z}_2 \wr D_5$	$\mathbb{Z}_2 \wr_* D_5$

7 Classification table

We are now in position of writing down a big table. We first recall the graph notations used in the paper.

Definition 7.1. We use the following notations.

1. Basic graphs: – the complete graph having n vertices is denoted K_n.
   – the disjoint union of n copies of X is denoted nX.
   – the prism having basis X is denoted $\texttt{Pr}(X)$.
2. Circulant graphs:
   – the n-cycle is denoted C_n.
   – the $2n$-cycle with diagonals is denoted C_{2n}^+.
   – the n-cycle with chords of length k is denoted $C_n(k)$.
3. Special graphs:
   – the triangle times itself is denoted `Torus`.
   – the Petersen graph is denoted `Petersen`.

As for quantum group notations, these have to be taken with care, because quantum groups do not really exist etc. Here they are.

Definition 7.2. We use the following notations.
– $\mathbb{Z}_n$, D_n, S_n are the cyclic, dihedral and symmetric groups.
– $\mathcal{Q}_n$ is the quantum permutation group.
– $\times, \wr, \wr_*$ are the product, wreath product and free wreath product.

The vertex-transitive graphs of order less than 11, modulo complementation, are given by the following table.

Order	Graph	Classical group	Quantum group
2	K_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	K_3	S_3	S_3
4	$2K_2$	$\mathbb{Z}_2 \wr \mathbb{Z}_2$	$\mathbb{Z}_2 \wr_* \mathbb{Z}_2$
4	K_4	S_4	$\mathcal{Q}_4$
5	C_5	D_5	D_5
5	K_5	S_5	$\mathcal{Q}_5$
6	C_6	D_6	D_6
6	$2K_3$	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr_* \mathbb{Z}_2$
6	$3K_2$	$\mathbb{Z}_2 \wr S_3$	$\mathbb{Z}_2 \wr_* S_3$
6	K_6	S_6	$\mathcal{Q}_6$
7	C_7	D_7	D_7
7	K_7	S_7	$\mathcal{Q}_7$
8	C_8, C_8^+	D_8	D_8
8	$\mathtt{Pr}(C_4)$	$S_4 \times \mathbb{Z}_2$	$\mathcal{Q}_4 \times \mathbb{Z}_2$
8	$2K_4$	$S_4 \wr \mathbb{Z}_2$	$\mathcal{Q}_4 \wr_* \mathbb{Z}_2$
8	$2C_4$	$(\mathbb{Z}_2 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$	$(\mathbb{Z}_2 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$
8	$4K_2$	$\mathbb{Z}_2 \wr S_4$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_4$
8	K_8	S_8	$\mathcal{Q}_8$
9	C_9, $C_9(3)$	D_9	D_9
9	`Torus`	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr \mathbb{Z}_2$
9	$3K_3$	$S_3 \wr S_3$	$S_3 \wr_* S_3$
9	K_9	S_9	$\mathcal{Q}_9$
10	C_{10}, $C_{10}(2)$, C_{10}^+, $\mathtt{Pr}(C_5)$	D_{10}	D_{10}
10	`Petersen`	S_5	?
10	$\mathtt{Pr}(K_5)$	$S_5 \times \mathbb{Z}_2$	$\mathcal{Q}_5 \times \mathbb{Z}_2$
10	$C_{10}(4)$	$\mathbb{Z}_2 \wr D_5$	$\mathbb{Z}_2 \wr_* D_5$
10	$2C_5$	$D_5 \wr \mathbb{Z}_2$	$D_5 \wr_* \mathbb{Z}_2$
10	$2K_5$	$S_5 \wr \mathbb{Z}_2$	$\mathcal{Q}_5 \wr_* \mathbb{Z}_2$
10	$5K_2$	$\mathbb{Z}_2 \wr S_5$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_5$
10	K_{10}	S_{10}	$\mathcal{Q}_{10}$
11	C_{11}, $C_{11}(2)$, $C_{11}(3)$	D_{11}	D_{11}
11	K_{11}	S_{11}	$\mathcal{Q}_{11}$

Here the first three columns are well-known, and can be found in various books or websites. The last one collects results in this paper.

By using the equality $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$, we reach the conclusion in the abstract: with one possible exception, all quantum groups in the right column can be obtained from $\mathbb{Z}_n$, S_n, $\mathcal{Q}_n$ by using the operations $\times$, $\rtimes$, $\wr$, $\wr_*$.

