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Tetravalent one-regular graphs of order $2pq$

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Abstract A graph is one-regular if its automorphism group acts regularly on the set of its arcs. In this article a complete classification of tetravalent one-regular graphs of order twice a product of two primes is given. It follows from this classification that with the exception of four graphs of orders 12 and 30, all such graphs are Cayley graphs on Abelian, dihedral, or generalized dihedral groups.

Keywords One-regular graph · Symmetric graph · Cayley graph

1 Introduction

For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X, and by C_n and K_n the cycle and the complete graph of order n, respectively. A graph X is said to be *vertex-transitive* and *arc-transitive* (or *symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$ and $A(X)$, respectively. In particular, if $\mathrm{Aut}(X)$ acts regularly on $A(X)$, then X is said to be *one-regular*.

A one-regular graph with each vertex having the same valency must be connected, and a graph of valency 2 is one-regular if and only if it is a cycle. The first example of cubic one-regular graph was constructed by Frucht [10] with 432 vertices, and much subsequent work was done in this line as part of a more general problem dealing with the investigation of cubic arc-transitive graphs (see [4–9, 26]). Tetravalent one-regular graphs have also received considerable attention. Chao [2] classified all tetravalent one-regular graphs of prime order, and Marušič [23] constructed an infinite family

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of tetravalent one-regular Cayley graphs on alternating groups. All tetravalent one-regular circulant graphs were classified in [37], and all tetravalent one-regular Cayley graphs on Abelian groups were classified in [36]. One may deduce a classification of tetravalent one-regular Cayley graphs on dihedral groups from Kwak and Oh [16] and Wang et al. [32, 33]. Malnič et al. [21] constructed an infinite family of infinite one-regular graphs, which steps into the important territory of symmetry in infinite graphs; see also [19, 31] for some more results related to this topic. Let p and q be primes. Clearly, every tetravalent one-regular graph of order p is a circulant graph. For a tetravalent one-regular graph X of order pq, if $|V(X)| = 2p$, then X is a circulant graph by [3] or [22]; if $p = q$, then X is a Cayley graph on an Abelian group of order p^2 (clearly $p \geq 3$ and any Sylow p-subgroup of $\mathrm{Aut}(X)$ has order p^2 which is regular on $V(X)$), and hence circulant by [36]; if $p > q > 2$, then X is not vertex-primitive by [29, 34], and then circulant by [27, 34], which can also be deduced from [24]. It follows that all tetravalent one-regular graphs of order p or pq are circulant, and a classification of such graphs can be easily deduced from [37]. In this paper we classify all tetravalent one-regular graphs of order $2pq$. It follows from the classification that with the exception of four graphs of orders 12 and 30, all such graphs are Cayley graphs on Abelian, or dihedral, or generalized dihedral groups. For more results on tetravalent symmetric graphs, see [11, 12, 17, 28].

We now introduce Cayley graph and coset graph. Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if G is transitive and semiregular on Ω. For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, for $x \in G$. Then, the homomorphism taking g to $R(g)$, for $g \in G$, is called the *right regular representation* of G, under which the image $R(G) = \{R(g) \mid g \in G\}$ of G is a regular permutation group on G. It is easy to see that $R(G)$ is isomorphic to G, which can therefore be regarded as a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Thus the Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Xu [38, Proposition 1.5] proved that $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$. A graph is called a *circulant graph*, in short, a *circulant* if it is a Cayley graph on a cyclic group.

Let X be a symmetric graph, and A an arc-transitive subgroup of $\mathrm{Aut}(X)$. Let $\{u, v\}$ be an edge of X. Assume that $H = A_u$ is the stabilizer of $u \in V(X)$ and that $g \in A$ interchanges u and v. It is easy to see that the core H_A of H in A (the largest normal subgroup of A contained in H) is trivial, and that HgH consists of all elements of A which maps u to one of its neighbors in X. By [18, 30], the graph X is isomorphic to the *coset graph* $X^* = \mathrm{Cos}(A, H, HgH)$, which is defined as the graph with vertex set $V(X^*) = \{Ha : a \in A\}$, the set of right cosets of H in A, and edge set $E(X^*) = \{\{Ha, Hda\} \bigm| a \in A, d \in HgH\}$. The valency of X^* is $|HgH|/|H| = |H : H \cap H^g|$, and X^* is connected if and only if HgH generates A. By right multiplication, every element in A induces an automorphism of X^*. Since $H_A = 1$, the induced action of

A on $V(X^*)$ is faithful, and hence one may view A as a group of automorphisms of X^*.

Throughout this paper we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, and by D_{2n} the dihedral group of order $2n$, respectively. For two groups M and N, $N \leq M$ means that N is a subgroup of M, and $N < M$ means that N is a proper subgroup of M.

2 Preliminaries

In this section, we introduce some preliminary results. The first is about transitive Abelian permutation groups.

Proposition 2.1 [35, Proposition 4.4] *Every transitive Abelian group G on a set Ω is regular.*

For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. The following proposition is due to Burnside.

Proposition 2.2 [15, Chapter IV, Theorem 2.6] *Let G be a finite group and P a Sylow p-subgroup of G. If $N_G(P) = C_G(P)$, then G has a normal subgroup N such that $G = NP$ with $N \cap P = 1$.*

Kwak and Oh [16, Theorem 3.1] classified tetravalent one-regular normal Cayley graphs on dihedral groups with a cyclic vertex stabilizer.

Proposition 2.3 *A tetravalent Cayley graph X on a dihedral group is one-regular and normal with cyclic vertex stabilizer if and only if X is isomorphic to* $\mathrm{Cay}(D_{2n}, \{b, ab, a^{\ell+1}b, a^{\ell^2+\ell+1}b\})$ *for some pair (n, ℓ) such that $\ell^3 + \ell^2 + \ell + 1 \equiv 0 (\mathrm{mod}\, n)$, $n \geq 10$, $\ell^2 - 1 \neq 0 (\mathrm{mod}\, n)$ and $(n, \ell) \neq (15, 2), (15, 8)$, where $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$.*

The following proposition can be extracted from Xu [37, Theorems 2 and 3], where tetravalent one-regular circulant graphs were classified.

Proposition 2.4 *Let p and q be primes and $G = \langle a \rangle \cong \mathbb{Z}_{2pq}$. A tetravalent Cayley graph $X = \mathrm{Cay}(G, S)$ on G is one-regular if and only if $p, q > 2$ and $S = a^H = \{a, a^{h_1}, a^{h_2}, a^{h_3}\}$ where $H = \{1, h_1, h_2, h_3\}$ is a subgroup of order 4 of $\mathbb{Z}_{2pq}^*$ such that $-1 \in H$. Furthermore, distinct subgroups of order 4 containing -1 give nonisomorphic one-regular graphs.*

The next proposition is a special case of [36, Theorem 3.5].

