J Algebr Comb (2009) 29: 261–268
DOI 10.1007/s10801-008-0136-1

A definition of the crystal commutor using Kashiwara's involution

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Received: 31 October 2006 / Accepted: 16 April 2008 / Published online: 29 April 2008

Abstract Henriques and Kamnitzer defined and studied a commutor for the category of crystals of a finite dimensional complex reductive Lie algebra. We show that the action of this commutor on highest weight elements can be expressed very simply using Kashiwara's involution on the Verma crystal.

Keywords Coboundary category · Crystals · Crystal commutor

1 Introduction

Let $\mathfrak{g}$ be a complex reductive Lie algebra. If A and B are crystals of representations of $\mathfrak{g}$, then $A \otimes B$ and $B \otimes A$ are isomorphic. However the map $(a, b) \mapsto (b, a)$ is not an isomorphism. In [1], following an idea of Berenstein, A. Henriques and the first author construct an explicit isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$, which they call the commutor. This commutor is involutive and satisfies the cactus relation, a certain axiom involving triple tensor products (see Section 5).

Consider the following alternative definition for a commutor. First notice that we only need to define $\sigma_{A,B}$ when A and B are irreducible. Also, a crystal isomorphism is uniquely defined by the images of highest weight elements. So, for each highest weight element $b_\lambda \otimes c \in B_\lambda \otimes B_\mu$, we need to specify its image $b_\mu \otimes b \in B_\mu \otimes B_\lambda$. We do this using Kashiwara's involution $*$ on B_∞. By the properties of $*$, if $b_\lambda \otimes c$

The second author was supported by the RTG grant DMS-0354321.

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is a highest weight element in $B_\lambda \otimes B_\mu$, then $*c \in B_\lambda$ (where we identify B_λ and B_μ with their images in B_∞), and $b_\mu \otimes *c$ is highest weight. Therefore we can define a crystal commutor by specifying that each highest weight element $b_\lambda \otimes c$ is taken to $b_\mu \otimes *c$. In this note we show that this definition gives the same commutor as that studied by Henriques and Kamnitzer.

The original definition of the commutor used the Schützenberger involution on each B_λ, while this definition uses Kashiwara's involution on B_∞. Thus one way to interpret our result is that it gives a non-trivial relationship between these two involutions. The Schützenberger involution does not exist for crystals of non-finite symmetrizable Kac-Moody Lie algebras, however Kashiwara's involution does. Hence this work extends the definition of the commutor to highest weight crystals of symmetrizable Kac-Moody Lie algebras.

2 Background

2.1 Notation and terminology

We include only a brief review of some basic facts about crystals. For the most part we follow the conventions from the review article [4], which we recommend for a more detailed overview of the subject.

- Let $\mathfrak{g}$ be a complex reductive Lie algebra.
- Let I denote the set of vertices of the Dynkin diagram of $\mathfrak{g}$.
- Let $\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in I}$ denote the positive roots and coroots of $\mathfrak{g}$.
- Let $\{s_i\}_{i\in I}$ denote the generators of the Weyl group.
- Let w_0 denote the long element of the Weyl group.
- Let $\langle\cdot,\cdot\rangle$ denote the pairing between weight space and coweight space.
- Let Λ denote the set of weights of $\mathfrak{g}$, $\{\Lambda_i\}_{i\in I}$ the set of fundamental weights, and Λ_+ the set of dominant weights.
- A crystal for $\mathfrak{g}$ is a finite set B along with maps $e_i, f_i : B \to B \sqcup 0$ for each $i \in I$, and a map $\mathrm{wt} : B \to \Lambda$, satisfying a certain set of axioms. These axioms may be found in [4, Sect. 7.2].
- For $\lambda \in \Lambda_+$, let B_λ denote the crystal corresponding to the irreducible representation V_λ of $\mathfrak{g}$.
- An element of a crystal is called highest (resp. lowest) weight if it is killed by all e_i (resp. all f_i). We use b_λ and b_λ^{low} to denote the unique highest and lowest weight elements of B_λ.
- For any crystal B and any $b \in B$, let $\varepsilon_i(b) = \max\{n : e_i^n(b) \neq 0\}$. Let $\varepsilon(b) \in \Lambda_+$ be the unique weight such that, for all $i \in I$, $\langle\varepsilon(b), \alpha_i^\vee\rangle = \varepsilon_i(b)$.
- Similarly, let $\varphi_i(b) = \max\{n : f_i^n(b) \neq 0\}$ and $\varphi(b) \in \Lambda_+$ be the unique weight such that, for all $i \in I$, $\langle\varphi(b), \alpha_i^\vee = \varphi_i(b)$.
- The weight of $b \in B$ is $\mathrm{wt}(b) = \varphi(b) - \varepsilon(b)$.
- There is a tensor product rule for crystals corresponding to the tensor product for representations of $\mathfrak{g}$. The underlying set of $A \otimes B$ is $A \times B$ (whose elements we

