

On automorphism groups of quasiprimitive 2-arc transitive graphs

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Abstract We characterize the automorphism groups of quasiprimitive 2-arc-transitive graphs of twisted wreath product type. This is a partial solution for a problem of Praeger regarding quasiprimitive 2-arc transitive graphs. The solution stimulates several further research problems regarding automorphism groups of edge-transitive Cayley graphs and digraphs.

Keywords Quasiprimitive · 2-arc transitive · Automorphisms · Cayley graphs

1 Introduction

In a graph Γ , an ordered pair of adjacent vertices is called an *arc*, and a sequence of distinct vertices (u, v, w) is called a *2-arc* if v is adjacent to both u and w . A graph Γ is called $(X, 2)$ -*arc transitive*, where $X \leq \text{Aut}\Gamma$, if X acts transitively on the set of 2-arcs of Γ .

The class of 2-arc transitive graphs is one of the central objects in algebraic graph theory, which have been intensively studied in the literature, see for example [1, 7, 12, 13, 17] and references therein. In particular, Praeger [13] gave a reduction for the study of finite non-bipartite 2-arc transitive graphs, which leads to the study of quasiprimitive 2-arc transitive graphs, defined later. The main purpose of this paper is to characterize the automorphism group for one type of quasiprimitive 2-arc transitive graph.

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Theorem 1.1 *Let Γ be a finite $(X, 2)$ -arc-transitive graph. Assume that X has a unique minimal normal subgroup G , which is non-Abelian and regular on the vertex set of Γ . Then Γ is not 3-arc transitive, and either*

- (i) Γ has valency 8, and $\text{Aut}\Gamma = (G \times \mathbb{Z}_2^3) \rtimes \text{GL}(3, 2)$, or
- (ii) G is the unique minimal normal subgroup of $\text{Aut}\Gamma$.

A permutation group G on a set Ω is said to be *quasiprimitive* if every non-trivial normal subgroup of G is transitive. Praeger [13] showed that only four of the eight types of quasiprimitive permutation groups can occur as automorphism groups of 2-arc transitive graphs, which are described below. Let X be a quasiprimitive permutation group on Ω , and let $\text{soc}(X)$ be the *socle* of X , that is the product of all minimal normal subgroups of X . The four types involved in the study of 2-arc transitive graphs are

- HA *Holomorph Affine type*: $\text{soc}(X)$ is elementary Abelian, regular on Ω .
- AS *Almost Simple type*: $\text{soc}(X)$ is a non-Abelian simple group.
- PA *Product Action type*: $\text{soc}(X)$ is a non-Abelian non-simple minimal normal subgroup of X , and $\text{soc}(X)$ has no normal subgroup which is regular on Ω .
- TW *Twisted Wreath product type*: $\text{soc}(X)$ is a non-Abelian non-simple minimal normal subgroup of X , and $\text{soc}(X)$ acts regularly on Ω .

A graph Γ is called *quasiprimitive $(X, 2)$ -arc transitive* if X is quasiprimitive on $V\Gamma$ and 2-arc transitive on Γ . A classification of quasiprimitive $(X, 2)$ -arc transitive graphs of type HA is given in [7]; a description of those of type TW is given in [1]; examples of type PA are given in [12]. In the study of such graphs, the problem of determining their automorphism groups naturally occurs, refer to [15] and [5, 7, 8]. Theorem 1.1 determines the automorphism groups for quasiprimitive 2-arc transitive graphs of twisted wreath product type.

2 Preliminary results

In this section we collect notation and preliminary results which will be used in the ensuing sections. For a group X and a subgroup $H < X$, denote by $N_X(H)$ and $C_X(H)$ the normalizer and the centralizer of H in X , respectively. Let $Z(X)$ be the center of X . We need some properties regarding 2-transitive permutation groups, see [3].