Proposition 2.5 *Let p be a prime and $G \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$. Then there exists a tetravalent one-regular Cayley graph on G if and only if $p - 1$ is a multiple of 4 and in this case, there are exactly two nonisomorphic tetravalent one-regular Cayley graphs on G.*

Let X be a connected symmetric graph, and let $G \leq \mathrm{Aut}(X)$ be arc-transitive on X. For a normal subgroup N of G, the *quotient graph* X_N of X relative to the set of orbits of N is defined as the graph with vertices the orbits of N in $V(X)$, and with two orbits adjacent if there is an edge in X between those two orbits. Let X be a tetravalent connected symmetric graph, and N an elementary Abelian p-group. A classification of tetravalent connected symmetric graphs was obtained for the case when N has at most two orbits in [11], and a characterization of such graphs was given for the case when X_N is a cycle in [12]. The following is a 'reduction' theorem.

Proposition 2.6 [11, Theorem 1.1] *Let X be a tetravalent connected symmetric graph, and let $G \leq \mathrm{Aut}(X)$ act arc-transitively on X. For each normal subgroup N of G, one of the following holds*:

(1) *N is transitive on $V(X)$*;
(2) *X is bipartite and N acts transitively on each part of the bipartition*;
(3) *N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and G induces the full automorphism group D_{2r} on X_N*;
(4) *N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-symmetric graph and X is a topological cover of X_N.*

3 Examples

In this section, we introduce some tetravalent one-regular graphs of order $2pq$, where p, q are primes. The first example is the graph $\mathcal{CG}_{2p^2}$ of order $2p^2$ $(p = q)$, which was defined by Gardiner and Praeger [11, Definition 4.3].

Example 3.1 Let p be a prime congruent to 1 mod 4, and let $\pm\varepsilon$ be the two elements of order 4 in $\mathbb{Z}_p^*$. The graph $\mathcal{CG}_{2p^2}$ is defined to have vertex set $\mathbb{Z}_2 \times (\mathbb{Z}_p \times \mathbb{Z}_p)$ with two vertices $(0, (x_1, y_1))$ and $(1, (x_2, y_2))$ being adjacent if and only if

$$(x_2, y_2) - (x_1, y_1) \in \{(1, 1), (-1, \varepsilon), (1, -1), (-1, -\varepsilon)\}.$$

Furthermore, the tetravalent graph $\mathcal{CG}_{2p^2}$ is one-regular.

Clearly, $\mathcal{CG}_{2p^2}$ is independent of the choice of ε. A group of automorphisms of $\mathcal{CG}_{2p^2}$ was given in [11, Definition 4.3] that is arc-transitive on $\mathcal{CG}_{2p^2}$, but the full automorphism group of $\mathcal{CG}_{2p^2}$ was not obtained there. We shall compute the full automorphism group of $\mathcal{CG}_{2p^2}$ in the following lemma, showing that $\mathcal{CG}_{2p^2}$ is one-regular.

Lemma 3.2 *Let $G(2p^2) = \langle a, b, c \mid a^p = b^p = c^2 = 1, cac = a^{-1}, cbc = b^{-1}, ab = ba\rangle$, the so called generalized dihedral group of order $2p^2$. Let p be congruent to 1 mod 4, and let ε, $-\varepsilon$ be the two elements of order 4 in $\mathbb{Z}_p^*$. Set $S = \{cab, ca^{-1}b^{\varepsilon}, cab^{-1}, ca^{-1}b^{-\varepsilon}\}$. Then $\mathcal{CG}_{2p^2} \cong \mathrm{Cay}(G(2p^2), S)$. Furthermore, $\mathcal{CG}_{2p^2}$ is one-regular and $\mathrm{Aut}(\mathcal{CG}_{2p^2}) \cong G(2p^2) \rtimes \mathbb{Z}_4$.*

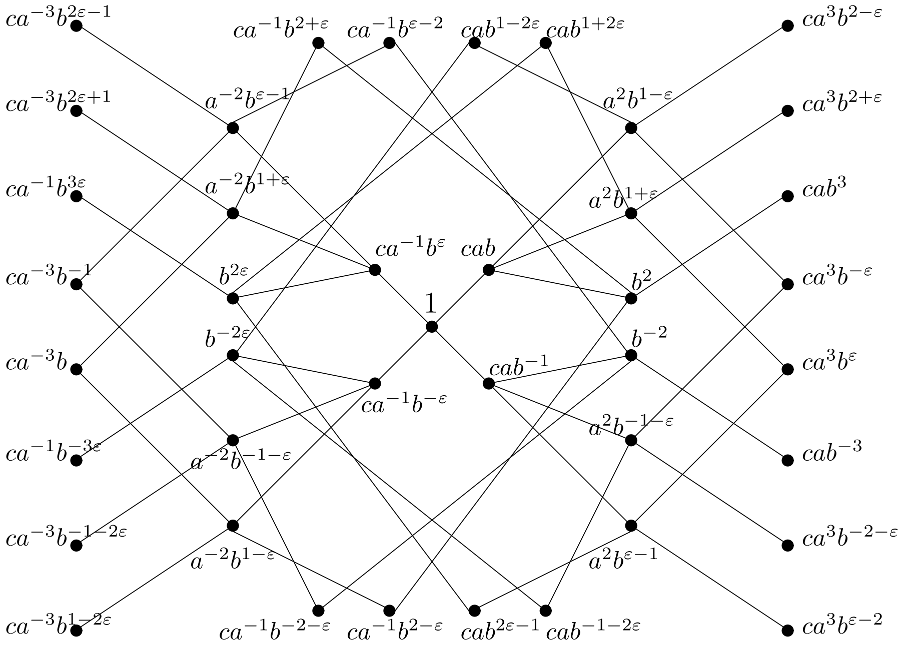

Fig. 1 An induced subgraph of $\mathcal{CG}_{2p^2}$

Proof Let $G = G(2p^2)$ and $X = \mathrm{Cay}(G, S)$. Note that the graph $\mathcal{CG}_{2p^2}$ has vertex set $\mathbb{Z}_2 \times (\mathbb{Z}_p \times \mathbb{Z}_p)$ with two vertices $(0, (x_1, y_1))$ and $(1, (x_2, y_2))$ adjacent if and only if $(x_2, y_2) - (x_1, y_1) \in \{(1, 1), (-1, \varepsilon), (1, -1), (-1, -\varepsilon)\}$. It is easy to see that the map defined by $(i, (x, y)) \mapsto c^i a^x b^y$, $(i, (x, y)) \in \mathbb{Z}_2 \times (\mathbb{Z}_p \times \mathbb{Z}_p)$, is an isomorphism from $\mathcal{CG}_{2p^2}$ to X. Let $A = \mathrm{Aut}(X)$, and let A_1 be the stabilizer of 1 in A. Since S generates G, X is connected. The map $c \mapsto c$, $a \mapsto a^{-1}$ and $b \mapsto b^{\varepsilon}$ induces an automorphism of G, denoted by α, which cyclically permutes the elements in S. This means $R(G) \rtimes \langle \alpha \rangle \leq A$, so that X is symmetric. To finish the proof, it suffices to show that $A = R(G) \rtimes \langle \alpha \rangle$.

For the case $p = 5$, it can be shown with the help of the computer software package MAGMA [1] that $\mathrm{Aut}(X)$ has order 200, which implies that $A = R(G) \rtimes \langle \alpha \rangle$.

