Generalized cluster complexes via quiver representations

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Abstract We give a quiver representation theoretic interpretation of generalized cluster complexes defined by Fomin and Reading. Using d-cluster categories defined by Keller as triangulated orbit categories of (bounded) derived categories of representations of valued quivers, we define a d-compatibility degree $(- \parallel -)$ on any pair of "colored" almost positive real Schur roots which generalizes previous definitions on the noncolored case and call two such roots compatible, provided that their d-compatibility degree is zero. Associated to the root system Φ corresponding to the valued quiver, using this compatibility relation, we define a simplicial complex which has colored almost positive real Schur roots as vertices and d-compatible subsets as simplices. If the valued quiver is an alternating quiver of a Dynkin diagram, then this complex is the generalized cluster complex defined by Fomin and Reading.

Keywords Colored almost positive real Schur root · Generalized cluster complex · d-cluster category · d-cluster tilting object · d-compatibility degree

Mathematics Subject Classification (2000) 05A15 · 16G20 · 16G70 · 17B20

1 Introduction

Generalized cluster complexes associated to finite root systems are introduced by Fomin and Reading [12]. They have some nice properties, see [2] and references

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therein. They are a generalization of cluster complexes (so-called generalized associahedra) associated to the same root systems introduced in [14, 15]. Cluster complexes describe the combinatorial structure of cluster algebras introduced by Fomin–Zelevinsky [13] in order to give an algebraic and combinatorial framework for the canonical basis, see [11] for a nice survey on this combinatorics and also cluster combinatorics of root systems. In [22], Marsh, Reineke and Zelevinsky use "decorated" quiver representations and tilting theory to give a quiver interpretation of cluster complexes. This connection between tilting theory and cluster combinatorics leads Buan, Marsh, Reineke, Reiten and Todorov [6] to introduce cluster categories for a categorical model for cluster algebras, see also [9] for type A_n. Cluster categories are the orbit categories $\mathcal{D}/\tau^{-1}[1]$ of derived categories of hereditary categories arising from the action of subgroup $\langle \tau^{-1}[1] \rangle$ of the automorphism group. They are triangulated categories [19] and now they have become a successful model for acyclic cluster algebras [5, 7, 8], see also the surveys [4, 24] and references therein for recent developments and background of cluster tilting theory.

d-cluster categories $\mathcal{D}/\tau^{-1}[d]$, as a generalization of cluster categories, were introduced by Keller [19] and Thomas [25] for $d \in \mathbf{N}$. They are studied by Keller and Reiten [20], Palu [1, 23]; see also [3] for a geometric description of d-cluster categories of type A_n. d-cluster categories are triangulated categories with Calabi–Yau dimension $d + 1$. When $d = 1$, the cluster categories are recovered.

The aim of this paper is to give not only a quiver representation theoretic interpretation of all key ingredients in defining generalized cluster complexes using d-cluster categories, but also a generalization of generalized cluster complexes to infinite root systems (compare Remark 3.13 in [12], where the authors asked whether there was such an extension). For the simply-laced Dynkin case, Thomas [25] gives a realization of generalized cluster complexes by defining the d-cluster categories.

The paper is organized as follows: In the first two parts, we recall the well-known facts on d-cluster categories and (generalized) cluster complexes of finite root systems. In particular, we recall and generalize the BGP-reflection functors for cluster categories [26, 27] to d-cluster categories. In the third part, we prove some properties of d-cluster tilting objects, including that any basic d-cluster tilting object contains exactly n indecomposable direct summands. In the final section, for any root system Φ, using a d-cluster category $\mathcal{C}_d(\mathcal{H})$, we define a d-compatibility degree on any pair of colored almost positive real Schur roots. Using the d-compatibility degree, we define a generalized cluster complex associated to Φ, which has colored almost positive real Schur roots as the vertices, and any subset forms a face if and only if any two elements of this subset are d-compatible. This simplicial complex is isomorphic to the cluster complex of d-cluster category $\mathcal{C}_d(\mathcal{H})$. If Φ is a finite root system, and if we take $\mathcal{H}_0$ to be the category of representations of an alternating quiver corresponding to Φ, then our generalized cluster complex is the usual generalized cluster complex $\Delta^d(\Phi)$ defined by Fomin and Reading [12].

2 Basics on d-cluster categories

In this section, we collect some basic materials and fix the notation which we will use later on.

A valued graph $(\Gamma, \mathbf{d})$ is a finite set of vertices $1, \ldots, n$, together with nonnegative integers d_{ij} for all pairs $i, j \in \Gamma$ such that $d_{ii} = 0$ and there exist positive integers $\{\varepsilon_i\}_{i \in \Gamma}$ satisfying

$$d_{ij}\varepsilon_j = d_{ji}\varepsilon_i \quad \text{for all } i, \, j \in \Gamma.$$

A pair $\{i, j\}$ of vertices is called an edge of $(\Gamma, \mathbf{d})$ if $d_{ij} \neq 0$. An orientation Ω of a valued graph $(\Gamma, \mathbf{d})$ is given by prescribing for each edge $\{i, j\}$ of $(\Gamma, \mathbf{d})$ an order (indicated by an arrow $i \to j$). For simplicity, we denote a valued graph by Γ and a valued quiver by (Γ, Ω).

Let (Γ, Ω) be a valued quiver. We always assume that the valued quiver (Γ, Ω) contains no oriented cycles. Such orientation Ω is called admissible. Let K be a field and $\mathbf{M} = (F_i, {}_iM_j)_{i,j \in \Gamma}$ a reduced K-species of (Γ, Ω); that is, for all $i, j \in \Gamma$, ${}_iM_j$ is an $F_i - F_j$-bimodule, where F_i and F_j are division rings which are finite-dimensional vector spaces over K and $\dim({}_iM_j)_{F_j} = d_{ij}$ and $\dim_K F_i = \varepsilon_i$. We denote by $\mathcal{H}$ the category of finite-dimensional representations of $(\Gamma, \Omega, \mathcal{M})$. It is a hereditary Abelian category [10]. Let Φ be the root system of the Kac–Moody Lie algebra corresponding to the graph Γ. We assume that $P_1, \ldots, P_n$ are nonisomorphic indecomposable projective representations in $\mathcal{H}$, $E_1, \ldots, E_n$ are simple representations with dimension vectors $\alpha_1, \ldots, \alpha_n$, and $\alpha_1, \ldots, \alpha_n$ are simple roots in Φ. We use $D(-)$ to denote $\mathrm{Hom}_K(-, K)$, which is a duality of $\mathcal{H}$.

Denote by $\mathcal{D} = D^b(\mathcal{H})$ the bounded derived category of $\mathcal{H}$ with shift functor [1].

2.1 d-cluster categories

The derived category $\mathcal{D}$ has Auslander–Reiten triangles, and the Auslander–Reiten translate τ is an automorphism of $\mathcal{D}$. Fix a positive integer d and denote $F_d = \tau^{-1}[d]$; it is an automorphism of $\mathcal{D}$. The d-cluster category of H is defined in [19, 25]:

We denote by $\mathcal{D}/F_d$ the corresponding factor category. The objects are by definition the F_d-orbits of objects in $\mathcal{D}$, and the morphisms are given by

$$\mathrm{Hom}_{\mathcal{D}/F_d}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(X, F_d^i Y).$$

Here X and Y are objects in $\mathcal{D}$, and $\widetilde{X}$ and $\widetilde{Y}$ are the corresponding objects in $\mathcal{D}/F_d$ (although we shall sometimes write such objects simply as X and Y).

Definition 2.1 ([19, 25]) The orbit category $\mathcal{D}/F_d$ is called the d-cluster category of $\mathcal{H}$ (or of (Γ, Ω)), which is denoted by $\mathcal{C}_d(\mathcal{H})$, sometimes denoted by $\mathcal{C}_d(\Omega)$.

By [19] the d-cluster category is a triangulated category with shift functor [1] which is induced by the shift functor in $\mathcal{D}$, the projection $\pi : \mathcal{D} \longrightarrow \mathcal{D}/F$ is a triangle functor. When $d = 1$, this orbit category is called the cluster category of $\mathcal{H}$, denoted by $\mathcal{C}(\mathcal{H})$ (sometimes denoted by $\mathcal{C}(\Omega)$).

$\mathcal{H}$ is a full subcategory of $\mathcal{D}$ consisting of complexes concentrated in degree 0, then passing to $\mathcal{C}_d(\mathcal{H})$ by the projection π, $\mathcal{H}$ is a (possibly, not full) subcategory of $\mathcal{C}_d(\mathcal{H})$. For any $i \in \mathbf{Z}$, we use $(\mathcal{H})[i]$ to denote the copy of $\mathcal{H}$ under the ith shift $[i]$ as a subcategory of $\mathcal{C}_d(\mathcal{H})$. In this way, we have that $(\mathrm{ind}\,\mathcal{H})[i] = \{M[i] \mid M \in \mathrm{ind}\,\mathcal{H}\}$.

For any object M in $\mathcal{C}_d(\mathcal{H})$, add M denotes the full subcategory of $\mathcal{C}_d(\mathcal{H})$ consisting of direct summands of direct sums of copies of M.

For $X, Y \in \mathcal{C}_d(\mathcal{H})$, we will use $\mathrm{Hom}(X, Y)$ to denote the Hom-space $\mathrm{Hom}_{\mathcal{C}_d(\mathcal{H})}(X, Y)$ in the d-cluster category $\mathcal{C}_d(\mathcal{H})$ throughout the paper. Define $\mathrm{Ext}^i(X, Y)$ to be $\mathrm{Hom}(X, Y[i])$.