denote $a \otimes b$) and the actions of e_i and f_i are given by the following rules:

$$e_i(a \otimes b) = \begin{cases} e_i(a) \otimes b, & \text{if } \varphi_i(a) \geq \varepsilon_i(b) \\ a \otimes e_i(b), & \text{otherwise} \end{cases} \tag{1}$$

$$f_i(a \otimes b) = \begin{cases} f_i(a) \otimes b, & \text{if } \varphi_i(a) > \varepsilon_i(b) \\ a \otimes f_i(b), & \text{otherwise.} \end{cases} \tag{2}$$

2.2 Kashiwara's involution on B_∞

For any dominant weights λ and γ, there is an inclusion of crystals $B_{\lambda+\gamma} \to B_\gamma \otimes B_\lambda$ which sends $b_{\lambda+\gamma}$ to $b_\lambda \otimes b_\gamma$. The following is immediate from the tensor product rule:

Lemma 2.1 *The image of the inclusion $B_{\lambda+\gamma} \to B_\lambda \otimes B_\gamma$ contains all elements of the form $b \otimes b_\gamma$ for $b \in B_\lambda$.* □

Lemma 2.1 defines a map $\iota_\lambda^{\lambda+\gamma} : B_\lambda \to B_{\lambda+\gamma}$ which is e_i equivariant and takes b_λ to $b_{\lambda+\gamma}$. These maps make $\{B_\lambda\}$ into a directed system, and the limit of this system is B_∞. There are e_i equivariant maps $\iota_\lambda^\infty : B_\lambda \to B_\infty$. When there is no danger of confusion we denote ι_λ^∞ simply by ι.

The infinite set B_∞ has additional combinatorial structure. In particular, we will need:

(i) The map $\tau : B_\infty \to \Lambda_+$ defined by $\tau(b) = \min\{\lambda : b \in \iota(B_\lambda)\}$.
(ii) The map $\varepsilon : B_\infty \to \Lambda_+$ given by, for any $b \in B_\infty$ and any λ such that $b \in \iota(B_\lambda)$, $\varepsilon(b) = \varepsilon(\iota^{-1}(b))$, where ε is defined on B_λ as in Section 2.1.
(iii) Kashiwara's involution $*$ (for the construction of this involution see [3, Theorem 2.1.1]).

These maps are related by the following result of Kashiwara.

Proposition 2.2 [4, Proposition 8.2] *Kashiwara's involution preserves weights and satisfies*

$$\tau(*b) = \varepsilon(b), \quad \varepsilon(*b) = \tau(b).$$

Remark 2.3 All of this combinatorial structure can be seen easily using the MV polytope model [2]. The inclusions ι correspond to translating polytopes. The maps τ and ε are given by counting the lengths of edges coming out of the top and bottom vertices. The involution $*$ corresponds to negating a polytope. From this description, the proof of the above proposition is immediate.

2.3 The commutor

We now recall the definition of the commutor from [1, Sect. 2.2]. Let $\theta : I \to I$ be the involution such that $-w_0 \cdot \alpha_i = \alpha_{\theta(i)}$. Recall that each crystal B_λ comes with an involution ξ_λ which acts by w_0 on weights and exchanges the action of e_i and $f_{\theta(i)}$.

These involutions can be extended to a map $\xi_B : B \to B$ for any crystal B and they lead to the definition of the commutor for crystals. Namely,

$$\begin{aligned}\sigma_{B,C} &: B \otimes C \to C \otimes B \\ &\quad b \otimes c \mapsto \xi_{C\otimes B}(\xi_C(c) \otimes \xi_B(b)) = \mathrm{Flip} \circ \xi_B \otimes \xi_C(\xi_{B\otimes C}(b \otimes c)).\end{aligned} \tag{3}$$

The second expression here is just the inverse of the first expression, and the equality is proved in [1, Proposition 2].