Now assume $p > 5$. We depict the subgraph of X induced by the vertices at distance less than 4 from 1 (see Fig. 1). Let A_1^* be the subgroup of A_1 fixing S pointwise. From Fig. 1 one may see that passing through cab^{-1}, 1 and cab, there is a 6-cycle passing through $a^2b^{1-\varepsilon}$ and another 6-cycle passing through $a^2b^{1+\varepsilon}$, but no 6-cycle passing through b^2. This implies that A_1^* fixes b^2, and similarly A_1^* fixes b^{-2}, $b^{2\varepsilon}$ and $b^{-2\varepsilon}$. Note that $b^{2\varepsilon}$ and $a^2b^{1+\varepsilon}$ have a common neighbor, but $b^{2\varepsilon}$ and $a^2b^{1-\varepsilon}$ have no common neighbor. It follows that A_1^* fixes $a^2b^{1-\varepsilon}$ and $a^2b^{1+\varepsilon}$. Thus, one may show that A_1^* fixes every vertex at distance 2 from 1 in X. By connectivity of X and transitivity of A on $V(X)$, we find that A_1^* fixes every vertex in X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$ and $|A| \leq 48p^2$.

Set $P = \langle R(a), R(b)\rangle$. Then P is a Sylow p-subgroup of A. Since $\alpha \in \mathrm{Aut}(G(2p^2))$, one has $x^{\alpha^{-1}R(a^ib^j)\alpha} = (x^{\alpha^{-1}}a^ib^j)^\alpha = xa^{i\alpha}b^{j\alpha} = x^{R(a^{-i}b^{j\varepsilon})}$ for any $x \in G$. Thus, $\alpha^{-1}R(a^ib^j)\alpha \in P$, that is, α normalizes P. Similarly, one may show that $\alpha R(c) = R(c)\alpha$. Since $R(c)$ normalizes P, $|N_A(P)|$ is divisible by $8p^2$. By Sylow theorem, the number of Sylow p-subgroups is $kp+1 = |A : N_A(P)|$ and is a divisor of 6. This forces $k = 0$ because $p > 5$ and hence P is normal in A.

Suppose that $A_1 \neq \langle\alpha\rangle$. Take $\beta \in A_1\setminus\langle\alpha\rangle$. Since $\langle\alpha\rangle \cong \mathbb{Z}_4$ and $A_1 \cong A_1^S \leq S_4$, one has $\langle\alpha, \beta\rangle \cong D_8$ or S_4. Since α permutes the elements in $S = \{cab, ca^{-1}b^\varepsilon, cab^{-1}, ca^{-1}b^{-\varepsilon}\}$ cyclicly, there exists an involution $\gamma \in A_1\setminus\langle\alpha\rangle$ such that γ fixes cab and cab^{-1} and interchanges $ca^{-1}b^\varepsilon$ and $ca^{-1}b^{-\varepsilon}$. Note that there is no 6-cycle passing through cab^{-1}, 1, cab and b^2. Then γ fixes b^2. On the other hand, since γ normalizes P, for any $x, y \in \langle a, b\rangle$ one has $(xy)^\gamma = 1^{R(xy)\gamma} = 1^{\gamma^{-1}(R(x)R(y))^\gamma} = 1^{R(x)^\gamma R(y)^\gamma} = R(x)^\gamma R(y)^\gamma = 1^{R(x)^\gamma}1^{R(y)^\gamma} = x^\gamma y^\gamma$, that is, γ induces an automorphism on $\langle a, b\rangle$. Thus, γ fixes $\langle b^2\rangle$ pointwise. In particular, γ fixes $b^{2\varepsilon}$. Since $ca^{-1}b^\varepsilon$ is the unique common neighbor of 1 and $b^{2\varepsilon}$ in X, γ fixes $ca^{-1}b^\varepsilon$, a contradiction. This means that $A_1 = \langle\alpha\rangle$, that is, $A = R(G(2p^2)) \rtimes \langle\alpha\rangle$. □

Xu [37, Theorems 2 and 3] classified all tetravalent one-regular circulant graphs, and one may deduce the following example.

Example 3.3 Let p and q be odd primes. Assume $p = q$. Then $\mathbb{Z}_{2pq}^*$ has a unique subgroup of order 4, say H_0, when $p \equiv 1 (\mathrm{mod}\ 4)$. Assume $p \neq q$. Then $\mathbb{Z}_{2pq}^*$ has a unique non-cyclic subgroup of order 4, say H_1, and if both $p-1$ and $q-1$ are divisible by 4, then $\mathbb{Z}_{2pq}^*$ has exactly two cyclic subgroups of order 4 containing -1, say H_2 and H_3. Let $G = \langle a\rangle \cong \mathbb{Z}_{2pq}$ and define $\mathcal{CC}_{2pq}^i = \mathrm{Cay}(G, a^{H_i})$, $i = 0, 1, 2, 3$. Note that $\mathcal{CC}_{2pq}^0 = \mathcal{CC}_{2p^2}^0$. The graphs $\mathcal{CC}_{2pq}^i$ $(i = 0, 1, 2, 3)$ are pairwise nonisomorphic tetravalent one-regular circulant graphs of order $2pq$ and $\mathrm{Aut}(\mathcal{CC}_{2pq}^i) \cong G \rtimes H_i$.

The following example follows from [36, Theorem 3.3 and Proposition 3.3(iv)].

Example 3.4 Let p be a prime congruent to 1 mod 4, and w an element of order 4 in $\mathbb{Z}_p^*$ with $1 \leq w \leq p-1$. Let $G = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$. Then the Cayley graphs $\mathcal{CA}_{4p}^0 = \mathrm{Cay}(G, \{a, a^{-1}, a^{w^2}b, a^{-w^2}b\})$ and $\mathcal{CA}_{4p}^1 = \mathrm{Cay}(G, \{a, a^{-1}, a^wb, a^{-w}b\})$ are nonisomorphic tetravalent one-regular graphs. Furthermore, $\mathrm{Aut}(\mathcal{CA}_{4p}^0) \cong G \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\mathrm{Aut}(\mathcal{CA}_{4p}^1) \cong G \rtimes \mathbb{Z}_4$.

Since $0 \leq w \leq p-1$, the graphs $\mathcal{CA}_{4p}^0$ and $\mathcal{CA}_{4p}^1$ are independent of the choice of w. The following example is a special part of [16, Theorem 3.1].

Example 3.5 Let p and q be odd primes with $p > q$, and let $D_{2pq} = \langle a, b \mid a^{pq} = b^2 = 1, bab = a^{-1}\rangle$.

(1) Let $p \equiv 1 (\mathrm{mod}\ 4)$ and $pq \neq 15$. Then there is exactly one subgroup of order 4, say $\langle s_0\rangle$, in $\mathbb{Z}_{pq}^*$ such that $s_0^2 + 1 \equiv 0\ (\mathrm{mod}\ p)$ and $s_0 + 1 \equiv 0\ (\mathrm{mod}\ q)$. Let $\mathcal{CD}_{2pq}^0 = \mathrm{Cay}(D_{2pq}, \{b, ab, a^{s_0+1}b, a^{s_0^2+s_0+1}b\})$.