We summarize some known facts about d-cluster categories [6, 19].

Proposition 2.2 (1) $\mathcal{C}_d(\mathcal{H})$ *has Auslander–Reiten triangles and Serre functor* $\Sigma = \tau[1]$, *where* τ *is the AR-translate in* $\mathcal{C}_d(\mathcal{H})$, *which is induced from AR-translate in* $\mathcal{D}$.

(2) $\mathcal{C}_d(\mathcal{H})$ *is a Calabi–Yau category of CY-dimension* $d + 1$.

(3) $\mathcal{C}_d(\mathcal{H})$ *is a Krull–Remark–Schmidt category.*

(4) $\mathrm{ind}\,\mathcal{C}_d(\mathcal{H}) = \bigcup_{i=0}^{i=d-1}(\mathrm{ind}\,\mathcal{H})[i] \cup \{P_j[d] \mid 1 \leq j \leq n\}$.

Proof (1) This is Proposition 1.3 of [6] and Corollary 1 in Sect. 8.4 of [19].

(2) It is proved in Corollary 1 in Sect. 8.4 of [19].

(3) This is proved in Proposition 1.2 of [6].

(4) The proof for $d = 1$ is given in Proposition 1.6 of [6], which can be modified for the general d.   □

From Proposition 2.2 we define the degree for every indecomposable object in $\mathcal{C}_d(\mathcal{H})$ as follows:

Definition 2.3 For any indecomposable object $X \in \mathcal{C}_d(\mathcal{H})$, we call the nonnegative integer $\min\{k \in \mathbf{Z}_{\geq 0} \mid X \cong M[k]$ in $\mathcal{C}_d(\mathcal{H})$ for some $M \in \mathrm{ind}\,\mathcal{H}\}$ the degree of X, denoted by $\deg X$.

By Definition 2.3 any indecomposable object X of degree k is isomorphic to $M[k]$ in $\mathcal{C}_d(\mathcal{H})$, where M is an indecomposable representation in $\mathcal{H}$; $0 \leq \deg X \leq d$, X has degree d if and only if $X \cong P[d]$ in $\mathcal{C}_d(\mathcal{H})$ for some indecomposable projective object $P \in \mathcal{H}$; and X has degree 0 if and only if $X \cong M[0]$ in $\mathcal{C}_d(\mathcal{H})$ for some indecomposable object $M \in \mathcal{H}$. Here $M[0]$ means regarding the object M of $\mathcal{H}$ as a complex concentrated in degree 0.

2.2 BGP-reflection functors

If T is a tilting object in $\mathcal{H}$, then the endomorphism algebra $A = \mathrm{End}_{\mathcal{H}}(T)$ is called a tilted algebra. The tilting functor $\mathrm{Hom}_{\mathcal{H}}(T, -)$ induces the equivalence $\mathrm{RHom}(T, -) : D^b(\mathcal{H}) \to D^b(A)$, where $\mathrm{RHom}(T, -)$ is the derived functor of $\mathrm{Hom}_{\mathcal{H}}(T, -)$.

Any standard triangle functor $G : D^b(\mathcal{H}) \to D^b(\mathcal{H}')$ induces a well-defined functor $\tilde{G} : \mathcal{C}_d(\mathcal{H}) \longrightarrow \mathcal{C}_d(\mathcal{H}')$ with the following commutative diagram [19, 26]:

$$
\begin{array}{ccc}
D^b(\mathcal{H}) & \xrightarrow{\ G\ } & D^b(\mathcal{H}') \\
\downarrow & & \downarrow \\
\mathcal{C}_d(\mathcal{H}) & \xrightarrow{\ \tilde{G}\ } & \mathcal{C}_d(\mathcal{H}')
\end{array}
$$

The following result is proved in [26, 27].

Proposition 2.4 *If $G : D^b(\mathcal{H}) \to D^b(\mathcal{H}')$ is a triangle equivalence, then $\tilde{G}$ is also an equivalence of triangulated categories.*

Let k be a vertex in the valued quiver (Γ, Ω); the reflection of (Γ, Ω) at k is the valued quiver $(\Gamma, s_k\Omega)$, where $s_k\Omega$ is the orientation of Γ obtained from Ω by reversing all arrows starting or ending at k. The corresponding category of representations of $(\Gamma, s_k\Omega, \mathcal{M})$ is denoted simply by $s_k\mathcal{H}$. If k is a sink in the valued quiver (Γ, Ω), then k is a source of $(\Gamma, s_k\Omega)$, and the reflection of $(\Gamma, s_k\Omega)$ at k is (Γ, Ω). Let k be a sink in (Γ, Ω). Then P_k is a simple projective representation, and $T = \oplus_{j \neq k} P_j \oplus \tau^{-1} P_k$ is a tilting representation in $\mathcal{H}$ [24]. The tilting functor $S_k^+ = \mathrm{Hom}_{\mathcal{H}}(T, -)$ is a so-called BGP-reflection functor, and its derived functor $\mathrm{RHom}(T, -)$ is a triangle equivalence from $D^b(\mathcal{H})$ to $D^b(s_k\mathcal{H})$, which is also denoted by S_k^+. Similarly, one has BGP-reflection functors S_k^- for sources k.

Definition 2.5 The induced functors $\widetilde{S_k^+} : \mathcal{C}_d(\mathcal{H}) \longrightarrow \mathcal{C}_d(s_k\mathcal{H})$ for sinks k and $\widetilde{S_k^-} : \mathcal{C}_d(\mathcal{H}) \longrightarrow \mathcal{C}_d(s_k\mathcal{H})$ for sources k are called BGP-reflection functors of d-cluster categories.

Remark 2.6 When $d = 1$, BGP-reflection functors are discussed in [26].

We remind the reader that $\mathcal{H}$ (or $\mathcal{H}'$) is the category of representations of the valued quiver (Γ, Ω) $((\Gamma, s_k\Omega)$, respectively); the P_i (respectively, the P_i') are the indecomposable projective representations in $\mathcal{H}$ (respectively, $\mathcal{H}'$), and the E_i (respectively, the E_i') are the corresponding simple representations which are the tops of the P_i (respectively, the P_i') for $i = 1, \ldots, n$.

We recall from Proposition 2.2 and Definition 2.3 that any indecomposable object Y in $\mathcal{C}_d(\mathcal{H})$ is isomorphic to $X[i]$, where $X \in \mathrm{ind}\,\mathcal{H}$, and i is the degree of Y. Keeping this notation, we have the following proposition which gives the images of indecomposable objects in $\mathcal{C}_d(\mathcal{H})$ under the BGP-reflection functor $\widetilde{S_k^+}$.

Proposition 2.7 *Let k be a sink of the valued quiver (Γ, Ω) and Y an indecomposable object in $\mathcal{C}_d(\mathcal{H})$ with degree i. Then $Y \cong X[i]$ for an indecomposable representation X in $\mathcal{H}$, and*

$$\widetilde{S_k^+}(X[i]) = \begin{cases} P_k'[d] & \text{if } X \cong P_k(\cong E_k) \text{ and } i = 0, \\ E_k'[i-1] & \text{if } X \cong P_k(\cong E_k) \text{ and } 0 < i \leq d, \\ P_j'[d] & \text{if } X \cong P_j \not\cong P_k \text{ and } i = d, \\ S_k^+(X)[i] & \text{otherwise.} \end{cases}$$

Proof The statement in the proposition was proved in [26, 27] when $d = 1$. The proof for the case $d > 1$ is the same as there. We give a sketch of the proof for the convenience of readers. The BGP-reflection functor $S_k^+ : \mathcal{H} \longrightarrow s_k\mathcal{H}$ induces a triangle equivalence $D^b(\mathcal{H}) \longrightarrow D^b(s_k\mathcal{H})$, denoted also by S_k^+. It induces an equivalence $\mathrm{ind}\,D^b(\mathcal{H}) \longrightarrow \mathrm{ind}\,D^b(s_k\mathcal{H})$. For any indecomposable object $X[i] \in \mathrm{ind}\,D^b(\mathcal{H})$, it is not hard to show that $S_k^+(X[i]) = S_k^+(X)[i]$ for $X \not\cong P_k$ (note that $P_k = E_k$, since

k is a sink in (Γ, Ω)), and $S_k^+(P_k[i]) = E_k'[i-1]$ for $i \in \mathbf{Z}$ (cf. [26] or [27]). Since E_k' is an injective representation in $s_k \mathcal{H}$, we have $\tau P_k'[i] = E_k'[i-1]$ in $D^b(s_k \mathcal{H})$. Now passing to the d-cluster category $\mathcal{C}_d(\mathcal{H})$ (which is an orbit category of the derived category $D^b(\mathcal{H})$), we get the images of indecomposable objects of $\mathcal{C}_d(\mathcal{H})$ under $\widetilde{S_k^+}$ as stated in the proposition. □

3 Cluster combinatorics of root systems

For a valued graph Γ, we denote by $\Phi = \Phi^+ \cup \Phi^-$ the set of roots of the corresponding Kac–Moody Lie algebra.

Definition 3.1 (1) The set of almost positive roots is

$$\Phi_{\geq -1} = \Phi^+ \cup \{-\alpha_i \mid i = 1, \ldots n\}.$$

(2) Denote by $\Phi_{\geq -1}^{\text{re}}$ the subset of $\Phi_{\geq -1}$ consisting of the positive real roots together with the negatives of the simple roots.

When Φ is of finite type, $\Phi_{\geq -1} = \Phi_{\geq -1}^{\text{re}}$.