The exceptional situation is that of the Petersen graph, which might give a new quantum group. We discuss it in the next section.

8 The Petersen graph

The techniques of the previous sections do not apply to the Petersen graph, which is not a circulant graph and cannot be written as a graph product. Also we could not carry a direct analysis similar to the one of the torus because of the complexity of some computations. The usual symmetry group is S_5, so in view of the results in our classification table, we have at least two natural candidates for the quantum symmetry group of the Petersen graph: S_5 and $\mathcal{Q}_5$.

Theorem 8.1. *The quantum automorphism group of the Petersen graph has an irreducible 5-dimensional representation. In particular it is not isomorphic to the quantum symmetric group $\mathcal{Q}_5$.*

Proof: Let G be the quantum automorphism group of the Petersen graph, denoted here P. We have an inclusion $S_5 \subset G$. It is well-known that

$$\mathrm{Sp}(\mathrm{P}) = \{3, -2, 1\}$$

and that the corresponding eigenspaces have dimensions $1, 4, 5$. These eigenspaces furnish representations of G and of S_5. It is straightforward to compute the character of the permutation representation of S_5 on $\mathbb{C}(\mathrm{P})$, and then using the character table of S_5 (see e.g. [10]), we see that $\mathbb{C}(\mathrm{P})$ is the direct sum of 3 irreducible representations of S_5. These have to be the previous eigenspaces, and in particular the 5-dimensional one is an irreducible representation of S_5, and of G. On the other hand, it is known from [1] that $\mathcal{Q}_5$ has no irreducible 5-dimensional representation. Thus the quantum groups G and $\mathcal{Q}_5$ are not isomorphic. □

The question now is: does the Petersen graph have quantum symmetry? In other words, is $A(\mathrm{P})$ commutative? The above result seems to indicate that if $A(\mathrm{P})$ is not commutative, we probably will have a new quantum permutation group.

References

1. T. Banica, "Symmetries of a generic coaction," *Math. Ann.* **314** (1999), 763–780.
2. T. Banica, "Quantum automorphism groups of small metric spaces," *Pacific J. Math.* **219** (2005), 27–51.
3. T. Banica, "Quantum automorphism groups of homogeneous graphs," *J. Funct. Anal.* **224** (2005), 243–280.
4. T. Banica and J. Bichon, "Free product formulae for quantum permutation groups," *J. Inst. Math. Jussieu*, to appear.
5. T. Banica and B. Collins, "Integration over compact quantum groups," *Publ. Res. Inst. Math. Sci.*, to appear.
6. T. Banica and S. Moroianu, "On the structure of quantum permutation groups," *Proc. Amer. Math. Soc.*, to appear.
7. J. Bichon, "Quantum automorphism groups of finite graphs," *Proc. Amer. Math. Soc.* **131** (2003), 665–673.
8. J. Bichon, "Free wreath product by the quantum permutation group," *Alg. Rep. Theory* **7** (2004), 343–362.
9. J. Bichon, A. De Rijdt, and S. Vaes, "Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups," *Comm. Math. Phys.* **262** (2006), 703–728.

10. W. Fulton and J. Harris, Representation Theory: A First Course, GTM 129, Springer, 1991.
11. J. Kustermans and S. Vaes, "Locally compact quantum groups," *Ann. Sci. Éc. Norm. Supér. (4)* **33**(6) (2000), 837–934.
12. A. Maes and A. Van Daele, "Notes on compact quantum groups," *Nieuw Arch. Wiskd. (4)* **16**(1–2) (1998), 73–112.
13. S. Vaes, "Strictly outer actions of groups and quantum groups," *J. Reine Angew. Math.* **578** (2005), 147–184.
14. S. Vaes and R. Vergnioux, "The boundary of universal discrete quantum groups, exactness and factoriality," `arxiv:math.OA/0509706`.
15. A. Van Daele and S. Wang, "Universal quantum groups," *Internat. J. Math.* **7** (1996), 255–264.
16. R. Vergnioux, "The property of rapid decay for discrete quantum groups," preprint.
17. S. Wang, "Free products of compact quantum groups," *Comm. Math. Phys.* **167** (1995), 671–692.
18. S. Wang, "Quantum symmetry groups of finite spaces," *Comm. Math. Phys.* **195** (1998), 195–211.
19. S.L. Woronowicz, "Compact matrix pseudogroups," *Comm. Math. Phys.* **111** (1987), 613–665.