(2) Let $q \equiv 1 \pmod 4$. Then there is exactly one subgroup of order 4, say $\langle s_1\rangle$, in $\mathbb{Z}_{pq}^*$ such that $s_1 + 1 \equiv 0 \pmod p$ and $s_1^2 + 1 \equiv 0 \pmod q$. Let $\mathcal{CD}_{2pq}^1 = \mathrm{Cay}(D_{2pq}, \{b, ab, a^{s_1+1}b, a^{s_1^2+s_1+1}b\})$.

The Cayley graphs $\mathcal{CD}_{2pq}^0$ and $\mathcal{CD}_{2pq}^1$ are independent of the choice of s_0 and s_1 respectively, and nonisomorphic tetravalent one-regular graphs. For $i = 0$ or 1, let α_i be the automorphism of D_{2pq} induced by the map $a \mapsto a^{s_i}$ and $b \mapsto ab$. Then $\mathrm{Aut}(\mathcal{CD}_{2pq}^0) = R(D_{2pq}) \rtimes \langle\alpha_0\rangle$ and $\mathrm{Aut}(\mathcal{CD}_{2pq}^1) = R(D_{2pq}) \rtimes \langle\alpha_1\rangle$.

The proof of uniqueness of $\langle s_0\rangle$ or $\langle s_1\rangle$ is straightforward (also this can be proved by the equation (2) in the proof of Lemma 3.1 in [20], which claims that for each $u \in \mathbb{Z}_p$ and $v \in \mathbb{Z}_q$ the equation $|(u + Q) \cap (v + P)| = 1$ holds in $\mathbb{Z}_{pq}$ where $Q = \{kq \mid k \in \mathbb{Z}_{pq}\}$ and $P = \{kp \mid k \in \mathbb{Z}_{pq}\}$). There are two elements of order 4 in $\langle s_0\rangle$, that is, s_0 and s_0^3. Since the automorphism of D_{2pq} induced by $a \mapsto a^{-s_0}$, $b \mapsto b$ maps $\{b, ab, a^{s_0+1}b, a^{s_0^2+s_0+1}b\}$ to $\{b, ab, a^{-s_0^2-s_0}b, a^{-s_0}b\} = \{b, ab, a^{s_0^3+1}b, a^{s_0^6+s_0^3+1}b\}$, the graph $\mathcal{CD}_{2pq}^0$ is independent of the choice of s_0 and similarly, the graph $\mathcal{CD}_{2pq}^1$ is independent of the choice of s_1. Let p and q be primes congruent to 1 mod 4. By Proposition 2.3, the Cayley graphs $\mathcal{CD}_{2pq}^0$ and $\mathcal{CD}_{2pq}^1$ are normal and one-regular, implying that $\mathrm{Aut}(\mathcal{CD}_{2pq}^0) = R(D_{2pq}) \rtimes \langle\alpha_0\rangle$ and $\mathrm{Aut}(\mathcal{CD}_{2pq}^1) = R(D_{2pq}) \rtimes \langle\alpha_1\rangle$. Clearly, $\mathrm{Aut}(\mathcal{CD}_{2pq}^0)$ and $\mathrm{Aut}(\mathcal{CD}_{2pq}^1)$ have unique normal Sylow p-subgroups, say P_0 and P_1. The group of automorphisms of P_0 induced by the conjugacy action of elements in $\mathrm{Aut}(\mathcal{CD}_{2pq}^0)$ is isomorphic to $\mathbb{Z}_4$, but the group of automorphisms of P_1 induced by the conjugacy action of elements in $\mathrm{Aut}(\mathcal{CD}_{2pq}^1)$ is isomorphic to $\mathbb{Z}_2$. Thus, $\mathrm{Aut}(\mathcal{CD}_{2pq}^0) \ncong \mathrm{Aut}(\mathcal{CD}_{2pq}^1)$, and hence $\mathcal{CD}_{2pq}^0$ and $\mathcal{CD}_{2pq}^1$ are nonisomorphic.

Example 3.6 Let A_5 and S_5 be the alternating group and the symmetric group of degree 5, respectively. Let $G = A_5 \times \langle x\rangle$ with $\langle x\rangle \cong \mathbb{Z}_2$.

(1) Set $H = \langle(1\ 2)(3\ 4), (1\ 3)(2\ 4)x\rangle$ and $D = H(1\ 3)(2\ 5)H$. Then $\mathrm{Cos}(G, H, D)$ is a tetravalent one-regular graph of order 30 with girth 5, denoted by $\mathcal{NC}_{30}^0$, and $\mathrm{Aut}(\mathcal{NC}_{30}^0) = G$.
(2) Set $H = \langle(1\ 2)(3\ 4), (1\ 3)(2\ 4)x\rangle$ and $D = H(1\ 3)(2\ 5)xH$. Then $\mathrm{Cos}(G, H, D)$ is a tetravalent one-regular graph of order 30 with girth 3, denoted by $\mathcal{NC}_{30}^1$, and $\mathrm{Aut}(\mathcal{NC}_{30}^1) = G$.
(3) Set $H = \langle(1\ 2)(3\ 4), (1\ 2)\rangle$ and $D = H(1\ 3)(2\ 5)H$. Then $\mathrm{Cos}(S_5, H, D)$ is a tetravalent one-regular graph of order 30 with girth 4, denoted by $\mathcal{NC}_{30}^2$, and $\mathrm{Aut}(\mathcal{NC}_{30}^2) = S_5$.

For the first case, $\mathrm{Aut}(\mathcal{NC}_{30}^0)$ contains G as a subgroup because $H_G = 1$, which acts regularly on arcs of the graph. With the help of MAGMA [1], we can show that $\mathrm{Aut}(\mathcal{NC}_{30}^0)$ has order 120, implying that $\mathrm{Cos}(G, H, D)$ is one-regular and $\mathrm{Aut}(\mathcal{NC}_{30}^0) = G$. One may have a similar proof for the second and the third cases.

Remark Guo [14] classified symmetric graphs of order 30, but the graphs given in Example 3.6 are missing from his classification.

Example 3.7 Let $A_4 = \langle a, b \mid a^3 = b^3 = (ab)^2 = 1\rangle$. Then the tetravalent Cayley graph $\mathcal{C}_{12} = \mathrm{Cay}(A_4, \{a, b, a^{-1}, b^{-1}\})$ is normal and one-regular with automorphism group isomorphic to $R(A_4) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Clearly, $(ab)^2 = 1$ if and only if $(ba)^2 = 1$. Thus, the map α induced by $a \mapsto b$, $b \mapsto a$, and the map β induced by $a \mapsto a^{-1}, b \mapsto b^{-1}$ are automorphisms of A_4. Set $S = \{a, b, a^{-1}, b^{-1}\}$. Then $\alpha, \beta \in \mathrm{Aut}(G, S)$, and hence $\mathcal{C}_{12}$ is arc-transitive. Again with the help of MAGMA [1], $\mathrm{Aut}(\mathcal{C}_{12})$ has order 48, implying that $\mathrm{Aut}(\mathcal{C}_{12}) \cong R(A_4) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ (also this can be obtained by [25, p.1114]).