Definition 3.2 Let s_i be the Coxeter generator of the Weyl group of Φ corresponding to $i \in \Gamma_0$. We call the following map the "truncated simple reflection" σ_i of $\Phi_{\geq -1}$ [14]:

$$\sigma_i(\alpha) = \begin{cases} \alpha, & \alpha = -\alpha_j, \ j \neq i, \\ s_i(\alpha), & \text{otherwise.} \end{cases}$$

It is easy to see that σ_i is an automorphism of $\Phi_{\geq -1}^{\text{re}}$.

3.1 Cluster complexes of finite root systems

In this first paragraph, we do not assume that Γ is a Dynkin diagram (i.e., of finite type). Let $i_1, \ldots, i_n$ be an admissible ordering of Γ with respect to Ω, i.e., i_t is a sink with respect to $s_{i_{t-1}} \cdots s_{i_2} s_{i_1} \Omega$ for any $1 \leq t \leq n$. Denote $R_\Omega = \sigma_{i_n} \cdots \sigma_{i_1}$. This is an automorphism of $\Phi_{\geq -1}$ and does not depend on the choice of admissible ordering of Γ with respect to Ω. It is the automorphism induced by the Auslander–Reiten translation τ in $\mathcal{C}(\mathcal{H})$ (cf. [26, 27]).

In the rest of this subsection, we always assume that Γ is a valued Dynkin graph, which is not necessarily connected. Fomin and Zelevinsky [15] associate a nonnegative integer $(\alpha \parallel \beta)$, known as the compatibility degree, to each pair α, β of almost positive roots.

This is defined in the following way: Let Ω_0 denote one of the alternating orientations of Γ, and Γ^+ (respectively, Γ^-) the set of sinks (respectively, sources) of (Γ, Ω_0). Define

$$\tau_\pm = \prod_{i \in \Gamma^\pm} \sigma_i.$$

Then $R_{\Omega_0} = \tau_- \tau_+$, which is simply denoted by R.

Denote by $n_i(\beta)$ the coefficient of α_i in the expansion of β in terms of the simple roots $\alpha_1, \ldots, \alpha_n$. Then $(\|)$ is uniquely defined by the following two properties:

$$(*) \quad (-\alpha_i \| \beta) = \max([\beta : \alpha_i], 0),$$
$$(**) \quad (\tau_\pm \alpha \| \tau_\pm \beta) = (\alpha \| \beta),$$

for any $\alpha, \beta \in \Phi_{\geq -1}$ and any $i \in \Gamma$.

Two almost positive roots α, β are called compatible if $(\alpha \| \beta) = 0$.

The cluster complex $\Delta(\Phi)$ associated to the finite root system Φ is defined in [14].

Definition 3.3 The cluster complex $\Delta(\Phi)$ is a simplicial complex on the ground set $\Phi_{\geq -1}$. Its faces are mutually compatible subsets of $\Phi_{\geq -1}$. The facets of $\Delta(\Phi)$ are called the (root-)clusters associated to Φ.

3.2 Generalized cluster complexes of finite root systems

At the beginning of this subsection, we assume that Γ is an arbitrary valued graph, which is not necessarily connected, except where we express specifically. As before, Φ denotes the set of roots of the corresponding Lie algebra, and $\Phi_{\geq -1}$ denotes the set of almost positive roots. Fix a positive integer d; for any $\alpha \in \Phi^+$, following [12], we call $\alpha^1, \ldots, \alpha^d$ the d "colored" copies of α.

Definition 3.4 ([12]) The set of colored almost positive roots is

$$\Phi^d_{\geq -1} = \{\alpha^i : \alpha \in \Phi_{>0}, i \in \{1, \ldots, d\}\} \cup \{(-\alpha_i)^1 : 1 \leq i \leq n\}.$$

When Γ is a Dynkin graph, the root system Φ of the corresponding Lie algebra is finite. In this case, the generalized cluster complex $\Delta^d(\Phi)$ is defined on the ground set $\Phi^d_{\geq -1}$ and using the binary compatibility relation on $\Phi^d_{\geq -1}$. This binary compatibility relation is a natural generalization of binary compatibility relation on $\Phi_{\geq -1}$, which we now recall from [12].

For a root $\beta \in \Phi_{\geq -1}$, let $t(\beta)$ denote the smallest t such that $R^t(\beta)$ is a negative root.

Definition 3.5 ([12]) Two colored roots $\alpha^k, \beta^l \in \Phi^d_{\geq -1}$ are called compatible if and only if one of the following conditions is satisfied:

(1) $k > l$. $t(\alpha) \leq t(\beta)$, and the roots $R(\alpha)$ and β are compatible (in the original "non-colored" sense).
(2) $k < l$. $t(\alpha) \geq t(\beta)$, and the roots α and $R(\beta)$ are compatible.
(3) $k > l$. $t(\alpha) > t(\beta)$, and the roots α and β are compatible.
(4) $k < l$. $t(\alpha) < t(\beta)$, and the roots α and β are compatible.
(5) $k = l$. And the roots α and β are compatible.

Now we are ready to recall the definition of generalized cluster complex $\Delta^d(\Phi)$ for a finite root system Φ.

Definition 3.6 ([12]) $\Delta^d(\Phi)$ has $\Phi^d_{\geq -1}$ as the set of vertices, its simplices are mutually compatible subsets of $\Phi^d_{\geq -1}$. The subcomplex of $\Delta^d(\Phi)$ which has $\Phi^d_{>0}$ as the set of vertices is denoted by $\Delta^d_+(\Phi)$

Now we generalize the definition of R_d [12] for a finite root system to an arbitrary root system.

Definition 3.7 Let (Γ, Ω) be a valued quiver. For $\alpha^k \in \Phi^d_{\geq -1}$, we set

$$R_{d,\Omega}\left(\alpha^k\right) = \begin{cases} \alpha^{k+1} & \text{if } \alpha \in \Phi_{>0} \text{ and } k < d, \\ (R_\Omega(\alpha))^1 & \text{otherwise.} \end{cases}$$

Remark 3.8 If (Γ, Ω_0) is a valued Dynkin graph with an alternating orientation, then the automorphism R of $\Phi_{\geq -1}$ defined by Fomin and Zelevinsky [14] is R_{Ω_0}; hence, R_{d,Ω_0} is the usual one (R_d) defined by Fomin and Reading [12].

Theorem 3.9 ([12]) *Let Φ be a finite root system. The compatibility relation on $\Phi^d_{\geq -1}$ has the following properties:*

(1) α^k *is compatible with β^l if and only if $R_d(\alpha^k)$ is compatible with $R_d(\beta^l)$.*
(2) $(-\alpha_i)^1$ *is compatible with β^l if and only if $n_i(\beta) = 0$.*

Moreover, conditions 1–2 uniquely determine this relation.

Now we generalize the "truncated simple reflections" of $\Phi_{\geq -1}$ to the colored almost positive roots. Let Φ be an arbitrary root system (not necessarily of finite type).

Definition 3.10 Let s_k be the Coxeter generator of the Weyl group of Φ corresponding to $k \in \Gamma_0$. We define the following map $\sigma_{k,d}$ of $\Phi^d_{\geq -1}$:

$$\sigma_{k,d}(\alpha^i) = \begin{cases} \alpha^d_k & \text{if } i = 1 \text{ and } \alpha = -\alpha_k, \\ \alpha^{i-1}_k & \text{if } 1 < i \leq d \text{ and } \alpha = \alpha_k, \\ (-\alpha_j)^1 & \text{if } i = 1 \text{ and } \alpha = -\alpha_j, \ j \neq k, \\ (s_k(\alpha))^i & \text{otherwise.} \end{cases}$$

$\sigma_{k,d}$ is a bijection of $\Phi^d_{\geq -1}$. We call it a *d-truncated simple reflection* of $\Phi^d_{\geq -1}$.

4 *d*-cluster tilting in *d*-cluster categories

Let $\mathcal{C}_d(\mathcal{H})$ be a *d*-cluster category of type $\mathcal{H}$, where $\mathcal{H}$ is the category of representations of the valued quiver (Γ, Ω). It is a Calabi–Yau triangulated category with CY-dimension $d + 1$.

Definition 4.1 (1) An object X in $\mathcal{C}_d(\mathcal{H})$ is called exceptional if $\text{Ext}^i(X, X) = 0$ for any $1 \leq i \leq d$.

(2) An object X is called a d-cluster tilting object if it satisfies the property: $Y \in \mathrm{add}(X)$ if and only if $\mathrm{Ext}^i(X, Y) = 0$ for $1 \leq i \leq d$.

(3) An object X is called almost complete tilting if there is an indecomposable object Y such that $X \oplus Y$ is a d-cluster tilting object. Such an indecomposable object Y is called a complement of X.

Proposition 4.2 (1) *For an object X in $\mathcal{H}$, X is exceptional in $\mathcal{H}$ i.e., $\mathrm{Ext}^1_{\mathcal{H}}(X, X) = 0$ if and only if $X[0]$ is exceptional in $\mathcal{C}_d(\mathcal{H})$.*

(2) Any indecomposable exceptional object X in $\mathcal{C}_d(\mathcal{H})$ is of the form $M[i]$ with M being an exceptional representation in $\mathcal{H}$ and $0 \leq i \leq d-1$ or of the form $P_j[d]$ for some $1 \leq j \leq n$. In particular, if Γ is a Dynkin graph, then any indecomposable object in $\mathcal{C}_d(\mathcal{H})$ is exceptional.

(3) Suppose that $d > 1$. Then $\mathrm{End}_{\mathcal{C}_d(\mathcal{H})} X$ is a division algebra for any indecomposable exceptional object X.