Lemma 2.1 *Let H be a 2-transitive permutation group on Ω . Assume that $N \triangleleft H$ is imprimitive on Ω . Then H is affine with $\text{soc}(H) = \mathbb{Z}_p^e$, where p is a prime and $e \geq 1$, and further, the following hold:*

- (i) *An imprimitive normal subgroup of H is of the form $\mathbb{Z}_p^e \cdot \mathbb{Z}_b$, where $b \mid p^{e'} - 1$ and e' is a proper divisor of e .*
- (ii) *If N is a subnormal subgroup of H , then either N is transitive on Ω , or $N < \text{soc}(H)$.*
- (iii) *H_ω has no non-trivial normal subgroup of p -power order, for $\omega \in \Omega$.*

Let Γ be an $(X, 2)$ -arc-transitive graph with vertex set V . For $v \in V$, let $\Gamma(v) = \{w \in V \mid (v, w) \text{ is an arc}\}$, the neighborhood of v . As usual, let $X_v^{[1]}$ be the kernel of X_v acting on $\Gamma(v)$, and let $X_v^{\Gamma(v)}$ be the permutation group induced by X_v on $\Gamma(v)$. Then $X_v^{\Gamma(v)} \cong X_v / X_v^{[1]}$. For $w \in \Gamma(v)$, let $X_{vw} = X_v \cap X_w$, and $X_{vw}^{[1]} = X_v^{[1]} \cap X_w^{[1]}$, the kernel of X_{vw} acting on $\Gamma(v) \cup \Gamma(w)$. We observe that $X_v \cong X_v^{[1]} \cdot X_v^{\Gamma(v)}$, and $(X_v^{[1]})^{\Gamma(w)} \cong (X_v^{[1]} X_w^{[1]}) / X_w^{[1]} \cong X_v^{[1]} / X_{vw}^{[1]}$. Thus $X_v^{[1]} \cong X_{vw}^{[1]} \cdot (X_v^{[1]})^{\Gamma(w)}$, and so

$$X_v \cong X_v^{[1]} \cdot X_v^{\Gamma(v)} \cong (X_{vw}^{[1]} \cdot (X_v^{[1]})^{\Gamma(w)}) \cdot X_v^{\Gamma(v)}. \tag{1}$$

In particular, if $X_{vw}^{[1]} = 1$, then

$$\begin{cases} X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(w)}, \\ X_v \cong (X_v^{[1]})^{\Gamma(w)} \cdot X_v^{\Gamma(v)}. \end{cases} \tag{2}$$

Here we have a fundamental theorem for studying 2-arc transitive graphs, which can be found in [17].

Theorem 2.2 *Let Γ be an $(X, 2)$ -arc transitive graph, and let $\{v, w\}$ be an edge of Γ . Then either $X_{vw}^{[1]} = 1$, or $X_{vw}^{[1]}$ is a p -group with p prime and $\text{PSL}(d, q) \triangleleft X_v^{\Gamma(v)}$ with $|\Gamma(v)| = \frac{q^d - 1}{q - 1}$.*

3 Proof of the main theorem

Let Γ be an $(X, 2)$ -arc transitive graph with vertex set V . Assume that X is quasi-primitive on V of type TW. Let $G = \text{soc}(X)$, the unique minimal normal subgroup of X , which is regular on V . Let $n = |V| = |G|$.

We first treat the case where the centralizer $C_{\text{Aut}\Gamma}(G)$ is non-trivial.

Lemma 3.1 *Assume that $C_{\text{Aut}\Gamma}(G) \neq 1$. Then Γ has valency 8, and $\text{Aut}\Gamma = (G \times \mathbb{Z}_2^3) \rtimes \text{GL}(3, 2) = G \rtimes \text{AGL}(3, 2)$.*

Proof Let $C = C_{\text{Aut}\Gamma}(G)$, $E = CG$, and $F = CX$. Since G is regular, we have that $E = C \times G$, and $G, E \triangleleft F$. Thus $X \cap E = G$, and so $X_v \cap E_v = 1$ and $F_v = E_v \rtimes X_v$. Since Γ is connected, F_v acts faithfully on $\Gamma(v)$, and hence $X_v \cong X_v^{\Gamma(v)} \leq F_v^{\Gamma(v)} \cong F_v$, and X_v and F_v are both 2-transitive permutation groups on $\Gamma(v)$. Inspecting the 2-transitive permutation groups, see [3], it is easily concluded that $E_v \cong \mathbb{Z}_2^3$, and $X_v \cong \text{GL}(3, 2) \cong \text{PSL}(2, 7)$, which is of degree 8. Thus $F_v = \mathbb{Z}_2^3 \rtimes \text{GL}(3, 2) = \text{AGL}(3, 2)$, and Γ has valency 8. Finally, $C \cong E/G \cong E_v \cong \mathbb{Z}_2^3$, and $C \times X_v \cong ((C \times G) \times X_v) / G = F/G \cong F_v = \text{AGL}(3, 2)$. Since $\text{AGL}(3, 2)$ acts irreducibly on its natural module \mathbb{Z}_2^3 , it follows that C is a minimal normal subgroup of F .