Lemma 3.8 *The graphs defined in Examples 3.1-3.7 are pairwise nonisomorphic.*

Proof The graphs $\mathcal{CG}_{2p^2}$ and $\mathcal{CC}^0_{2p^2}$, defined in Examples 3.1 and 3.3, are not isomorphic because a Sylow p-subgroup of $\mathrm{Aut}(\mathcal{CG}_{2p^2})$ is elementary Abelian, and a Sylow p-subgroup of $\mathrm{Aut}(\mathcal{CC}_{2p^2})$ is cyclic. The graphs $\mathcal{CC}^i_{2pq}$, $\mathcal{CD}^0_{2pq}$ and $\mathcal{CD}^1_{2pq}$, defined in Examples 3.3 and 3.5, are not isomorphic one another because there is an involution in the center of $\mathrm{Aut}(\mathcal{CC}^i_{2pq})$ $(0 \le i \le 4)$, but neither $\mathrm{Aut}(\mathcal{CD}^0_{2pq})$ nor $\mathrm{Aut}(\mathcal{CD}^1_{2pq})$ has such an involution. In fact, if there was an involution in the center of $\mathrm{Aut}(\mathcal{CD}^0_{2pq})$, say γ, then $P = \langle\gamma\rangle \times \langle\alpha_0\rangle$ would be a Sylow 2-subgroup of $\mathrm{Aut}(\mathcal{CD}^0_{2pq})$. Recall that $\mathrm{Aut}(\mathcal{CD}^0_{2pq}) = R(D_{2pq}) \rtimes \langle\alpha_0\rangle$, and that α_0 is the automorphism of D_{2pq} induced by the map $a \mapsto a^{s_0}$ and $b \mapsto ab$, where $s_0^2 + 1 \equiv 0 \pmod p$ and $s_0 + 1 \equiv 0 \pmod q$. Clearly, $|P \cap R(D_{2pq})| = 2$. Assume that $P \cap R(D_{2pq}) = \langle R(a^i b)\rangle$ for some $0 \le i \le pq - 1$. Then $P = \langle R(a^i b)\rangle \times \langle\alpha_0\rangle$, and hence $\gamma = R(a^i b)$ or $R(a^i b)\alpha_0^2$. Since $R(a)^{R(a^i b)} = R(a)^{-1}$, one has $\gamma = R(a^i b)\alpha_0^2$. Thus, for any $x \in D_{2pq}$, $x^{R(a)} = x^{\gamma R(a)\gamma} = x^{R(a^i b)\alpha_0^2 R(a)\alpha_0^2 R(a^i b)} = ((xa^i b)^{\alpha_0^2} a)^{\alpha_0^2} a^i b = xa^i ba^{s_0^2} a^i b = x^{R(a^{-s_0^2})}$. It follows that $R(a) = R(a^{-s_0^2})$, implying $s_0^2 + 1 \equiv 0 \pmod{pq}$ and hence $s_0^2 + 1 \equiv 0 \pmod q$. Since $s_0 + 1 \equiv 0 \pmod q$, one has $q = 2$, a contradiction. Similarly, one may show that there is no involution in the center of $\mathrm{Aut}(\mathcal{CD}^1_{2pq})$. □

4 Classification

In this section, we shall classify tetravalent one-regular graphs of order $2pq$, where p and q are primes. First we consider the case where the automorphism group is nonsolvable.

Lemma 4.1 *Let p and q be primes. Then a tetravalent one-regular graph X of order $2pq$ has a nonsolvable automorphism group if and only if $X \cong \mathcal{NC}^i_{30}$, $i = 0, 1, 2$.*

Proof By Example 3.6 the graphs $\mathcal{NC}^i_{30}$, $i = 0, 1, 2$, are one-regular with nonsolvable automorphism groups and the sufficiency follows. To prove the necessity, let X be a tetravalent one-regular graph of order $2pq$ with $G = \mathrm{Aut}(X)$ a nonsolvable group. By the definition of coset graph, one may let $X = \mathrm{Cos}(G, H, HdH)$, where H is the stabilizer of $v \in V(X)$ in G, and d interchanges v and one of its neighbors.

Then $d^2 = 1$ because of the one-regularity of X. As $|V(X)| = 2pq$, one further has $|G| = 8pq$, and hence $|H| = 4$. It follows that $H \cap H^d = 1$ because X has valency 4. The connectivity of X implies $\langle HdH \rangle = G$. To finish the proof, it suffices to show that $X \cong \mathcal{NC}^i_{30}$, $i = 0, 1, 2$.

Since G is nonsolvable, G has a nonsolvable composition factor, say G_1/G_2, which is a non-Abelian simple group. Since $|G| = 8pq$, by [13, P.12-14] one has $G_1/G_2 \cong A_5$ or $\mathrm{PSL}(2,7)$. If $G_1/G_2 \cong \mathrm{PSL}(2,7)$, then $G = \mathrm{PSL}(2,7)$ because $|G| = 8pq$. However, by using MAGMA [1], we find that there is no tetravalent one-regular graph with $G = \mathrm{PSL}(2,7)$ as full automorphism group, a contradiction.

Thus, $G_1/G_2 \cong A_5$. In this case, X has order 30 and G has order 120. By elementary group theory, up to isomorphism, there are three nonsolvable groups of order 120 which are $\mathrm{SL}(2,5)$, $A_5 \times \mathbb{Z}_2$ and S_5. Suppose $G = \mathrm{SL}(2,5)$. Then, by [15, Theorem 8.10], G has a unique involution, say a. This means that $a \in H \cap H^d$, contrary to $H^d \cap H = 1$. It follows that $G = A_5 \times \mathbb{Z}_2$ or S_5.

Let $G = A_5 \times \mathbb{Z}_2 = A_5 \times \langle x \rangle$. Then, $G = A_5 \times \langle x \rangle \leq S_5 \times \langle x \rangle$ ($A_5 \leq S_5$). Since $|H| = 4$, one has either $H \leq A_5$ or $H \cap A_5 \cong \mathbb{Z}_2$. For the former, by MAGMA [1], we can show that $\mathrm{Cos}(G, H, HdH)$ is not tetravalent one-regular for any involution $d \in G$. Thus, $H \cap A_5 \cong \mathbb{Z}_2$. Since all involutions of A_5 are conjugate and $H^d \cap H = 1$, one may assume $H \cap A_5 = \langle (1\ 2)(3\ 4) \rangle$ and $H = \langle (1\ 2)(3\ 4), zx \rangle$ for some involution $z \in A_5$. Since both zx and x commute with $(1\ 2)(3\ 4)$, z commutes with $(1\ 2)(3\ 4)$, which implies that $H = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4)x, (1\ 4)(2\ 3)x\}$. Clearly, $d = y$ or yx for some involution $y \in A_5$. By using MAGMA [1], there are 8 such y's such that $\mathrm{Cos}(G, H, HdH)$ is tetravalent and one-regular. Furthermore, all such y's are conjugate under $N_{S_5}(H)$. By Example 3.6, one may assume that $y = (1\ 3)(2\ 5)$, and hence $X \cong \mathcal{NC}^0_{30}$ or $\mathcal{NC}^1_{30}$, corresponding to $d = y$ or yx, respectively.