(4) Suppose that $d > 1$. Let P be a projective representation in $\mathcal{H}$ and X a representation in $\mathcal{H}$. Then, for any $-d \leq i \leq d$, $\mathrm{Ext}^1(P, X[i]) = 0$ except possibly for $i \in \{-1, d-1, d\}$.

Proof (1) Let $X \in \mathcal{H}$ be exceptional. We will prove that $\mathrm{Ext}^i(X, X) = 0$ for any $i \in \{1, \ldots, d\}$. By definition we have that $\mathrm{Ext}^i(X, X) = \bigoplus_{k \in \mathbf{Z}} \mathrm{Ext}^i_{\mathcal{D}}(X, \tau^{-k}X[kd]) = \mathrm{Ext}^i_{\mathcal{D}}(X, X) \oplus \mathrm{Ext}^i_{\mathcal{D}}(X, \tau X[-d])$. In this sum, the first summand $\mathrm{Ext}^i_{\mathcal{D}}(X, X) = 0, \forall i \geq 1$, while the second summand $\mathrm{Ext}^i_{\mathcal{D}}(X, \tau X[-d]) \cong \mathrm{Hom}_{\mathcal{D}}(X, \tau X[i - d])$, which is zero when $i < d$ and is isomorphic to $\mathrm{Ext}^1_{\mathcal{D}}(X, X) = 0$ when $i = d$. This proves that X is exceptional in $\mathcal{C}_d(\mathcal{H})$. The proof for the converse directly follows from the definition.

(2) The statements follow from Proposition 2.2(4) and Definition 4.1, also using Part 1 and the fact that the shift is an autoequivalence.

(3) Let X be an indecomposable exceptional representation in $\mathcal{H}$, and suppose that $d > 1$. From the definition of the orbit category It follows that $\mathrm{End}_{\mathcal{C}_d(H)} X \cong \bigoplus_{m \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(X, \tau^{-m}X[dm]) \cong \mathrm{End}_{\mathcal{H}} X$. The last isomorphism holds due to the facts: $\mathrm{Hom}_{\mathcal{D}}(X, \tau^m X[-md]) \cong \mathrm{Hom}_{\mathcal{D}}(X[md], \tau^m X) \cong \mathrm{Ext}^1_{\mathcal{D}}(\tau^{m-1}X, X[md]) = 0$ for any positive integer m and $\mathrm{Hom}_{\mathcal{D}}(X, \tau^{-m}X[md]) \cong \mathrm{Hom}_{\mathcal{D}}(\tau^m X, X[md])$, which is also zero, since $md > 1$ (we use the assumption $d > 1$ here) for any positive integer m. Then $\mathrm{End}_{\mathcal{C}_d(H)} X$ is a division algebra, since $\mathrm{End}_{\mathcal{H}} X$ is a division algebra. Since any indecomposable exceptional object M in $\mathcal{C}_d(H)$ is some shift $X[i]$ of an indecomposable exceptional representation X in $\mathcal{H}$, $\mathrm{End}_{\mathcal{C}_d(H)} M = \mathrm{End}_{\mathcal{C}_d(H)} X[i] \cong \mathrm{End}_{\mathcal{C}_d(H)} X$ is a division algebra.

(4) Suppose that $d > 1$. Let P be a projective representation in $\mathcal{H}$ and X a representation in $\mathcal{H}$. Then, for any $-d \leq i \leq d$, $\mathrm{Ext}^1(P, X[i]) = \bigoplus_{k \in \mathbf{Z}} \mathrm{Ext}^1_{\mathcal{D}}(P, \tau^{-k}X[dk + i]) \cong \mathrm{Ext}^1_{\mathcal{D}}(P, \tau X[-d + i]) \oplus \mathrm{Ext}^1_{\mathcal{D}}(P, X[i])$. Now if $i \neq -1, d-1, d$, then $\mathrm{Ext}^1_{\mathcal{D}}(P, \tau X[-d + i]) = 0 = \mathrm{Ext}^1_{\mathcal{D}}(P, X[i])$. Then, for any $-d \leq i \leq d$, $\mathrm{Ext}^1(P, X[i]) = 0$ except for $i = -1, d-1$, and d. $\qquad\square$

Remark 4.3 Any basic (i.e., multiplicity-free) exceptional object contains at most $(d + 1)n$ nonisomorphic indecomposable direct summands.

Proof Let X be a basic exceptional object in $\mathcal{C}_d(\mathcal{H})$. Then any indecomposable direct summand of X is exceptional; hence, by Proposition 4.2(2), we write M as $M = \bigoplus_{k=0}^{k=d} \bigoplus_{i \in I_k} M_{i,k}[k]$ with $M_{i,k}$ being an indecomposable exceptional representation. Therefore, $\bigoplus_{i \in I_k} M_{i,k}$ is an exceptional object in hereditary category $\mathcal{H}$; hence, the number of direct summands is at most n, i.e., $|I_k| \leq n$. Then the number of indecomposable direct summands of M is at most $(d + 1)n$. □

For any pair of objects T, X in $\mathcal{C}_d(\mathcal{H})$, due to the Calabi–Yau property of $\mathcal{C}_d(\mathcal{H})$, we have that $\mathrm{Ext}^i(X, T) = 0$ for $1 \leq i \leq d$ if and only if $\mathrm{Ext}^i(T, X) = 0$ for $1 \leq i \leq d$. Hence, by Remark 4.3 and Definition 4.1, T is a d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$ if and only if add T is a maximal d-orthogonal subcategory of $\mathcal{C}_d(\mathcal{H})$ in the sense of [17]: i.e., add T is contravariantly finite and covariantly finite in $\mathcal{C}_d(\mathcal{H})$ and satisfies the following property: $X \in$ add T if and only if $\mathrm{Ext}^i(X, T) = 0$ for $1 \leq i \leq d$ if and only if $\mathrm{Ext}^i(T, X) = 0$ for $1 \leq i \leq d$. In the following, we will prove that any basic d-cluster tilting object contains exactly n indecomposable direct summands. First of all, we recall some results from [17] which hold in any $(d + 1)$-Calabi–Yau triangulated category.

Theorem 4.4 (Iyama) *Let X be an almost complete tilting object in $\mathcal{C}_d(\mathcal{H})$ and X_0 a complement of X. Then there are $d + 1$ triangles:*

$$(*) \quad X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \xrightarrow{\sigma_i} X_{i+1}[1],$$

where f_i is the minimal right add X-*approximation of X_i and g_i minimal left* add X-*approximation of X_{i+1}, all X_i are indecomposable and complements of X, $i = 0, \ldots, d$.*

For the convenience of readers, we sketch the proof; for details, see [18].

Proof We suppose that $d > 1$; the same statement for $d = 1$ was proved in [6]. For the complement X_0 of X, we consider the minimal right add X-approximation $f_0 : B_0 \to X_0$ of X_0, extend f_0 to the triangle $X_1 \xrightarrow{g_0} B_0 \xrightarrow{f_0} X_0 \xrightarrow{\sigma_0} X_1[1]$. It is easy to see that X_1 is indecomposable, g_0 is the minimal left add X-approximation of X_1, and $X \oplus X_1$ is an exceptional object in $\mathcal{C}_d(\mathcal{H})$ (cf. [6]). From Theorem 5.1 in [18] it follows that $X \oplus X_1$ is a d-cluster tilting object. Continuing this step, one can get complements $X_1, \ldots, X_{d+1}$ with triangles $X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \xrightarrow{\sigma_i} X_i[1]$ for $0 \leq i \leq d$, where f_i (g_i) is the minimal right (left, resp.) add X-approximation of X_i (X_{i+1}, resp.), and $X \oplus X_i$ is a d-cluster tilting object. □

Corollary 4.5 *With the notation of Theorem 4.4, we have that $\sigma_d[d]\sigma_{d-1}[d-1]\cdots \sigma_1[1]\sigma_0 \neq 0$. In particular, $\mathrm{Hom}(X_i, X_j[j-i]) \neq 0$ and $X_i \not\cong X_j, \forall 0 \leq i < j \leq d$.*

Proof From Theorem 4.4 we have that $\sigma_0 \neq 0$, since the triangle $(*)$ at $i = 0$ in Theorem 4.4 is nonsplitting. Suppose that $\sigma_d[d]\sigma_{d-1}[d-1]\cdots \sigma_1[1]\sigma_0 = 0$; then $\sigma_{d-1}[d-1]\cdots \sigma_1[1]\sigma_0 : X_0 \to X_d[d]$ factors through $f_d[d] : B_d[d] \to X_d[d]$, since we have a triangle $X_{d+1}[d] \xrightarrow{g_d[d]} B_d[d] \xrightarrow{f_d[d]} X_d[d] \xrightarrow{\sigma_d[d]} X_{d+1}[d+1]$.

Since $\mathrm{Hom}(X_0, B_d[d]) = \mathrm{Ext}^d(X_0, B_d) = 0$, $\sigma_{d-1}[d-1]\cdots\sigma_1[1]\sigma_0 = 0$. Similarly, $\sigma_{d-2}[d-2]\cdots\sigma_1[1]\sigma_0 = 0$ and, finally, $\sigma_0 = 0$, a contradiction. Now we prove the final statement: we have that $\sigma_{j-1}[j-1]\cdots\sigma_i \in \mathrm{Hom}(X_i, X_j[j-i])$ and $\sigma_{j-1}[j-1]\cdots\sigma_i \neq 0$. Otherwise $\sigma_{j-1}[j-1]\cdots\sigma_i = 0$, and hence $\sigma_{d-1}[d-1]\cdots\sigma_1[1]\sigma_0 = 0$, a contradiction. Now suppose that $X_i \cong X_j$ for some $i < j$. Then $\mathrm{Ext}^k(X_i, X_j) = 0$ for $1 \leq k \leq d$, a contradiction. Then $X_i \not\cong X_j$. $\qquad\square$

Now we state our main result of this section.