Let $Y = N_{\text{Aut}\Gamma}(G)$. Then $F_v \leq Y_v$, and Y_v is a 2-transitive permutation group on $\Gamma(v)$. Suppose that $F_v < Y_v$. It follows since $F_v = \text{AGL}(3, 2)$ that $Y_v = \text{A}_8$ or S_8 . However, since $G \triangleleft Y$ and G is non-Abelian and characteristically simple, $C = C_Y(G) \cong CG/G \triangleleft Y/G \cong Y_v$, which is a contradiction. Therefore, $F_v = Y_v$ and so $F = Y = N_{\text{Aut}\Gamma}(G)$.

Suppose that $F < \text{Aut}\Gamma$. Let Z be a group such that $F < Z \leq \text{Aut}\Gamma$, and F is a maximal subgroup of Z . Then $\text{AGL}(3, 2) = F_v^{\Gamma(v)} \leq Z_v^{\Gamma(v)} \leq \text{Sym}(\Gamma(v)) \cong \text{S}_8$. It follows that $Z_v^{\Gamma(v)} = \text{AGL}(3, 2)$, A_8 or S_8 , and $Z_{vw}^{\Gamma(w)} \cong \text{GL}(3, 2)$, A_7 or S_7 , respectively. Since $(Z_v^{[1]})^{\Gamma(w)} \leq Z_{vw}^{\Gamma(w)}$, we have that $(Z_v^{[1]})^{\Gamma(w)} = 1$, $\text{GL}(3, 2)$, A_7 or S_7 . By Theorem 2.2, we conclude that $Z_{vw}^{[1]} = 1$, and hence $Z_v \cong (Z_v^{[1]})^{\Gamma(w)}.Z_v^{\Gamma(v)}$. It then follows that either $Z_v \cong Z_v^{\Gamma(v)}$, or $Z_v \cong Z_{vw}^{\Gamma(w)} \times Z_v^{\Gamma(v)}$. Since $Z_v > F_v$ and $F_v \cong \text{AGL}(3, 2)$, A_8 or S_8 , it is easily shown that Z_v is one of the following:

$$\text{A}_8, \quad \text{S}_8, \quad \text{GL}(3, 2) \times \text{AGL}(3, 2), \quad \text{A}_7 \times \text{A}_8, \quad \text{A}_7 \times \text{S}_8, \quad \text{S}_7 \times \text{S}_8.$$

As F and Z are transitive on V , $\frac{|Z|}{|F|} = \frac{|Z_v|}{|F_v|}$ is one of the following numbers:

$$2^3 \cdot 3 \cdot 7, \quad 2^3 \cdot 3^2 \cdot 5 \cdot 7, \quad 2^4 \cdot 3^2 \cdot 5 \cdot 7, \quad 3 \cdot 5, \quad 2 \cdot 3 \cdot 5, \quad 2^3 \cdot 3^3 \cdot 5^2 \cdot 7, \quad 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \quad \text{or} \quad 2^5 \cdot 3^3 \cdot 5^2 \cdot 7.$$

Since G is a characteristic subgroup of F and G is not normal in Z , it follows that F is not normal in Z . Let $\Omega = [Z : F]$, the set of right cosets of F in Z . Since F is a maximal subgroup of Z , the action of Z on Ω is primitive. As G is not normal in Z , and as G, C are the only minimal normal subgroups of F , it follows that the kernel of this Z -action on Ω is trivial or equals $C \cong \mathbb{Z}_2^3$. If the kernel equals C , then it follows that Z centralizes C . However, X does not centralizes C and $Z > X$, which is a contradiction. Therefore, Z is a primitive permutation group on Ω . Then since $|\Omega| = |Z : F|$ is not a proper power of an integer, it follows from the O’Nan-Scott theorem that Z is an almost simple group. Since $Z = GZ_v$, by [2, Theorem 1.4], $\text{soc}(Z) \cong \text{A}_n$ where $n = |G|$, and so Γ is a complete graph, which is a contradiction.