Let $G = S_5$. Then $P = \langle (1\ 3\ 2\ 4), (1\ 2) \rangle$ is a Sylow 2-subgroup of G. Since Sylow 2-subgroups of G are conjugate, one may assume $H \leq P$, implying $H = \langle (1\ 3\ 2\ 4) \rangle$, $\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$ or $\langle (1\ 2)(3\ 4), (1\ 2) \rangle$. If $H = \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$ or $\langle (1\ 3\ 2\ 4) \rangle$, then by using MAGMA [1], there is no involution d such that $\mathrm{Cos}(G, H, HdH)$ is tetravalent and one-regular. Thus, $H = \langle (1\ 2)(3\ 4), (1\ 2) \rangle$. Again by using MAGMA [1], there are 8 such d's such that $\mathrm{Cos}(G, H, HdH)$ is tetravalent and one-regular. Furthermore, all such d's are conjugate under $N_G(H)$. By Example 3.6, one may assume that $d = (1\ 3)(2\ 5)$, and hence $X \cong \mathcal{NC}^2_{30}$. □

Let X and Y be two graphs. The *lexicographic product* $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X[Y])$, u is adjacent to v in $X[Y]$ whenever $\{x_1, x_2\} \in E(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$. The following is the main result of this paper.

Theorem 4.2 *Let p and q be primes. A tetravalent graph X of order $2pq$ is one-regular if and only if it is isomorphic to one of the graphs in Table* 1. *Furthermore, all the graphs in Table* 1 *are pairwise nonisomorphic.*

Proof By Examples 3.1-3.7 and Lemmas 3.2 and 3.8, all graphs in Table 1 are pairwise nonisomorphic tetravalent one-regular graphs. Let X be a tetravalent one-regular

Table 1 Tetravalent one-regular graphs of order $2pq$

X	$\lvert X\rvert$	$\mathrm{Aut}(X)$	References
$\mathcal{CG}_{2p^2}$	$2p^2$, $p\equiv 1(\mathrm{mod}\ 4)$	$G(2p^2)\rtimes\mathbb{Z}_4$	Example 3.1
$\mathcal{CC}^0_{2p^2}$	$2p^2$, $p\equiv 1(\mathrm{mod}\ 4)$	$\mathbb{Z}_{2p^2}\rtimes\mathbb{Z}_4$	Example 3.3
$\mathcal{CC}^1_{2pq}$	$2pq$, $p>q>2$	$\mathbb{Z}_{2pq}\rtimes(\mathbb{Z}_2^2)$	Example 3.3
$\mathcal{CC}^2_{2pq}$	$2pq$, $p\equiv 1(\mathrm{mod}\ 4)$ $q\equiv 1(\mathrm{mod}\ 4)$, $p>q$	$\mathbb{Z}_{2pq}\rtimes\mathbb{Z}_4$	Example 3.3
$\mathcal{CC}^3_{2pq}$	$2pq$, $p\equiv 1(\mathrm{mod}\ 4)$ and $q\equiv 1(\mathrm{mod}\ 4)$, $p>q$	$\mathbb{Z}_{2pq}\rtimes\mathbb{Z}_4$	Example 3.3
$\mathcal{CA}^0_{4p}$	$4p$, $p\equiv 1(\mathrm{mod}\ 4)$	$(\mathbb{Z}_{2p}\times\mathbb{Z}_2)\rtimes(\mathbb{Z}_2^2)$	Example 3.4
$\mathcal{CA}^1_{4p}$	$4p$, $p\equiv 1(\mathrm{mod}\ 4)$	$(\mathbb{Z}_{2p}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4$	Example 3.4
$\mathcal{CD}^0_{2pq}$	$2pq$, $p\equiv 1(\mathrm{mod}\ 4)$, $p>q>2$ and $pq\neq 15$	$D_{2pq}\rtimes\mathbb{Z}_4$	Example 3.5
$\mathcal{CD}^1_{2pq}$	$2pq$, $q\equiv 1(\mathrm{mod}\ 4)$, $p>q>2$	$D_{2pq}\rtimes\mathbb{Z}_4$	Example 3.5
$\mathcal{C}_{12}$	12	$A_4\rtimes(\mathbb{Z}_2^2)$	Example 3.7
$\mathcal{NC}^0_{30}$	30	$A_5\times\mathbb{Z}_2$	Example 3.6
$\mathcal{NC}^1_{30}$	30	$A_5\times\mathbb{Z}_2$	Example 3.6
$\mathcal{NC}^2_{30}$	30	S_5	Example 3.6

graph of order $2pq$. To finish the proof, it suffices to show that X is one of the graphs listed in Table 1. Set $A=\mathrm{Aut}(X)$, and let A_v be the stabilizer of $v\in V(X)$ in A. By the one-regularity of X, $|A|=8pq$. If A is nonsolvable, then by Lemma 4.1, one has $X\cong\mathcal{NC}^i_{30}$, $i=0,1,2$, which correspond to the last three rows in Table 1. Thus, we assume that A is solvable. Set $n=pq$ and let B be a normal subgroup of A. First we prove three claims.

Claim 1 *If n is odd and B is a 2-subgroup, then $|B|=2$.*

Consider the quotient graph X_B of X relative to the set of orbits of B. Then each orbit of B on $V(X)$ has length 2, and $|X_B|=n>2$. By Proposition 2.6, X_B has valency 2 or 4. If X_B has valency 2, then X is isomorphic to $C_n[2K_1]$ which is not one-regular, a contradiction. Thus, X_B has valency 4, and by Proposition 2.6, $B\cong\mathbb{Z}_2$.

Claim 2 *If B is an Abelian group of odd order n, then there is an involution $\alpha\in$ $\mathrm{Aut}(X)$ such that $\langle B,\alpha\rangle$ acts regularly on $V(X)$ with $b^\alpha=b^{-1}$ for each $b\in B$.*

Clearly, B acts semiregularly on $V(X)$ with two orbits, say Δ and Δ'. Set $\Delta=\{\Delta(b)\mid b\in B\}$ and $\Delta'=\{\Delta'(b)\mid b\in B\}$. One may assume that the actions of B on Δ and Δ' are just by right multiplication, that is, $\Delta(b)^g=\Delta(bg)$ and $\Delta'(b)^g=\Delta'(bg)$ for any $b,g\in B$. By symmetry of X, there is no edge in Δ and Δ', implying that X is bipartite. Let the neighbors of $\Delta(1)$ be $\Delta'(b_1)$, $\Delta'(b_2)$, $\Delta'(b_3)$ and $\Delta'(b_4)$, where $b_1,b_2,b_3,b_4\in B$. Note that B is Abelian. Then for any $b\in B$, the neighbors of $\Delta(b)$ are $\Delta'(bb_1)$, $\Delta'(bb_2)$, $\Delta'(bb_3)$ and $\Delta'(bb_4)$, and furthermore, the neighbors of $\Delta'(b)$ are $\Delta(bb_1^{-1})$, $\Delta(bb_2^{-1})$, $\Delta(bb_3^{-1})$ and $\Delta(bb_4^{-1})$. The map α defined by $\Delta(b) \mapsto \Delta'(b^{-1})$, $\Delta'(b) \mapsto \Delta(b^{-1})$ for any $b \in B$, is an automorphism of X of order 2. For any $b', b \in B$, one has $\Delta(b')^{\alpha b \alpha} = \Delta(b'b^{-1}) = \Delta(b')^{b^{-1}}$ and $\Delta'(b')^{\alpha b \alpha} = \Delta'(b'b^{-1}) = \Delta'(b')^{b^{-1}}$, implying that $b^\alpha = b^{-1}$. It follows that $|\langle B, \alpha\rangle| = 2pq$ and hence $\langle B, \alpha\rangle$ acts regularly on $V(X)$.