Theorem 4.6 *Any basic d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$ contains exactly n indecomposable direct summands.*

To prove the theorem, we need some technical lemmas.

Lemma 4.7 *Let $d > 1$, and let $X = M[i]$, $Y = N[j]$ be indecomposable objects of degrees i, j, respectively, in $\mathcal{C}_d(\mathcal{H})$. Suppose that $\mathrm{Hom}(X, Y) \neq 0$. Then one of the following holds*:

(1) *We have $i = j$ or $j - 1$ (provided that $j \geq 1$).*
(2) *We have $i = 0$, $i = d$ (and $M = P$) or $d - 1$ (provided that $j = 0$).*

Proof Let $d > 1$. Firstly we note that, for any indecomposable object $X \in \mathcal{C}_d(\mathcal{H})$, $0 \leq \deg X \leq d$, $\deg X = d$ if and only if $X = P_i[d]$ for an indecomposable projective representation P_i. This implies that $-d \leq \deg Y - \deg X \leq d$ for indecomposable objects $X, Y \in \mathcal{C}_d(\mathcal{H})$. Let $X = M[i]$, $Y = N[j]$ be indecomposable objects of degrees i, j, respectively, in $\mathcal{C}_d(\mathcal{H})$. We have $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(M, N[j-i]) = \bigoplus_{k \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(M, \tau^{-k}N[j-i+kd]) = \mathrm{Hom}_{\mathcal{D}}(M, \tau N[j-i-d]) \oplus \mathrm{Hom}_{\mathcal{D}}(M, N[j-i]) \oplus \mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1}N[j-i+d])$. The last equality holds due to $-d \leq j - i \leq d$, and $\mathrm{Hom}_{\mathcal{D}}(M, \tau^{-k}N[j-i+kd]) = 0$ for $k \neq -1, 0, 1$. We divide the calculation of $\mathrm{Hom}(X, Y)$ into three cases:

(1) The case $-d < j - i < d$. We have that $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(M, N[j-i]) \oplus \mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1}N[j-i+d])$. The first summand is zero when $j - i \neq 0, 1$, while the second is zero when $d + j - i \neq 1$ (equivalently, $d + j - i > 1$, since $0 < d + j - i < 2d$).
(2) The case $j - i = -d$. Then $j = 0$, $i = d$ ($M = P$). Then $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(P, \tau^{-1}N)$.
(3) The case $j - i = d$. Then $j = d$ ($N = P$), $i = 0$. Then $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(M, \tau P) \oplus \mathrm{Hom}_{\mathcal{D}}(M, P[d]) \oplus \mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1}P[2d]) = 0$.

Therefore, if $\mathrm{Hom}(X, Y) \neq 0$, then $\mathrm{Hom}(M[0], N[j-i]) \neq 0$. Proof of (1). Suppose that $j \geq 1$. Then combining with Case 3, we have that $-d < j - i < d$. We want to prove that if $j - i \neq 0, 1$, then $\mathrm{Hom}(X, Y) = 0$, and this will finish the proof of (1). Under the condition $j - i \neq 0, 1$, from Case 1 we have that $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1}N[j-i+d])$, which is zero for $d + j - i \neq 1$. But if $d + j - i = 1$, i.e., $i = d$, then $M = P$ and $j = 1$. Then $\mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1}N[j-i+d]) = \mathrm{Hom}_{\mathcal{D}}(P, \tau^{-1}N[1]) = 0$. We have finished the proof of (1).

Proof of (2) Suppose that $j = 0$. Then $-d \leq j - i \leq 0$. From Cases 1–2 it follows that $i = 0$, $i = d$ ($M = P$), or $i = d - 1$. This finishes the proof of (2). $\qquad\square$

Lemma 4.8 *If $d > 2$, then* $\text{Ext}^2(M[i], N[i]) = 0$ *for objects* $M, N \in \mathcal{H}$ *and any* i.

Proof It is sufficient to prove that $\text{Ext}^2(M[0], N[0]) = 0$. From the definition of the orbit category $\mathcal{D}/\tau^{-1}[d]$ we have that

$$\text{Ext}^2(M[0], N[0]) = \text{Hom}(M[0], N[2]) = \bigoplus_{k \in \mathbf{Z}} \text{Hom}_{\mathcal{D}}(M, \tau^{-k} N[kd + 2]),$$

where each summand $\text{Hom}_{\mathcal{D}}(M, \tau^{-k} N[kd + 2])$ equals 0, since $kd + 2 \geq 2$ or $kd + 2 \leq -1$ by the condition $d > 2$. Hence, $\text{Ext}^2(M[0], N[0]) = 0$. $\square$

Lemma 4.9 *Let $d > 1$ and $M, N \in \mathcal{H}$. Then* $\text{Ext}^1(M[0], N[0]) \cong \text{Ext}^1_{\mathcal{H}}(M, N)$. *Furthermore, any non-split triangle between $M[0]$ and $N[0]$ in $\mathcal{C}_d(\mathcal{H})$ is induced from a non-split exact sequence between M and N in $\mathcal{H}$.*

Proof Under the condition $d > 1$, it is easy to see that $\text{Ext}^1(M[0], N[0]) = \bigoplus_{k \in \mathbf{Z}} \text{Ext}^1_{\mathcal{D}}(M, \tau^{-k} N[2k]) = \text{Ext}^1_{\mathcal{D}}(M, N) = \text{Ext}^1_{\mathcal{H}}(M, N)$. This proves the first statement. Since $\mathcal{H} \subset \mathcal{C}_d(\mathcal{H})$ is a (not necessarily full) embedding and any exact short sequence in $\mathcal{H}$ induces a triangle in $\mathcal{C}_d(\mathcal{H})$, the final statement then follows from the first statement. $\square$

Proof (of Theorem 4.6) We assume that $d > 1$, since it was proved in [6] for $d = 1$. Let $M = \bigoplus_{i \in I} M_i[k_i]$ be a d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$, where all M_i are indecomposable representations in $\mathcal{H}$, $0 \leq k_i \leq d$ (when $k_i = d$, M_i is projective). One can assume that one of k_i is 0, otherwise one can replace M by a suitable shift of M. Denote $\nu(M) = \max\{|k_i - k_j| \mid \forall i, j\}$. We prove that $|I| = n$ by induction on $\nu(M)$, where $|I|$ denotes the cardinality of I. If $\nu(M) = 0$, i.e., $k_i = 0$ for all i, then $\bigoplus_{i \in I} M_i[0]$ is a d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$ and hence a tilting object in $\mathcal{H}$. Then $|I| = n$. Now assume that $\nu(M) = m > 0$. Without loss of generality, we assume that $k_1 = \cdots = k_t = m$ and $k_j < m$ for $j > t$. From the complement $X_0 = M_1[k_1]$ of $X = M \setminus M_1[k_1]$ (here we use $X \setminus X_1$ to denote a complement of X_1 in X for a direct summand X_1 of X), by Theorem 4.4, we have at least $d + 1$ complements X_j, $j = 0, \ldots, d$, which form the triangles $(*)$ in Theorem 4.4. In these triangles, it is easy to see that $f_i = 0$ if and only if $B_i = 0$ if and only if $g_i = 0$. We will prove that there is at least one of complements X_j with smaller degree than m. At first, we prove this statement for the special case $m = 1$. We claim that the degree of X_1 is 0 or 1 in this case. Otherwise $X_1 = P[d]$ for some indecomposable projective representation P or $X_1 = Y[d - 1]$ for some indecomposable representation Y. Write X_0 as $Z[1]$, where Z is an indecomposable representation in $\mathcal{H}$. If $X_1 = P[d]$, then $\text{Hom}(X_1, X_0[d]) = \text{Hom}(P[d], X_0[d]) \cong \text{Hom}(P, Z[1]) = 0$, a contradiction to the fact that $\text{Hom}(X_1, X_0[d]) \cong \text{Hom}(X_0, X_1[1])$ is not zero by Theorem 4.4 or Corollary 4.5. If $X_1 = Y[d - 1]$, then X_1 has degree 1 when $d = 2$, and $\text{Hom}(X_1, X_0[d]) = \text{Hom}(Y[d - 1], Z[d + 1]) \cong \text{Ext}^2(Y, Z) = 0$ by Lemma 4.8 when $d > 2$, which also contradicts to the fact that $\text{Hom}(X_1, X_0[d]) \cong \text{Hom}(X_0, X_1[1])$ is not zero. This proves the statement that X_1 has degree 0 or 1. Now if there are no complements X_j of X with degree 0, then all X_j have degree 1. We prove that any three successive complements, say X_0, X_1, X_2, cannot have the same degree. If all

degrees of X_i, $i = 0, 1, 2$, are the same, we can assume that all X_i have degree 0. By Lemma 4.9, we have non-split short exact sequences in $\mathcal{H}$:

$$0 \longrightarrow X_1 \longrightarrow B_0 \longrightarrow X_0 \longrightarrow 0,$$
$$0 \longrightarrow X_2 \longrightarrow B_1 \longrightarrow X_1 \longrightarrow 0.$$

From the first short exact sequence we have $\mathrm{Ext}^1_{\mathcal{H}}(X_0, X_1) \ncong 0$. Applying $\mathrm{Hom}_{\mathcal{H}}(X_0, -)$ to the second exact sequence, we have the long exact sequence

$$\cdots \rightarrow \mathrm{Ext}^1_{\mathcal{H}}(X_0, X_2) \rightarrow \mathrm{Ext}^1_{\mathcal{H}}(X_0, B_1) \rightarrow \mathrm{Ext}^1_{\mathcal{H}}(X_0, X_1)$$
$$\rightarrow \mathrm{Ext}^2_{\mathcal{H}}(X_0, X_2) \rightarrow \mathrm{Ext}^2_{\mathcal{H}}(X_0, B_1).$$

Since $X \oplus X_0$ is a d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$ and $B_1 \in \mathrm{add}\, X$, $\mathrm{Ext}^1(X_0, B_1) = 0$. Hence we have that $\mathrm{Ext}^1_{\mathcal{H}}(X_0, B_1) = 0$ by Lemma 4.9. It follows that $\mathrm{Ext}^1_{\mathcal{H}}(X_0, X_1) = 0$, since $\mathrm{Ext}^2_{\mathcal{H}}(X_0, X_2) = 0$ due to $\mathcal{H}$ being hereditary. It is a contradiction. This finishes the proof for $m = 1$.