Therefore, $\text{Aut}\Gamma = F = G \rtimes \text{AGL}(3, 2)$. □

From now on, we assume that

$$\text{C}_{\text{Aut}\Gamma}(G) = 1.$$

Suppose that G is not normal in $\text{Aut}\Gamma$. Then $X \neq \text{Aut}\Gamma$, and there exists $Z \leq \text{Aut}\Gamma$ such that X is a maximal subgroup of Z . Let N be a minimal normal subgroup of Z . Then $N \not\leq X$. Write $N = T_1 \times \dots \times T_l$, where $l \geq 1$ and $T_1 \cong \dots \cong T_l$ are simple groups. We use a series of steps to derive a contradiction.

Step 1. N is non-Abelian.

Suppose that N is Abelian, say $N \cong \mathbb{Z}_p^l$ with p prime. Let $E = NG = N \rtimes G$, and let $F = NX = N \rtimes X$. Since G is regular on V , we have that $|E_v| = |N| = p^l$, and E_v is a p -group. It follows that $E_v^{\Gamma(v)}$ is a non-trivial p -group. Since $E \triangleleft F$, we have $1 \neq E_v^{\Gamma(v)} \triangleleft F_v^{\Gamma(v)}$, and since $F_v^{\Gamma(v)}$ is a 2-transitive permutation group, $F_v^{\Gamma(v)}$ is affine. It then follows that $E_v \cong E_v^{\Gamma(v)}$, and $\text{soc}(F_v^{\Gamma(v)}) = E_v^{\Gamma(v)} \cong \mathbb{Z}_p^l$. By Theorem 2.2, $F_{vw}^{[1]} = 1$, and hence by Formula (2),

$$p^l = |N| = \frac{|F|}{|X|} = \frac{|F_v|}{|X_v|} = \frac{|F_v^{\Gamma(v)}|}{|X_v^{\Gamma(v)}|} \cdot |(F_v^{[1]})^{\Gamma(w)}|.$$

Now $X_v^{\Gamma(v)}$ and $F_v^{\Gamma(v)}$ are affine 2-transitive permutation groups of degree p^l such that $X_v^{\Gamma(v)}$ is a subgroup of $F_v^{\Gamma(v)}$ of index dividing p^l , and $(F_v^{[1]})^{\Gamma(w)}$ is a normal subgroup of $F_{vw}^{\Gamma(w)}$ of order dividing p^l . Inspecting the classification of 2-transitive permutation groups of degree p^l (see [3]), we conclude that $F_v^{\Gamma(v)} = X_v^{\Gamma(v)}$ and $(F_v^{[1]})^{\Gamma(v)} = 1$, which is a contradiction. Thus N is non-Abelian.

Step 2. N is transitive on V , and either $G < N$, or each prime divisor r of $|T|$ is such that $r^l \nmid |V|$.

If $G < N$, then N is transitive on V . Suppose that $G \not< N$. Since G is a minimal normal subgroup of X and $G \cap N$ is normalized by X , it follows that $G \cap N = 1$. Since $C_{\text{Aut}\Gamma}(G) = 1$ and G is insoluble, G and so X permutes by conjugation $\{T_1, \dots, T_l\}$ non-trivially. As G is a minimal normal subgroup of X , it follows that G permutes by conjugation the collection $\{T_1, \dots, T_l\}$, and so $G \leq \text{Sym}(l)$. In particular, $|G|$ divides $l!$.

Suppose that there exists a prime r dividing $|T|$ such that r^l divides $|V|$. Then r^l divides $|G|$, and so r^l divides $l!$. However, it is known that the highest power of r dividing $l!$ is at most $r^{\frac{l-1}{r-1}}$. Hence $r^l \leq r^{\frac{l-1}{r-1}}$, and so $l \leq \frac{l-1}{r-1} \leq l - 1$, which is not possible. Thus, if a prime $r \mid |T|$, then $r^l \nmid |V|$. In particular, N is not semiregular on V . Since Γ is not bipartite, it follows that N is transitive on V .

Step 3. Z is quasiprimitive on V of type PA, and $G < N$.