3 Main theorem

A crystal isomorphism $B_\lambda \otimes B_\mu \to B_\mu \otimes B_\lambda$ is uniquely defined by the images of the highest weight elements in $B_\lambda \otimes B_\mu$. These are all of the form $b_\lambda \otimes c$, and must be sent to highest weight elements of $B_\mu \otimes B_\lambda$, which in turn are of the form $b_\mu \otimes b$. It follows from the tensor product rule for crystal that $b_\lambda \otimes c$ is highest weight in $B_\lambda \otimes B_\mu$ if and only if $\varepsilon(c) \le \lambda$.

As in Section 2.2, B_λ and B_μ embed in B_∞. Let $b_\lambda \otimes c$ be a highest weight element in $B_\lambda \otimes B_\mu$. Since $\varepsilon(c) \le \lambda$, by Proposition 2.2, $\tau(*c) \le \lambda$, or equivalently $*c \in \iota(B_\lambda)$. For this reason $*c$ can be considered an element of B_λ. Also $\tau(c) \le \mu$, which implies that $\varepsilon(*c) \le \mu$, and so $b_\mu \otimes *c$ is highest weight. So there is a unique isomorphism of crystals $B_\lambda \otimes B_\mu \to B_\mu \otimes B_\lambda$ which takes each highest weight element $b_\lambda \otimes c$ to $b_\mu \otimes *c$. The following shows that this isomorphism is equal to the crystal commutor.

Theorem 3.1 *If $b_\lambda \otimes c$ is a highest weight element in $B_\lambda \otimes B_\mu$, then $\sigma_{B_\lambda, B_\mu}(b_\lambda \otimes c) = b_\mu \otimes *c$.*

4 Proof

One of the main tools we will need is the notion of Kashiwara data (also called string data), first studied by Kashiwara (see for example [4] Section 8.2). Fix a reduced word $\mathbf{i}$ for w_0, by which we mean $\mathbf{i} = (i_1, \dots, i_m)$, where each i_k is a node of the Dynkin diagram, and $w_0 = s_{i_1} \cdots s_{i_m}$. The downward Kashiwara data for $b \in B_\lambda$ with respect to $\mathbf{i}$ is the sequence of non-negative integers $(p_1, \dots p_m)$ defined by

$$p_1 := \varphi_{i_1}(b), \quad p_2 := \varphi_{i_2}(f_{i_1}^{p_1} b), \quad \dots, \quad p_m := \varphi_{i_m}(f_{i_{m-1}}^{p-1} \dots f_{i_1}^{p_1} b).$$

That is, we apply the lowering operators in the direction of i_1 as far as we can, then in the direction i_2, and so on. The following result is due to Littelmann [5, Section 1].

Lemma 4.1 *After we apply these steps, we reach the lowest element of the crystal. That is:*

$$f_{i_m}^{p_m} \dots f_{i_1}^{p_1} b = b_\lambda^{low}.$$

Moreover, the map $B_\lambda \to \mathbb{N}^m$ taking $b \to (p_1, \dots p_m)$ is injective.

Similarly, the upwards Kashiwara data for $b \in B_\lambda$ with respect to $\mathbf{i}$ is the sequence $(q_1, \ldots q_m)$ defined by

$$q_1 := \varepsilon_{i_1}(b), \quad q_2 := \varepsilon_{i_2}(e_{i_1}^{q_1} b), \quad \ldots, \quad q_m := \varepsilon_{i_m}(e_{i_{m-1}}^{q-1} \ldots e_{i_1}^{q_1} b).$$

We introduce the notation $w_k^{\mathbf{i}} := s_{i_1} \cdots s_{i_k}$.

Lemma 4.2 *In the crystal B_λ, we have the following*:

(i) *The downward Kashiwara data for b_λ is given by $p_k = \langle w_{k-1}^{\mathbf{i}} \cdot \alpha_{i_k}^\vee, \lambda \rangle$.*
(ii) *For each k, $\varepsilon_{i_k}(f_{i_{k-1}}^{p_{k-1}} \ldots f_{i_1}^{p_1} b_\lambda) = 0$.*