Now suppose that $m > 1$. We will prove that there is at least one of complements X_j with smaller degree than m. We divide the proof into two cases:

Case 1. All maps f_i (equivalently g_i) are nonzero. Now we assume that there are no complements of X with smaller degree than m. Then by Lemma 4.7 the degrees of all X_i are m. If $d > 2$, then $\mathrm{Ext}^2(X_0, X_2) = 0$ by Lemma 4.8, a contradiction to Corollary 4.5. If $d = 2$, then the same proof as above shows that $\mathrm{Ext}^1(X_0, X_1) = 0$, which contradicts to Corollary 4.5. Therefore, there is a complement of X with smaller degree than m.

Case 2. There are some i such that $f_i = 0$ (equivalently $g_i = 0$). Then $X_i \cong X_{i+1}[1]$ for such i. It follows that X_{i+1} has smaller degree than X_i if X_i has a strictly positive degree. Therefore, we have a complement of X, say X_s, such that the degree k'_1 of X_s is smaller than $m = k_1$. Now we replace X by $X' = (X \setminus X_0) \oplus X_s$, which is, by Theorem 4.4, a d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$ containing $|I|$ indecomposable direct summands. The number of indecomposable direct summands of X' with the (maximal) degree $m (= \nu(M))$ is $t - 1$. We repeat the step for the complement $M_2[k_2]$ of almost complete tilting object $X' \setminus M_2[k_2]$, getting a d-cluster tilting object X'' containing $|I|$ indecomposable direct summands, and the number of indecomposable direct summands of X'' with the (maximal) degree $m (= \nu(M))$ is $t - 2$. Repeating such a step t times, one can get a (basic) d-cluster tilting object T containing $|I|$ indecomposable direct summands and $\nu(T) < \nu(M)$. By induction, T contains exactly n indecomposable direct summands. Then $|I| = n$. □

Remark 4.10 Theorem 4.6 is proved by Thomas [25] for a simply-laced Dynkin quiver (Γ, Ω_0), using the fact that $\mathrm{ind}\, D^b(K\vec{\Delta}) \approx \mathbf{Z}\vec{\Delta}$ for a Dynkin quiver $\vec{\Delta}$. This fact does not hold for non-Dynkin quivers. Our proof is more categorical.

Denote by $\mathcal{E}(\mathcal{H})$ the set of isomorphism classes of indecomposable exceptional representations in $\mathcal{H}$. The set $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ of isoclasses of indecomposable exceptional objects in $\mathcal{C}_d(\mathcal{H})$ is the (disjoint) union of subsets $\mathcal{E}(\mathcal{H})[i]$, $i = 0, 1, \ldots, d - 1$, with $\{P_j[d] | 1 \leq j \leq n\}$. A subset $\mathcal{M}$ of $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ is called exceptional if, for any

$X, Y \in \mathcal{M}$, $\mathrm{Ext}^i(X, Y) = 0$ for all $i = 1, \ldots, d$. Denote by $\mathcal{E}_+(\mathcal{C}_d(\mathcal{H}))$ the subset of $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ consisting of all indecomposable exceptional objects other than $P_1[d], \ldots, P_n[d]$.

Now we are ready to define a simplicial complex associated to the d-cluster category $\mathcal{C}_d(\mathcal{H})$, which is a generalization of the classical cluster complexes of cluster categories [6, 24, 26].

Definition 4.11 The cluster complex $\Delta^d(\mathcal{H})$ of $\mathcal{C}_d(\mathcal{H})$ is a simplicial complex which has $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ as the set of vertices and has exceptional subsets in $\mathcal{C}_d(\mathcal{H})$ as its simplices. The positive part $\Delta^d_+(\mathcal{H})$ is the subcomplex of $\Delta^d(\mathcal{H})$ on the subset $\mathcal{E}_+(\mathcal{C}_d(\mathcal{H}))$.

By the definition, the facets (maximal simplices) are exactly the d-cluster tilting subsets (i.e., the sets of indecomposable objects of $\mathcal{C}_d(\mathcal{H})$ (up to isomorphism) whose direct sum is a d-cluster tilting object).

Proposition 4.12 (1) $\Delta^d(\mathcal{H})$ and $\Delta^d_+(\mathcal{H})$ are pure of dimension $n - 1$.

(2) For any sink (or source) k, the BGP-reflection functor $\tilde{S}^+_k$ (resp. $\tilde{S}^-_k$) induces an isomorphism between $\Delta^d(\mathcal{H})$ and $\Delta^d(s_k\mathcal{H})$. In particular, if Γ is a Dynkin diagram and Ω and Ω' are two orientations of Γ, then $\Delta^d(\mathcal{H})$ and $\Delta^d(\mathcal{H}')$ are isomorphic.

Proof (1) From Theorem 4.6 it follows that any d-cluster tilting subset contains exactly n elements. Hence $\Delta^d(\mathcal{H})$ is pure of dimension $n - 1$. Now suppose that $M = \oplus^{n-1}_{i=1} M_i$ is an exceptional object in $\mathcal{C}_d(\mathcal{H})$ and that none of the M_i are isomorphic to $P_j[d]$ for any j. In the proof of Theorem 4.6, we proved that not all complements of an almost complete tilting objects have the same degrees. Then M has a complement in $\mathcal{E}_+(\mathcal{C}_d(\mathcal{H}))$. This proves that $\Delta^d_+(\mathcal{H})$ is pure of dimension $n - 1$.

(2) Since $\tilde{S}^+_k$ is a triangle equivalence from the d-cluster category $\mathcal{C}_d(\mathcal{H})$ to $\mathcal{C}_d(s_k\mathcal{H})$, it sends (indecomposable) exceptional objects to (indecomposable) exceptional objects. Thus it induces an isomorphism from $\Delta^d(\mathcal{H})$ to $\Delta^d(s_k\mathcal{H})$. The second statement follows from the first statement together with the fact that, for two orientations Ω, Ω' of a Dynkin graph Γ, there is a admissible sequence with respect to sinks $i_1, \ldots, i_n$ such that $\Omega' = s_{i_n} \cdots s_{i_1} \Omega$. $\square$

5 Cluster combinatorics of d-cluster categories

We now define a map $\gamma^d_{\mathcal{H}}$ from $\mathrm{ind}\,\mathcal{C}_d(\mathcal{H})$ to $\Phi^d_{\geq -1}$. Note that any indecomposable object X of degree i in $\mathcal{C}_d(\mathcal{H})$ has the form $M[\overline{i}]$ with $M \in \mathrm{ind}\,\mathcal{H}$, and if $i = d$, then $M = P_j$, an indecomposable projective representation.

Definition 5.1 Let $\gamma^d_{\mathcal{H}}$ be defined as follows. Let $M[i] \in \mathrm{ind}\,\mathcal{C}_d(\mathcal{H})$, where $M \in \mathrm{ind}\,H$ and $i \in \{1, \ldots, d\}$ (note that if $i = d$, then $M = P_j$ for some j). We set

$$\gamma^d_{\mathcal{H}}(M[i]) = \begin{cases} (\underline{\dim}\,M)^{i+1} & \text{if } M[i] \in \mathrm{ind}\,\mathcal{H}[i] \text{ for some } 0 \leq i \leq d - 1; \\ (-\alpha_j)^1 & \text{if } M[i] = P_j[d]. \end{cases}$$

This map is a kind of extension of correspondence in Gabriel–Kac's Theorem between the indecomposable representations of quivers and positive roots of corresponding Lie– Kac–Moody algebras. It is a bijection if Γ is a Dynkin diagram.

We denote by $\Phi_{>0}^{sr}$ the set of real Schur roots of (Γ, Ω), i.e.,

$$\Phi_{>0}^{sr} = \left\{ \underline{\dim} M \mid M \in \mathrm{ind}\,\mathcal{E}(\mathcal{H}) \right\}.$$

Then the map $M \mapsto \underline{\dim} M$ gives a 1-1 correspondence between $\mathcal{E}(\mathcal{H})$ and $\Phi_{>0}^{sr}$ [24].

If we denote by $\Phi_{\geq -1}^{sr,d}$ the set of colored almost positive real Schur roots, which by definition consists of d copies of the set $\Phi_{>0}^{sr}$, together with one copy of the negative simple roots, then the map $\gamma_{\mathcal{H}}^d$ gives a bijection from $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ to $\Phi_{\geq -1}^{sr,d}$. $\Phi_{\geq -1}^{sr,d}$ contains a subset $\Phi_{>0}^{sr,d}$ consisting of all colored positive real Schur roots. The restriction of $\gamma_{\mathcal{H}}^d$ gives a bijection from $\mathcal{E}_+(\mathcal{C}_d(\mathcal{H}))$ to $\Phi_{>0}^{sr,d}$.