By Step 2, any minimal normal subgroup of Z is transitive on V , and so Z is quasiprimitive on V , which has an irregular minimal normal subgroup N . By [13], Z is of type HA, AS, TW or PA. Since N is not regular, Z is neither of type HA nor of type TW. If $Z = GZ_v$ is almost simple, then by [2, Theorem 1.4], $\text{soc}(Z) \geq A_n$ where $n = |V|$ and Z is 2-transitive on V . Thus Γ is a complete graph, and so X is 2-transitive on V , which is not possible. Thus Z is of type PA, and so there exists a prime $r \mid |T|$ such that $r^l \mid |V|$. By Step 2, $G < N$.

Step 4. The socle $\text{soc}(X_v) \leq N_v$.

Suppose $\text{soc}(X_v) \not\leq N_v$. Let $Y = NX$. Then $N \cap X_v = 1$, and $N \triangleleft Y$. Hence $N_v \triangleleft Y_v$, and $Y_v/N_v \cong Y/N \cong X_v$. So $Y_v = N_v \rtimes X_v$; in particular, $|\Gamma(v)|^2$ divides $|Y_v|$. Suppose that $Y_{vw}^{[1]} = 1$. Then by Formula (2), $Y_v \cong (Y_v^{[1]})^{\Gamma(w)} \cdot Y_v^{\Gamma(v)}$. Noting that $(Y_v^{[1]})^{\Gamma(w)} \triangleleft G_{vw}^{\Gamma(w)}$, it is easily shown that Y_v does not have the form $N_v \rtimes X_v$ such that both N_v and X_v are transitive on $\Gamma(v)$, which is not possible. Thus $Y_{vw}^{[1]} \neq 1$, and by Theorem 2.2, $\text{soc}(Y_v^{\Gamma(v)}) = \text{PSL}(d, q)$ with $|\Gamma(v)| = \frac{q^d - 1}{q - 1}$. Since $Y_{vw}^{[1]}$ is a p -group and $(Y_v^{[1]})^{\Gamma(w)} \triangleleft G_{vw}^{\Gamma(w)}$, it follows that $|\Gamma(v)|^2$ does not divide $|Y_v|$, which is a contradiction. Thus, $\text{soc}(X_v) \leq N_v$.

Let $G = S_1 \times \dots \times S_k$, where $k \geq 2$ and $S_1 \cong \dots \cong S_k$ are non-Abelian simple groups. Then X_v acts by conjugation transitively on $\{S_1, \dots, S_k\}$, and $\text{soc}(X_v)$ is half-transitive on $\{S_1, \dots, S_k\}$, that is, all orbits of $\text{soc}(X_v)$ on $\{S_1, \dots, S_k\}$ have equal size.

Step 5. $k = l$, and for each $i \in \{1, 2, \dots, l\}$, $T_i \cap G = S_j$ for some $1 \leq j \leq k$.

Let $H = T_1 \times \dots \times T_{k-1}$. Since Z is of type PA, the proper normal subgroup H is intransitive on V . Let \mathcal{B} be the set of H -orbits on V , and let $B \in \mathcal{B}$ contain v . Then $N^{\mathcal{B}} \cong N/H \cong T$, and $G^{\mathcal{B}} \cong G/(G \cap H) \cong S^m$ where $m \geq 1$. In particular, $|S|$ divides $|T|$. For convenience, write $\bar{T} = N^{\mathcal{B}} \cong T$. Now $G^{\mathcal{B}}$ is a transitive subgroup of $N^{\mathcal{B}}$, so $T \cong \bar{T} = G^{\mathcal{B}}\bar{T}_B \cong S^m\bar{T}_B$.

Suppose that $m > 1$. By [2, Theorem 1.4], $\bar{T} \cong A_n$ and $\bar{T}_B \cong A_{n-1}$ with $n \geq 10$. Let r be the largest prime which is less than n . Then r exactly divides $|\bar{T}|$ and $|\bar{T}_B|$. So r does not divide $|S|$. Further, r^l exactly divides $|N|$, and so r^l divides $|N_v|$. If N_v is primitive on $\Gamma(v)$, then H is semiregular on V , and $N_v \cong \bar{T}_B$, which is a contradiction. Hence N_v is imprimitive on $\Gamma(v)$. By Lemma 2.1, N_v is soluble, and so $\bar{T}_B \cong N_v/H_v$ is soluble, which is again a contradiction.