Proof Let $(p_1, \ldots p_m)$ be the downwards Kashiwara data for b_λ, and let μ_k be the weight of $f_{i_k}^{p_k} \ldots f_{i_1}^{p_1} b_\lambda$. Since $f_{i_k}^{p_k} \ldots f_{i_1}^{p_1} b_\lambda$ is the end of an α_{i_k} root string, we see that

$$\mu_k = s_{i_k} \cdot \mu_{k-1} - a_k \alpha_{i_k}, \tag{4}$$

where $a_k = \varepsilon_{i_k}(f_{i_{k-1}}^{p_{k-1}} \ldots f_{i_1}^{p_1} b_\lambda)$. Using this fact at each step,

$$\mu_m = w_0 \cdot \lambda - \sum_{k=1}^{m} a_k s_{i_m} \ldots s_{i_{k+1}} \cdot \alpha_{i_k}.$$

By Lemma 4.1, we know that $f_{i_m}^{p_m} \ldots f_{i_1}^{p_1} b_\lambda = b_\lambda^{low}$, so that $\mu_m = w_0 \cdot \lambda$. Hence

$$\sum_{k=1}^{m} a_k s_{i_m} \ldots s_{i_{k+1}} \cdot \alpha_{i_k} = 0.$$

Now, $s_{i_m} \cdots s_{i_k}$ is a reduced word for each k, which implies that $s_{i_m} \ldots s_{i_{k+1}} \alpha_{i_k}$ is a positive root. Thus each a_k is zero, proving part (ii).

Equation (4) now shows that $\mu_k = s_{i_k} \ldots s_{i_1} \cdot \lambda$, for all k. In particular that $f_{i_k}^{p_k}$ must perform the reflection s_{i_k} on the weight $s_{i_{k-1}} \ldots s_{i_1} \cdot \lambda$. Therefore,

$$p_k = \langle \alpha_{i_k}^\vee, s_{i_{k-1}} \ldots s_{i_1} \cdot \lambda \rangle = \langle w_{k-1}^{\mathbf{i}} \alpha_{i_k}^\vee, \lambda \rangle.$$

□

Lemma 4.3 *Let $b_\lambda \otimes c$ be a highest weight element of $B_\lambda \otimes B_\mu$. Let $b \otimes b_\mu^{low}$ be the lowest weight element of the component containing $b_\lambda \otimes c$. Let $(p_1, \ldots p_m)$ be the downward Kashiwara data for c with respect to $\mathbf{i}$, and $(q_1, \ldots q_m)$ the upward Kashiwara data for b with respect to $\mathbf{i}^{\text{rev}} := (i_m, \ldots, i_1)$. Then, for all k, $p_k + q_{m-k+1} = \langle w_{k-1}^{\mathbf{i}} \cdot \alpha_{i_k}^\vee, \nu \rangle$, where $\nu = \text{wt}(b_\lambda \otimes c)$.*

Proof Let $r_k = \langle w_{k-1}^{\mathbf{i}} \cdot \alpha_{i_k}^\vee, \nu \rangle$. By part (i) of Lemma 4.2, $(r_1, \ldots r_m)$ is the downward Kashiwara data for $b_\lambda \otimes c$. Define $b_k \in B_\lambda$ and $c_k \in B_\mu$ by $b_k \otimes c_k = f_{i_k}^{r_k} \ldots$

$f_{i_1}^{r_1}(b_\lambda \otimes c)$. Part (ii) of Lemma 4.2, along with the definition of Kashiwara data, shows that, for each $1 \le k \le m$,

$$e_{i_k}(b_{k-1} \otimes c_{k-1}) = 0 \quad \text{and} \quad f_{i_k}(b_k \otimes c_k) = 0.$$

In particular, the tensor product rule for crystals implies

$$e_{i_k} b_{k-1} = 0 \quad \text{and} \quad f_{i_k} c_k = 0.$$

Define p_k to be the number of times f_{i_k} acts on c_{k-1} to go from $b_{k-1} \otimes c_{k-1}$ to $b_k \otimes c_k$, and q_{m-k+1} to be the number of times f_{i_k} acts on b_{k-1}. Since $f_{i_k} c_k = 0$, we see that $\varphi_{i_k}(c_{k-1}) = p_k$. Hence, by definition $(p_1, \dots p_m)$ is the downward Kashiwara data for c with respect to $\mathbf{i}$. Similarly, $e_{i_k} b_{k-1} = 0$, so $\varepsilon_{i_k}(c_k) = q_{m-k+1}$. By Lemma 4.1, $b_m = b$, so this implies that $(q_1, \dots q_m)$ is the upward Kashiwara data for b with respect to $\mathbf{i}^{\text{rev}}$. Since $p_k + q_{m-k+1}$ is the number of times that f_{i_k} acts on $b_{k-1} \otimes c_{k-1}$ to reach $b_k \otimes c_k$, we see that $p_k + q_{m-k+1} = r_k$. □

Let $b_\lambda \otimes c$ be a highest weight element in $B_\lambda \otimes B_\mu$. As discussed in Sect. 3.1, $*c$ can be considered as an element of B_λ.