Claim 3 *If $p > q > 2$, then A contains a cyclic normal subgroup of order $n = pq$.*

Let T be a minimal normal subgroup of A. By solvability of A, T is elementary Abelian, and by Claim 1, $T \cong \mathbb{Z}_2$, $\mathbb{Z}_p$ or $\mathbb{Z}_q$. Assume $T \cong \mathbb{Z}_2$ and let L/T be a minimal normal subgroup of A/T. Again by Claim 1, $L/T \cong \mathbb{Z}_p$ or $\mathbb{Z}_q$. The Sylow p- or q-subgroup of L is normal and so characteristic in L. By the normality of L in A, A has a normal subgroup of order p or q. Thus, A always has a normal subgroup of order p or q, say N. Then $|N| = p$ or q. Set $C = C_A(N)$. Clearly, $C \trianglelefteq A$ and $N \le C$. Furthermore, A/C is isomorphic to a subgroup of $\mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$ or $\mathbb{Z}_{q-1}$.

Assume $N \cong \mathbb{Z}_p$. Let X_N be the quotient graph of X relative to the set of orbits of N, and K the kernel of A acting on $V(X_N)$. Then $N \le K$. Suppose that A/N is Abelian. Then its quotient A/K is Abelian, and by Proposition 2.1, A/K acts regularly on $V(X_N)$, which is impossible because the arc-transitivity of A on X implies that A/K is arc-transitive on X_N. Thus, A/N is non-Abelian. Since $N \le C$ and A/C is Abelian, N is a proper subgroup of C. Let M/N be a minimal normal subgroup of A/N contained in C/N. Then $M/N \cong \mathbb{Z}_q$ or M/N is a 2-group. For the former, $M \cong \mathbb{Z}_{pq}$ is a cyclic normal subgroup of order pq, as claimed. For the latter, $M = N \times R$, where R is a Sylow 2-subgroup of M. It follows that $R \trianglelefteq A$. By Claim 1, $R \cong \mathbb{Z}_2$, and hence $M \cong \mathbb{Z}_{2p}$. If $C_A(M) = M$, then A/M is isomorphic to a subgroup of $\mathrm{Aut}(M) \cong \mathbb{Z}_{p-1}$. Since $M/N \cong \mathbb{Z}_2$ is normal in A/N, M/N is in the center of A/N, and since $(A/N)/(M/N) \cong A/M$ is cyclic, A/N is Abelian, a contradiction. It follows that $M < C_A(M)$. Let T/M be a minimal normal subgroup of A/M contained in $C_A(M)/M$. Then T/M is a 2-group or q-group. If T/M is a 2-group, then Sylow 2-subgroups of T have orders greater than 2, which are normal in A, contrary to Claim 1. Thus, $T/M \cong \mathbb{Z}_q$ and hence $T \cong \mathbb{Z}_{2pq}$. It follows that T has a cyclic subgroup of order pq which is normal in A, as claimed.

Assume $N \cong \mathbb{Z}_q$. Since $p > q$, one has $p \mid |C|$, and hence $N < C$. Let M/N be a minimal normal subgroup of A/N contained in C/N. Then $M/N \cong \mathbb{Z}_p$ or M/N is a 2-subgroup. For the former, $M \cong \mathbb{Z}_{pq}$ is a cyclic normal subgroup of order pq, as claimed. For the latter, $M = N \times R$, where $R \cong \mathbb{Z}_2$ by Claim 1. Thus, $R \lhd A$. Let P be a Sylow p-subgroup of C. Then R and N normalize P, implying $|N_A(P)| \ge 2pq$ and $|A : N_A(P)| \le 4$. Note that P is also a Sylow p-subgroup of A. The number of Sylow p-subgroups of A is $kp + 1 = |A : N_A(P)| \le 4$, forcing $k = 0$, that is, $P \trianglelefteq A$. Thus, A has a cyclic normal subgroup PN of order pq, as claimed.

In what follows we consider three cases: $p = q$, $p > q = 2$ and $p > q > 2$, respectively.

Case 1: $p = q$

By [25, p.1111–1112, p.1144–1146], there exist no tetravalent one-regular graphs of orders 8 and 18. Thus, $p > 3$. Let P be a Sylow p-subgroup of A. Then

$|P| = p^2$ and hence P is Abelian. Suppose that P is not normal in A. Remember that $|A| = 8p^2$. By Sylow theorem, the number of Sylow p-subgroups of A is $kp + 1 = |A : N_A(P)|$ that divides 8. Since $p > 3$, one has $p = 7$ and $k = 1$, that is, $|A : N_A(P)| = 8$. Thus, $C_A(P) = P = N_A(P)$ and by Proposition 2.2, A has a normal Sylow 2-subgroup, contrary to Claim 1. It follows that P is normal in A. Since $|P| = p^2$, one has $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$. If $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $X \cong \mathcal{CG}_{2p^2}$ by [11, Theorem 1.3] and Lemma 3.2, which corresponds to the second row in Table 1. If $P \cong \mathbb{Z}_{p^2}$, then by Claim 2, there exists an $\alpha \in \mathrm{Aut}(X)$ such that $G = \langle P, \alpha\rangle$ acts regularly on $V(X)$. Thus, $A = GA_v$, where A_v is the stabilizer of $v \in V(X)$ in X. Let Q be a Sylow 2-subgroup of A such that $A_v \leq Q$. Then $Q = A_v \rtimes (Q \cap G)$, implying that Q is not cyclic. If $P = C_A(P)$, then $Q \cong A/P$, which is isomorphic to a subgroup of $\mathrm{Aut}(P) \cong \mathbb{Z}_{p(p-1)}$, a contradiction. It follows that $P < C_A(P)$ and by Claim 1, $C_A(P) \cong \mathbb{Z}_{2p^2}$. It is easy to see that $C_A(P)$ acts regularly on $V(X)$, that is, X is a tetravalent one-regular circulant graph of order $2p^2$. By Proposition 2.4 and Example 3.3, $X \cong \mathcal{CC}^0_{2p^2}$, which corresponds to the third row in Table 1.