Thus $m = 1$, and so $S^{k-1} < T^{l-1}$. Since $G \cap H \triangleleft G$, we may assume that $S_1 \times \dots \times S_{k-1} < H$. If $S_k \neq T_l$, then as $T_l \cap G \triangleleft G$ we have $T_l \cap G = 1$. Thus $S^k = G < N/T_k \cong T^{l-1}$, which is not possible. Hence $S_k < T_l$, and $T_l \cap G = S_k$. It follows that for each i , $T_i \cap G = S_j$ for some j , and $k = l$.

Step 6. *It is not possible for G not to be normal in $\text{Aut}\Gamma$.*

It follows from Step 5 that $|N_v| = |T : S|^k$. Since $\text{soc}(X_v) \leq N_v$, it follows that $\text{soc}(X_v)$ acts trivially on $\{S_1, \dots, S_k\}$, and so $\text{soc}(X_v)$ normalizes each S_j . Since $C_X(G) = 1$, we conclude that $\text{soc}(X_v) \leq \text{Out}(S_j)$. In particular, $\text{soc}(X_v)$ is soluble, by ‘Schreier’s Conjecture’, and so X_v is an affine 2-transitive permutation group on $\Gamma(v)$, say degree $|\Gamma(v)| = p^e$ with p prime. Inspecting outer-automorphism groups of simple groups, we conclude that either $p^e = 2^3$, or $e \leq 2$. It follows that $N_v^{\Gamma(v)} \geq \text{soc}(Y_v^{\Gamma(v)}) = \mathbb{Z}_p^e, A_{p^e}$, or $\text{PSL}(d, q)$ with either $e = 1$ and $\frac{q^d-1}{q-1} = p$ or $p^e = 2^3$. For the former two cases, $N_{vw}^{[1]} \triangleleft Y_{vw}^{[1]} = 1$, and hence $N_v \cong (N_v^{[1]})^{\Gamma(w)} \cdot N_v^{\Gamma(v)}$. It is now easily shown that $|N_v|$ is not a proper power of an integer, which is a contradiction. Thus $N_v^{\Gamma(v)} \geq \text{soc}(Y_v^{\Gamma(v)}) = \text{PSL}(d, q)$ or $\text{PSL}(2, 7)$; in particular, Γ is $(N, 2)$ -arc transitive. If $\text{soc}(Y_v^{\Gamma(v)}) = \text{PSL}(d, q)$, then $p = \frac{q^d-1}{q-1}$ divides $|N_v|$, but p^2 does not, so $|N_v|$ is not a proper power of an integer, which is a contradiction. Thus $\text{soc}(Y_v^{\Gamma(v)}) = \text{PSL}(2, 7)$. It follows that $|N_v|$ is not a proper power of an integer, which is again a contradiction.

Now we summarize the argument for proving Theorem 1.1 as follows.

Proof of Theorem 1.1: By Steps 1-6, we have that G is normal in $\text{Aut}\Gamma$. If $C_{\text{Aut}\Gamma}(G) \neq 1$, then by Lemma 3.1, part (i) of Theorem 1.1 holds. If $C_{\text{Aut}\Gamma}(G) = 1$, then G is the unique minimal normal subgroup of $\text{Aut}\Gamma$, as in part (ii) of Theorem 1.1. Finally, by [9, Proposition 2.3], Γ is not 3-arc transitive. □

4 Some related problems

Let Γ be an X -arc transitive graph with vertex set $V\Gamma$. Let $N \triangleleft X$ have at least three orbits on $V\Gamma$, and let \mathcal{B} be the set of N -orbits in $V\Gamma$. We define a graph Γ_N to have

vertex set \mathcal{B} such that $B, C \in \mathcal{B}$ are adjacent if and only if some $u \in B$ is adjacent in Γ to some $v \in C$, called the *normal quotient* of Γ induced by N . If Γ and Γ_N have equal valency, then Γ is a *normal cover* of Γ_N . An $(X, 2)$ -arc transitive graph Γ is always a normal cover of Γ_N , and choosing N to be maximal, each non-trivial normal subgroup of X/N is transitive on $V\Gamma_N$, that is, X/N is quasiprimitive on $V\Gamma_N$. This is the reduction for the study of 2-arc transitive graphs to the quasiprimitive case, given by Praeger [13].

A graph Γ with vertex set V is called a *Cayley graph* if there exist a group G and a subset $S \subset G$ with $S = S^{-1} = \{s^{-1} \mid s \in S\}$ such that V is identified with G and $x, y \in G$ are adjacent if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\text{Cay}(G, S)$.