Lemma 4.4 *Define $\nu = wt(b_\lambda \otimes c)$. Let $(p_1, \dots p_m)$ be the downward Kashiwara data for $c \in B_\mu$ with respect to $\mathbf{i}$. Let $(q_1, \dots q_m)$ be the downward Kashiwara data for $*c \in B_\lambda$ with respect to the decomposition $\theta(\mathbf{i}^{\text{rev}}) := (\theta(i_m), \dots, \theta(i_1))$ of w_0. Then, for all k, $p_k + q_{m-k+1} = \langle w_{k-1}^{\mathbf{i}} \cdot \alpha_{i_k}^\vee, \nu\rangle$.*

Proof The proof will depend on results from [2] on the MV polytope model for crystals. In particular, within this model it is easy to express Kashiwara data and the Kashiwara involution.

Let $P = P(M_\bullet)$ be the MV polytope of weight $(\nu - \lambda, \mu)$ corresponding to c. Then by Theorem 6.6 of [2],

$$p_k = M_{w_{k-1}^{\mathbf{i}} \cdot \Lambda_{i_k}} - M_{w_k^{\mathbf{i}} \cdot \Lambda_{i_k}}.$$

Now, consider P as a stable MV polytope (recall that this means that we only consider it up to translation). Then by Theorem 6.2 of [2], we see that $*(P) = -P$.

The element $\iota_\mu^{-1} * \iota_\lambda(c) \in B_\lambda$ corresponds to the MV polytope $\nu - P$ and hence has BZ datum $N_\bullet$, where $M_\bullet$ and $N_\bullet$ are related by

$$M_\gamma = \langle \gamma, \nu\rangle + N_{-\gamma}.$$

Let $\mathbf{i}' = \theta(\mathbf{i}^{\text{rev}})$. Then,

$$-w_{k-1}^{\mathbf{i}} \cdot \Lambda_{i_k} = w_{m-k+1}^{\mathbf{i}'} \cdot \Lambda_{i'_{m-k+1}} \quad \text{and} \quad -w_k^{\mathbf{i}} \cdot \Lambda_{i_k} = w_{m-k}^{\mathbf{i}'} \cdot \Lambda_{i'_{m-k+1}}.$$

Combining the last 3 equations, we see that

$$\begin{aligned} p_k &= \langle w_{k-1}^{\mathbf{i}} \cdot \Lambda_{i_k}, \nu\rangle + N_{-w_{k-1}^{\mathbf{i}} \cdot \Lambda_{i_k}} - \langle w_k^{\mathbf{i}} \cdot \Lambda_{i_k}, \nu\rangle - N_{-w_k^{\mathbf{i}} \cdot \Lambda_{i_k}} \\ &= N_{w_{m-k+1}^{\mathbf{i}'} \cdot \Lambda_{i'_{m-k+1}}} - N_{w_{m-k}^{\mathbf{i}'} \cdot \Lambda_{i'_{m-k+1}}} + \langle w_{k-1}^{\mathbf{i}} \cdot \Lambda_{i_k} - w_k^{\mathbf{i}} \cdot \Lambda_{i_k}, \nu\rangle. \end{aligned} \tag{5}$$

Applying Theorem 6.6 of [2] again,

$$q_k = N_{w^{\mathbf{i}'}_{k-1}\cdot\Lambda_{i'_k}} - N_{w^{\mathbf{i}'}_{k}\cdot\Lambda_{i'_k}}. \tag{6}$$

We now add equation (5) and (6), substituting $m-k+1$ for k in the second equation, to get

$$\begin{aligned} p_k + q_{m-k+1} &= \langle w^{\mathbf{i}}_{k-1}\cdot\Lambda_{i_k} - w^{\mathbf{i}}_{k}\cdot\Lambda_{i_k}, \nu\rangle \\ &= \langle w^{\mathbf{i}}_{k-1}\cdot(\Lambda_{i_k} - s_{i_k}\cdot\Lambda_{i_k}), \nu\rangle = \langle w^{\mathbf{i}}_{k-1}\cdot\alpha^\vee_{i_k}, \nu\rangle. \end{aligned}$$ □

Proof of Theorem 3.1 We know that $\sigma_{B_\lambda,B_\mu}(b_\lambda\otimes c) = b_\mu\otimes b$ for some $b\in B_\lambda$. By the definition of σ_{B_λ,B_μ} (see Section 2.3), we see that

$$\xi(b_\lambda\otimes c) = (\xi\circ\xi)(\mathrm{Flip}(\sigma(b_\lambda\otimes c))) = \xi(b)\otimes b_\mu^{low}.$$

In particular, $\xi(b)\otimes b_\mu^{low}$ is the lowest weight element of the component of $B_\lambda\otimes B_\mu$ containing $b_\lambda\otimes c$.