Since $\mathcal{E}(\mathcal{H}) \longrightarrow \Phi_{>0}^{sr} : M \mapsto \underline{\dim} M$ is a bijection, we use M_β to denote the unique indecomposable exceptional representation in $\mathcal{H}$ whose dimension vector is β. From Proposition 4.2 it follows that $\gamma_{\mathcal{H}}^d(M_\beta[i]) = \beta^{i+1}$ for any $0 \leq i \leq d - 1$. We sometimes use $M_{\beta^{i+1}}$ to denote the unique preimage of a colored almost positive real Schur root β^{i+1} under $\gamma_{\mathcal{H}}^d$.

We now prepare to define a simplicial complex $\Delta^{d,\mathcal{H}}(\Phi)$ associated with any root system Φ, which turns out to be isomorphic to the cluster complex $\Delta^d(\mathcal{H})$ of the d-cluster category $\mathcal{C}_d(\mathcal{H})$. When Γ is a Dynkin graph, taking an alternating orientation Ω_0 of Γ, this complex $\Delta^{d,\mathcal{H}_0}(\Phi)$ is the generalized cluster complex $\Delta^d(\Phi)$ defined by Fomin and Reading [12].

First of all, we define the "d-compatibility degree" on any pair of colored almost positive real Schur roots.

Definition 5.2 For any pair of colored almost positive real Schur roots α, β, the d-compatibility degree of α, β is defined as follows:

$$(\alpha \parallel \beta)_{d,\mathcal{H}} = \dim_{\mathrm{End}\,M_\alpha} \left(\mathrm{Ext}^1 \left(M_\alpha, \bigoplus_{i=0}^{i=d-1} M_\beta[i] \right) \right),$$

where $\dim_{\mathrm{End}\,M_\alpha}(\mathrm{Ext}^1(M_\alpha, \bigoplus_{i=0}^{i=d-1} M_\beta[i]))$ denotes the length of $\mathrm{Ext}^1(M_\alpha, \bigoplus_{i=0}^{i=d-1} M_\beta[i])$ as a right $\mathrm{End}\,M_\alpha$-module. When $d > 1$, $\mathrm{End}\,M_\alpha$ is a division algebra by Proposition 4.2(3), and this length equals the dimension of $\mathrm{Ext}^1(M_\alpha, \bigoplus_{i=0}^{i=d-1} M_\beta[i])$ over the division algebra $\mathrm{End}\,M_\alpha$.

Remark 5.3 When Γ is a Dynkin diagram with trivial valuation and Ω_0 is an alternating orientation of Γ, this compatibility degree is defined in [25]. When $d = 1$ and Γ is a Dynkin diagram, we recover the classical compatibility degree defined in [6, 27].

Theorem 5.4 (1) *For any pair of colored almost positive real Schur roots α, β, we have*:

(a) $(\alpha \parallel \beta)_{d,\mathcal{H}} = (\sigma_{k,d}(\alpha) \parallel \sigma_{k,d}(\beta))_{d,s_k\mathcal{H}}$ *if k is a sink (or a source)*.

(b) $(\alpha \parallel \beta)_{d,\mathcal{H}} = (R_{d,\Omega}(\alpha) \parallel R_{d,\Omega}(\beta))_{d,\mathcal{H}}$.

(c) $(\alpha \parallel \beta)_{d,\mathcal{H}} = 0$ if and only if $(\beta \parallel \alpha)_{d,\mathcal{H}} = 0$.

(2) *For any almost positive real Schur root β, $((-\alpha_i)^1 \parallel (\beta)^l)_{d,\mathcal{H}} = 0$ if and only if $\max\{n_i(\beta), 0\} = 0$, where $n_i(\beta)$ is the coefficient of α_i in the expansion of β in terms of the simple roots $\alpha_1, \ldots, \alpha_n$.*

Proof (1) Let α, β be two colored almost positive real Schur roots.

(a) We prove it for the case k is a sink, the proof for source is similar. It is easy to check that the following diagram is commutative:

$$
\begin{array}{ccc}
\operatorname{ind} \mathcal{C}_d(\mathcal{H}) & \xrightarrow{\tilde{S}_k^+} & \operatorname{ind} \mathcal{C}_d(s_k \mathcal{H}) \\
\gamma_{\mathcal{H}}^d \downarrow & & \downarrow \gamma_{s_k \mathcal{H}}^d \\
\Phi_{\geq -1}^d & \xrightarrow{\sigma_{k,d}} & \Phi_{\geq -1}^d
\end{array}
$$

Hence we have that

$$
\left(\sigma_{k,d}(\alpha) \parallel \sigma_{k,d}(\beta) \right)_{d, s_k \mathcal{H}} = \dim_{\operatorname{End} \tilde{S}_k^+(M_\alpha)} \operatorname{Ext}^1\left(\tilde{S}_k^+(M_\alpha), \bigoplus_{i=0}^{i=d-1} \tilde{S}_k^+(M_\beta)[i] \right)
$$

$$
= \dim_{\operatorname{End} M_\alpha} \operatorname{Ext}^1\left(M_\alpha, \bigoplus_{i=0}^{i=d-1} M_\beta[i] \right) = (\alpha \parallel \beta)_{d,\mathcal{H}}.
$$

(b) As we mentioned before, the shift functor [1] of $\mathcal{C}_d(\mathcal{H})$ is an auto-equivalence. We now check that the following diagram commutes:

$$
\begin{array}{ccc}
\operatorname{ind} \mathcal{C}_d(\mathcal{H}) & \xrightarrow{[1]} & \operatorname{ind} \mathcal{C}_d(\mathcal{H}) \\
\gamma_{\mathcal{H}}^d \downarrow & & \downarrow \gamma_{\mathcal{H}}^d \\
\Phi_{\geq -1}^d & \xrightarrow{R_{d,\Omega}} & \Phi_{\geq -1}^d
\end{array}
$$

By Proposition 2.2, any indecomposable object in $\mathcal{C}_d(\mathcal{H})$ is of the form $X[i]$ with X an indecomposable representation in $\mathcal{H}$ and with $0 \leq i \leq d - 1$ or of the form $P_j[d]$. Denote $\underline{\dim}\, X = \alpha$. If $i \leq d - 2$, then $R_{d,\Omega}\gamma_{\mathcal{H}}^d(X[i]) = R_{d,\Omega}((\alpha)^{i+1}) = (\alpha)^{i+2} = \gamma_{\mathcal{H}}^d[1](X[i])$. We will prove the equality for other indecomposable objects in $\mathcal{C}_d(\mathcal{H})$. Firstly, we have that $R_{d,\Omega}\gamma_{\mathcal{H}}^d(P_j[d - 1]) = R_{d,\Omega}((\underline{\dim}\, P_j)^d) = (-\alpha_j)^1$ and $\gamma_{\mathcal{H}}^d[1](P_j[d - 1]) = (-\alpha_j)^1$. Hence $R_{d,\Omega}\gamma_{\mathcal{H}}^d(P_j[d - 1]) = \gamma_{\mathcal{H}}^d[1](P_j[d - 1])$. Secondly, for any $X[d - 1]$ with X not being projective, we have $\tau X \in \operatorname{ind} \mathcal{H}$. We have that $R_{d,\Omega}\gamma_{\mathcal{H}}^d(X[d - 1]) = R_{d,\Omega}((\alpha)^d) = (R_\Omega(\alpha))^1$ and $\gamma_{\mathcal{H}}^d[1](X[d - 1]) = \gamma_{\mathcal{H}}^d(X[d]) = \gamma_{\mathcal{H}}^d(\tau^{-1}[d]\tau X) = \gamma_{\mathcal{H}}^d(\tau X) = (R_\Omega(\alpha))^1$. The last equality holds, since $\underline{\dim}\, \tau X = R_\Omega(\underline{\dim}\, X)$ (cf. Sect. 3.1). This proves that $R_{d,\Omega}\gamma_{\mathcal{H}}^d(X[d - 1]) = \gamma_{\mathcal{H}}^d[1](X[d - 1])$. For $P_j[d]$, the proof is similar. We finish the proof of the commutativity of the diagram.

It follows that

$$\left(R_{d,\Omega}(\alpha) \parallel R_{d,\Omega}(\beta)\right)_{d,\mathcal{H}} = \dim_{\operatorname{End}M_\alpha[1]}\operatorname{Ext}^1\left(M_\alpha[1], \bigoplus_{i=0}^{i=d-1} M_\beta[1][i]\right)$$

$$= \dim_{\operatorname{End}M_\alpha}\operatorname{Ext}^1\left(M_\alpha, \bigoplus_{i=0}^{i=d-1} M_\beta[i]\right) = (\alpha \parallel \beta)_{d,\mathcal{H}},$$

where the second equality follows from the fact that [1] is an equivalence.

(c) Let $X, Y \in \mathcal{C}_d(\mathcal{H})$ with $\operatorname{Ext}^i(X, Y) = 0$ for any $1 \le i \le d$. Then by the Calabi–Yau property of $\mathcal{C}_d(\mathcal{H})$ we have that, for any $1 \le j \le d$, $\operatorname{Ext}^j(Y, X) \cong \operatorname{Ext}^{d-j+1}(X, Y) = 0$. This proves (c).