A Cayley graph $\Gamma = \text{Cay}(G, S)$ has an automorphism group

$$\hat{G} = \{\hat{g} : x \mapsto xg \text{ for all } x \in G \mid g \in G\},$$

consisting of right multiplications of elements of G . The subgroup \hat{G} acts regularly on the vertex set of Γ ; in particular, Γ is vertex-transitive. Let

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\},$$

the subgroup of automorphisms of G and fix S setwise. It is easily shown that all elements of $\text{Aut}(G, S)$ are automorphisms of Γ fixing the vertex of Γ corresponding to the identity of G . If Γ is connected, or equivalently $\langle S \rangle = G$, then $\text{Aut}(G, S)$ acts faithfully on S . Further, $\text{Aut}(G, S)$ normalizes \hat{G} , and

$$\mathbf{N}_{\text{Aut}\Gamma}(\hat{G}) = \hat{G} \rtimes \text{Aut}(G, S) \leq \text{Aut}\Gamma.$$

Thus, if $\text{Aut}(G, S)$ is 2-transitive on S then Γ is $(\mathbf{N}_{\text{Aut}\Gamma}(\hat{G}), 2)$ -arc-transitive.

Much structural information of Γ is contained in the full automorphism group $\text{Aut}\Gamma$, such as the degree of symmetry of Γ , and the isomorphism class of Γ among Cayley graphs of G (refer to [10]). Generally, $\text{Aut}\Gamma$ is larger than $\hat{G} \rtimes \text{Aut}(G, S)$, see for example [4, 6, 11, 16]. Theorem 1.1 can be restated in the Cayley graph version.

Theorem 4.1 *Let G be a non-Abelian characteristically simple group, and let $\Gamma = \text{Cay}(G, S)$ be a connected Cayley graph. Assume that there exists a subgroup $X \leq \text{Aut}\Gamma$ such that \hat{G} is a unique minimal normal subgroup of X , and assume further that Γ is $(X, 2)$ -arc-transitive. Then $\text{Aut}\Gamma = \hat{G} \rtimes \text{Aut}(G, S)$, Γ is not 3-arc transitive, and further either*

- (i) Γ has valency 8, and $\text{Aut}\Gamma = (\hat{G} \times \mathbb{Z}_2^3) \rtimes \text{GL}(3, 2)$; or
- (ii) \hat{G} is the unique minimal normal subgroup of $\text{Aut}\Gamma$.

A natural question is whether the conclusion of Theorem 4.1 is always true for arbitrary groups G . Here is a counter-example.

Example 4.2 Let $G = N \rtimes \langle z \rangle \cong \mathbb{Z}_p^e \rtimes \mathbb{Z}_2$ with p odd prime and $e \geq 1$ such that $x^z = x^{-1}$ for all $x \in N$. Let $S = G \setminus N$, and let $\Gamma = \text{Cay}(G, S)$. Then S consists of all involutions of G which are conjugate, and $\Gamma = \mathbf{K}_{p^e, p^e}$, a complete bipartite

graph. It is easily shown that $\text{Aut}(G, S) = \text{Aut}(G) \cong \text{AGL}(e, p) = \mathbb{Z}_p^e \rtimes \text{GL}(e, p)$ and $\text{Aut}(G, S)$ acts 2-transitively on S . However, $\text{Aut}\Gamma \cong \text{S}_{p^e} \wr \text{S}_2$ is much bigger than $\mathbf{N}_{\text{Aut}\Gamma}(\hat{G}) = \hat{G} \rtimes \text{AGL}(e, p)$.

However, \mathbf{K}_{p^e, p^e} and their normal covers are the only known examples such that $\text{Aut}(G, S)$ is 2-transitive and \hat{G} is not normal in $\text{Aut}\Gamma$. This motivates the following conjecture:

Conjecture 4.3 For a connected Cayley graph $\Gamma = \text{Cay}(G, S)$, if $\text{Aut}(G, S)$ is 2-transitive on S , then either Γ is a normal cover of \mathbf{K}_{p^e, p^e} with p odd prime, or $\text{Aut}\Gamma = \hat{G} \rtimes \text{Aut}(G, S)$.