Fix a reduced expression $\mathbf{i} = (i_1,\dots,i_m)$ for w_0. Let $(p_1,\dots,p_m)$ be the downward Kashiwara data for c with respect to $\mathbf{i}$, and let $(q_1,\dots q_m)$ be the downward Kashiwara data for b with respect to $\theta(\mathbf{i}^{\mathrm{rev}}) := (\theta(i_m),\cdots,\theta(i_1))$. Notice that $(q_1,\dots q_m)$ is also the upward Kashiwara data for $\xi(b)$ with respect to $\mathbf{i}^{\mathrm{rev}} := (i_m,\dots i_1)$, since ξ interchanges the action of f_i and $e_{\theta(i)}$. Hence by Lemma 4.3,

$$p_k + q_{m-k+1} = \langle w^{\mathbf{i}}_{k-1}\cdot\alpha^\vee_{i_k}, \nu\rangle \tag{7}$$

for all k, where ν is the weight of $b_\lambda\otimes c$.

As discussed in Section 3, $*c\in\iota(B_\mu)$, and so can be considered as an element of B_μ. Let $(q'_1,\dots q'_m)$ by the downward Kashiwara data for $*c\in B_\mu$ with respect to $\theta(\mathbf{i}^{\mathrm{rev}})$. By Lemma 4.4 we have

$$p_k + q'_{m-k+1} = \langle w^{\mathbf{i}}_{k-1}\cdot\alpha^\vee_{i_k}, \nu\rangle. \tag{8}$$

Comparing equations (7) and (8), $q_k = q'_k$ for each $1\le k\le m$. That is, the downward Kashiwara data for b and $*c$ with respect to $\theta(\mathbf{i}^{\mathrm{rev}})$ are identical. Hence by Lemma 4.1, $b = *c$. □

5 Questions

The involution $*$ gives B_∞ an additional crystal structure, defined by $f_i^*\cdot b := *\circ f_i\circ *(b)$. Let B^i_∞ denote the crystal with vertex set $\mathbb{Z}_{\ge 0}$, where e_j, f_j act trivially for $j\ne i$ and e_i, f_i act as they do on the usual B_∞ for $\mathfrak{sl}_2$. Kashiwara [3, Theorem 2.2.1] showed that the map

$$\begin{aligned} B_\infty &\to B_\infty\otimes B^i_\infty \\ b &\mapsto (e_i^*)^{\varepsilon_i(*b)}(b)\otimes\varepsilon_i(*b) \end{aligned}$$

is a morphism of crystals with respect to the usual crystal structures on each side. We can think of this fact as an additional property of $*$.

On the other hand, the commutor σ also has an additional property, which is called the cactus relation. This relation states that if A, B, C are crystals, then

$$\sigma_{A,C\otimes B}\circ(1\otimes\sigma_{B,C})=\sigma_{B\otimes A,C}\circ(\sigma_{A,B}\otimes 1).$$

(See [1, Theorem 3]).

Question 1 *Is there a relation between this additional property of Kashiwara's involution $*$ and the cactus relation for the commutor σ?*

Another direction is to consider the generalization beyond finite dimensional reductive Lie algebras. We can define a crystal commutor for the crystals of highest weight representations of any symmetrizable Kac-Moody algebra by $\sigma(b_\lambda\otimes c)=b_\mu\otimes *c$ whenever $b_\lambda\otimes c$ is a highest weight element. This will be well defined by the analysis given in Section 3.

Question 2 *Does this commutor satisfy the cactus relation?*[1]

Acknowledgements We thank A. Berenstein, M. Haiman, and M. Vazriani for useful discussions and the referee for helpful comments. The first author thanks the American Institute of Mathematics for support and UC Berkeley for hospitality.

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[1] This question has recently been answered in the affirmative by Savage [6, Theorem 6.4].