(2) We first prove the necessity: Let β be an almost positive real Schur root with $((-\alpha_i)^1 \parallel (\beta)^l)_{d,\mathcal{H}} = 0$. If β is a negative simple root and $l = 1$, we easily have that $\max\{n_i(\beta), 0\} = 0$. Now we assume that β is a positive real Schur root. From the condition $((-\alpha_i)^1 \parallel (\beta)^l)_{d,\mathcal{H}} = 0$ we have $\operatorname{Ext}^j(P_i[d], M_{\beta^l}) = 0$, i.e., $\operatorname{Ext}^j(P_i[d], M_\beta[l-1]) = 0$ for any $1 \le j \le d$. Since $1 \le l \le d$, we have $1 \le j \le d$, where $j = d + 1 - l$. Now we have that $0 = \operatorname{Ext}^j(P_i[d], M_\beta[l-1]) \cong \operatorname{Hom}(P_i[d], M_\beta[l+j-1]) \cong \operatorname{Hom}(P_i, M_\beta)$. Hence $n_i(\beta) = \dim_{\operatorname{End}P_i}\operatorname{Hom}(P_i, M_\beta) = 0$.

Now we prove the other direction. Suppose that β is an almost positive real Schur root with $\max\{0, n_i(\beta)\} = 0$. Firstly, if β is the negative of a simple root, say $(-\alpha_j)^1$, then

$$\left((-\alpha_i)^1 \parallel (-\alpha_j)^1\right)_{d,\mathcal{H}} = \dim_{\operatorname{End}(P_i[d])}\operatorname{Ext}^1\left(P_i[d], \bigoplus_{k=0}^{k=d-1} P_j[d][k]\right)$$

$$= \dim_{\operatorname{End}(P_i[d])}\operatorname{Ext}^1\left(P_i, \bigoplus_{k=0}^{k=d-1} P_j[k]\right)$$

$$= \dim_{\operatorname{End}(P_i[d])}\operatorname{Ext}^1\left(P_i, P_j[d-1]\right),$$

the last equality following from Proposition 4.2(4). But $\operatorname{Ext}^1(P_i, P_j[d-1]) \cong \operatorname{Hom}(P_i, P_j[-1][d+1]) \cong D\operatorname{Hom}(P_j[-1], P_i) \cong D\operatorname{Ext}^1(P_j, P_i) = 0$. This proves that $((-\alpha_i)^1 \parallel (-\alpha_j)^1)_{d,\mathcal{H}} = 0$. Now we assume that β is a positive real Schur root and l is a positive integer not exceeding d. We will prove that $((-\alpha_i)^1 \parallel (\beta)^l)_{d,\mathcal{H}} = 0$ under the condition that $n_i(\beta) = 0$. We can assume that $d > 1$, since, for $d = 1$, the corresponding result is proved in [26]. From the condition $n_i(\beta) = 0$ it follows that $\operatorname{Hom}_{\mathcal{H}}(P_i, M_\beta) = 0$ and then $\operatorname{Hom}(P_i, M_\beta) = 0$. Hence $\operatorname{Ext}^1(P_i[d], M_\beta[d-1]) \cong \operatorname{Hom}(P_i, M_\beta) = 0$. We will prove that $\operatorname{Ext}^j(P_i[d], M_{\beta^l}) = 0$ for $1 \le j \le d$. Now given such j, $\operatorname{Ext}^j(P_i[d], M_{\beta^l}) = \operatorname{Ext}^j(P_i[d], M_\beta[l-1]) = \operatorname{Ext}^1(P_i[d], M_\beta[l+j-2]) \cong \operatorname{Ext}^1(P_i, M_\beta[l+j-d-2])$. Since $1 \le l \le d$, $1 \le j \le d$, we have $-d \le l+j-d-2 \le d-2$. Then we have that $\operatorname{Ext}^j(P_i[d], M_{\beta^l}) = 0$, which follows from Proposition 4.2(4) for $l+j-d-2 \ne -1$ and from the fact that $\operatorname{Ext}^1(P_i, M_\beta[-1]) \cong \operatorname{Hom}(P_i, M_\beta) = 0$ for $l+j-d-2 = -1$. □

Definition 5.5 Let Φ be the root system corresponding to Γ and $\mathcal{H}$ the category of representations of the valued quiver (Γ, Ω).

(1) Any pair α, β of almost positive real Schur roots is called d-compatible if $(\alpha \parallel \beta)_{d,\mathcal{H}} = 0$; a subset of $\Phi^{sr,d}_{\geq -1}$ is called d-compatible if any two elements of this subset are compatible.
(2) The simplicial complex $\Delta^{d,\mathcal{H}}(\Phi)$ associated to Φ and $\mathcal{H}$ is a complex which has $\Phi^{sr,d}_{\geq -1}$ as the set of vertices. Its simplices are d-compatible subsets of $\Phi^{sr,d}_{\geq -1}$. The subcomplex of $\Delta^{d,\mathcal{H}}(\Phi)$ which has $\Phi^{sr,d}_{>0}$ as the set of vertices is denoted by $\Delta^{d,\mathcal{H}}_{+}(\Phi)$. We call $\Delta^{d,\mathcal{H}}(\Phi)$ the generalized cluster complex associated to Φ and $\mathcal{H}$.

Remark 5.6 Given a graph Γ, we have the corresponding root system Φ. Since the set of real Schur roots of Φ depends on the category ind $\mathcal{H}$, equivalently, on the orientation Ω of Γ, the generalized cluster complexes $\Delta^{d,\mathcal{H}}(\Phi)$ are possibly nonisomorphic for different orientations of Γ, but they are isomorphic to each other if Γ is a Dynkin diagram by Proposition 4.12(2) and the following theorem.

Theorem 5.7 (1) *Let Γ be a valued graph and Φ the corresponding root system. Let Ω be an admissible orientation of Γ. Then $\gamma^d_{\mathcal{H}}$ provides an isomorphism from the simplicial complex $\Delta^d(\mathcal{H})$ to the generalized cluster complex $\Delta^{d,\mathcal{H}}(\Phi)$, which sends vertices to vertices and k-faces to k-faces.*

(2) The restriction of $\gamma^d_{\mathcal{H}}$ to $\Delta^d_+(\mathcal{H})$ gives an isomorphism from $\Delta^d_+(\mathcal{H})$ to $\Delta^{d,\mathcal{H}}_+(\Phi)$.

(3) If Γ is a Dynkin graph, and Ω_0 is an alternating orientation of Γ, then $\Delta^{d,\mathcal{H}_0}(\Phi)$ is the generalized cluster complex $\Delta^d(\Phi)$ defined by Fomin and Reading in [12].

Proof (1) $\gamma^d_{\mathcal{H}}$ provides a bijection from the vertices of $\Delta^d(\mathcal{H})$ to that of $\Delta^{d,\mathcal{H}}(\Phi)$. For any pair of colored almost positive real Schur roots α^k, β^l, they are d-compatible if and only if $M_{\alpha^k} \oplus M_{\beta^l}$ is an exceptional object, where M_{α^k} and M_{β^l} are the exceptional objects corresponding to α^k, β^l respectively under the map $\gamma^d_{\mathcal{H}}$. Hence $\gamma^d_{\mathcal{H}}$ is an isomorphism from $\Delta^d(\mathcal{H})$ to $\Delta^{d,\mathcal{H}}(\Phi)$.

(2) This is a direct consequence of (1).
(3) This is a direct consequence of Theorems 3.9 and 5.4.   □

From Theorem 5.7 one can translate results from each side. For example, one gets the number of d-cluster tilting objects in $\mathcal{C}_d(\mathcal{H})$ from the number of facets of generalized cluster complexes of finite root systems [12].

Corollary 5.8 (1) *The generalized cluster complex $\Delta^{d,\mathcal{H}}(\Phi)$ and its subcomplex $\Delta^{d,\mathcal{H}}_+(\Phi)$ are pure of dimension $n-1$.*

(2) Let (Γ, Ω) be a connected Dynkin quiver and Φ the root system corresponding to Γ. Then the number of d-cluster tilting objects of $\mathcal{C}_d(\mathcal{H})$ is $\prod_i \frac{dh+e_i+1}{e_i+1}$, where h is the Coxeter number of Φ, and $e_1, \ldots, e_n$ are the exponents of Φ.

(3) *Let (Γ, Ω) be a connected Dynkin quiver and Φ the corresponding root system. Then the number of complements of any almost complete tilting object in $\mathcal{C}_d(\mathcal{H})$ is $d+1$.*

Proof (1) It follows from Proposition 4.12(1) and Theorem 5.7(1).

(2) From Theorem 5.7(1) and Proposition 8.4 in [12] it follows that the statement holds for the d-cluster category $\mathcal{C}_d(\mathcal{H}_0)$ of Ω_0. Then by Proposition 4.12(2) the statement holds for a d-cluster category $\mathcal{C}_d(\mathcal{H})$ corresponding to an arbitrary orientation Ω.

(3) From Theorem 5.7(1) and Proposition 3.10 in [12] it follows that the number of complements of any almost complete tilting object in $\mathcal{C}_d(\mathcal{H}_0)$ is $d+1$. Hence by Proposition 4.12(2) the number of complements of any almost complete tilting object in $\mathcal{C}_d(\mathcal{H})$ is $d+1$. □

Remark 5.9 Corollary 5.8(1) generalizes Theorem 2.9 in [12] to infinite root systems.

Remark 5.10 (1) From Corollary 5.8(2) for $d=1$, combining with the result in [6] (see also [21]), in which the cluster tilting subcategories in $\mathcal{D}$ are proved to be in one-to-one correspondence with the cluster tilting modules in cluster categories by the projection π, we have an explanation on why the number of cluster tilting subcategories (i.e., Ext-configurations in [16]) in $\mathcal{D}$ is the same as the number of facets of $\Delta(\Phi)$.

(2) Corollary 5.8(3) is proved by Thomas [25] for an alternating simply-laced Dynkin quiver (Γ, Ω_0), using a different approach.

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