For a group G , a Cayley graph $\Gamma = \text{Cay}(G, S)$ is called *normal* if \hat{G} is normal in $\text{Aut}\Gamma$, see [18]. Then Conjecture 4.3 means that most Cayley graphs $\text{Cay}(G, S)$ are normal provided that $\text{Aut}(G, S)$ is 2-transitive on S .

More generally, we would like to propose several further problems regarding automorphism groups of Cayley graphs. We believe their solutions would be interesting for a better understanding of Cayley graphs.

Naturally, one would ask whether the condition in Conjecture 4.3 that $\text{Aut}(G, S)$ is 2-transitive on S can be weakened to the condition that $\text{Aut}(G, S)$ is only transitive on S , refer to [4, 11, 16, 18].

Problem 4.4 Characterize connected Cayley graphs $\Gamma = \text{Cay}(G, S)$ such that $\text{Aut}(G, S)$ is transitive on S and $\text{Aut}\Gamma \neq \hat{G} \rtimes \text{Aut}(G, S)$. Especially, do this for the case $\text{Aut}(G, S)$ is primitive on S .

By [16, Theorem 1], Problem 4.4 was solved for the case where G is non-Abelian simple and Γ is cubic, that is, it was shown that if G is a non-Abelian simple group and $\text{Aut}(G, S)$ is transitive on S with $|S| = 3$ then $\text{Aut}\Gamma = \hat{G} \rtimes \text{Aut}(G, S)$.

Example 4.5 For a Fermat prime $p = 2^d - 1$, the complete graph $\Gamma = \mathbf{K}_{2^d}$ of 2^d vertices is a Cayley graph $\text{Cay}(G, S)$ of $G = \mathbb{Z}_2^d$ such that $\text{Aut}(G, S) \cong \text{GL}(d, 2)$, which is primitive on S , and $\text{Aut}\Gamma = \text{Sym}(2^d)$. In particular, for $d \geq 3$, \hat{G} is not normal in $\text{Aut}\Gamma$.

Cayley graphs $\text{Cay}(G, S)$ with the property that $\text{Aut}(G, S)$ is transitive on S are called *normal arc-transitive Cayley graphs*. A study of such Cayley graphs was initiated in [16]. We wonder for a 2-arc-transitive Cayley graph $\text{Cay}(G, S)$ whether the transitivity of $\text{Aut}(G, S)$ on S implies the 2-transitivity:

Question 4.6 Does there exist a 2-arc-transitive Cayley graph $\text{Cay}(G, S)$ such that $\text{Aut}(G, S)$ is transitive but not 2-transitive on S ?

Cayley graphs defined above are undirected. Of course one may define directed Cayley graphs and ask similar questions regarding their automorphism groups to the undirected case. We are inclined to conjecture for a directed Cayley graph

$\Gamma = \text{Cay}(G, S)$, if $\text{Aut}(G, S)$ is 2-transitive on S , then $\text{Aut}(G, S)$ is also very close to $(\text{Aut}\Gamma)_v$ where v is the vertex of Γ corresponding to the identity of the group G .

Question 4.7 Let $\Gamma = \text{Cay}(G, S)$ be a connected directed Cayley graph, and assume further that $\text{Aut}(G, S)$ is 2-transitive on S . Does $\text{Aut}\Gamma$ equal $\hat{G} \rtimes \text{Aut}(G, S)$?

Praeger [14] gave a description of the bipartite 2-arc-transitive graphs. Let Γ be an $(X, 2)$ -arc-transitive bipartite graph with parts Δ and Δ' , where $X \leq \text{Aut}\Gamma$. Let $X^+ = X_\Delta = X_{\Delta'}$. The graph Γ is said to be X -bi-quasiprimitive if each non-trivial normal subgroup has at most two orbits and at least one has two orbits. It is shown in [14] that the bi-quasiprimitive case is an important case for understanding bipartite 2-arc-transitive graphs, and if X^+ is bi-quasiprimitive then X^+ is of type HA, AS, TW or PA. Motivated by Theorem 4.1, we propose

Conjecture 4.8 Let Γ be a connected $(X, 2)$ -arc-transitive bipartite graph with parts Δ and Δ' , where $X \leq \text{Aut}\Gamma$. Assume further that X^+ is quasiprimitive on Δ of type HA or type TW. Then either $\Gamma = \mathbf{K}_{p^e, p^e}$ with p prime, or $\text{soc}(X^+)$ is normal in $(\text{Aut}\Gamma)^+$.

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