Case 2: $p > q = 2$

In this case, $|A| = 16p$. If $p = 3$, then $|X| = 12$ and by [25, P.1114], there is a unique tetravalent one-regular graph of order 12, implying $X \cong \mathcal{C}_{12}$ by Example 3.7, which corresponds to the eleventh row in Table 1. Now assume $p > 3$. Let $N = O_2(A)$ be the largest normal 2-subgroup of A and P a Sylow p-subgroup of A. We shall prove that $P \lhd A$. Since $|A| = 16p$, A/N has a unique minimal normal subgroup, that is, PN/N. Thus, $PN \lhd A$. Consider the quotient graph X_N of X relative to the set of orbits of N, and let K be the kernel of A acting on $V(X_N)$. Then $N \leq K$ and A/K is arc-transitive on X_N, forcing $2 \mid |A/K|$. It follows that $|N| \mid 8$. If $|N| \mid 4$, then by Sylow theorem, P is normal in PN because $p \geq 5$. Then P is characteristic in PN and so normal in A. Similarly, P is normal in A when $|N| = 8$ and $p \neq 7$. Now assume $|N| = 8$ and $p = 7$. Then $N \cong D_8$, Q_8 (the quaternion group), $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2^3$. Let $C = C_A(N)$. Then, $C \trianglelefteq A$ and A/C is isomorphic to a subgroup of $\mathrm{Aut}(N)$. If $N \ncong \mathbb{Z}_2^3$, then $7 \nmid |\mathrm{Aut}(N)|$ and hence $7 \mid |C|$, implying $P \leq C$. It follows that P is characteristic in PN and hence normal in A. If $N \cong \mathbb{Z}_2^3$, then $N \leq C$ and $\mathrm{Aut}(N) \cong \mathrm{PSL}(2, 7)$. Note that $|A/N| = 14$. By [15, II,Theorem 8.27], $\mathrm{Aut}(N)$ has no subgroups of order 14, implying $C \neq N$. Since $|N| = |O_2(A)| = 8$, one has $|C| \neq 16$. Thus, $7 \mid |C|$. It follows that $P \leq C$ and $P \trianglelefteq A$.

Let X_P be the quotient graph of X relative to the set of orbits of P, and K the kernel of A acting on $V(X_P)$. Then $|V(X_P)| = 4$, $P \leq K$ and A/K acts arc-transitively on X_P. By Proposition 2.6, $X_P \cong C_4$ and hence $A/K \cong D_8$, forcing $|K| = 2p$. It follows that A/P is non-Abelian because A/K is a quotient group of A/P. Moreover, K is not semiregular on $V(X)$ because $|K| = 2p$. Let $v \in V(X)$. Then $K_v \cong \mathbb{Z}_2$. Set $C = C_A(P)$. Then, $C \trianglelefteq A$ and A/C is isomorphic to a subgroup of $\mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$, and since A/P is non-Abelian, P is a proper subgroup of C. If $C \cap K \neq P$, then $C \cap K = K$ ($|K| = 2p$). Since K_v is a Sylow 2-subgroup of K, K_v is characteristic in K and so normal in A, implying that $K_v = 1$, a contradiction. Thus, $C \cap K = P$ and $1 \neq C/P = C/(C \cap K) \cong CK/K \trianglelefteq A/K \cong D_8$. If $C/P \cong \mathbb{Z}_2$, then C/P is in the center of A/P and since $(A/P)/(C/P) \cong A/C$ is cyclic, A/P is Abelian, a contradiction. It follows that $|C/P| = 4$ or 8, and hence C/P has a characteristic subgroup of order 4, say H/P. Thus, $|H| = 4p$ and $H/P \lhd A/P$, implying $H \lhd A$. And

H is Abelian because $H \le C = C_A(P)$. Clearly, $|H_v| = 4$, 2 or 1. Suppose $|H_v| = 4$. Then H_v is a Sylow 2-subgroup of H, implying H_v is characteristic in H. The normality of H in A implies that $H_v \trianglelefteq A$, forcing that $H_v = 1$, a contradiction. Suppose $|H_v| = 2$. Let Q be a Sylow 2-subgroup of H. Then $Q \trianglelefteq A$ and $Q_v = H_v$. Consider the quotient graph X_Q of X relative to the set of orbits of Q. Since $|Q| = 4$ and $Q_v \cong \mathbb{Z}_2$, Proposition 2.6 implies that $X_Q \cong C_{2p}$ and hence $X \cong C_{2p}[2K_1]$, contrary to the one-regularity of X. Thus, $H_v = 1$ and since $|H| = 4p$, H is regular on $V(X)$. It follows that X is a Cayley graph on the Abelian group H. By Proposition 2.4, there is no tetravalent one-regular Cayley graph on $\mathbb{Z}_{4p}$. Thus, $H \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$, and by Proposition 2.5 and Example 3.4, $X \cong \mathcal{CA}^0_{4p}$ or $\mathcal{CA}^1_{4p}$, which correspond to the seventh and eighth rows in Table 1.

Case 3: $p > q > 2$

By Claim 3, A has a cyclic normal subgroup of order pq, say M. Clearly, $M \le C_A(M)$. If $M < C_A(M)$, then a Sylow 2-subgroup of $C_A(M)$ is characteristic in $C_A(M)$ and hence normal in A. By Claim 1, $C_A(M) \cong \mathbb{Z}_{2pq}$. It is easy to see that $C_A(M)$ acts regularly on $V(X)$, that is, X is a tetravalent one-regular Cayley graph on $\mathbb{Z}_{2pq}$. By Proposition 2.4 and Example 3.3, X is isomorphic to one of the graphs $\mathcal{CC}^i_{2pq}$ for $1 \le i \le 3$, which correspond to the fourth, fifth and sixth rows in Table 1. Thus, in what follows we assume $M = C_A(M)$. By Claim 2, there exists an involution $\alpha \in \mathrm{Aut}(X)$ such that $\langle M, \alpha\rangle$ acts regularly on $V(X)$ and $m^\alpha = m^{-1}$ for each $m \in M$. Thus, $\langle M, \alpha\rangle$ is dihedral, and one may assume $X = \mathrm{Cay}(G, S)$, where $G = \langle a, b \mid a^{pq} = b^2 = 1, bab = a^{-1}\rangle$. One may further assume that $\langle M, \alpha\rangle = R(G)$ and $M = \langle R(a)\rangle$. Recall that $A = R(G)A_1$. Since $A/M = A/C_A(M) \le \mathrm{Aut}(M) \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$, $R(G)/M$ is normal in A/M and hence $R(G)$ is normal in A, *i.e.,* $\mathrm{Cay}(G, S)$ is normal. Let P be a Sylow 2-subgroup of A such that $A_1 \le P$. Then $P \cong A/M \le \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$ and hence $P \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_8$. Noting that $P = P \cap A = P \cap R(G)A_1 = A_1 \times (P \cap R(G))$, one has $P \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $A_1 \cong \mathbb{Z}_4$. By Proposition 2.3, one may assume $S = \{b, ab, a^{\ell+1}b, a^{\ell^2+\ell+1}b\}$ for some pair (pq, ℓ) such that $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{pq}$, $\ell^2 - 1 \ne 0 \pmod{pq}$ and $(pq, \ell) \ne (15, 2), (15, 8)$. Note that the conditions $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{pq}$ and $(pq, \ell) \ne (15, 2), (15, 8)$ imply that $pq \ne 15$. Let β be the automorphism of G induced by $a \mapsto a^\ell$ and $b \mapsto ab$. Then β permutes the elements in $\{b, ab, a^{\ell+1}b, a^{\ell^2+\ell+1}b\}$ cyclicly, and by the normality of X, $A_1 = \langle\beta\rangle$ and $A = R(G) \rtimes A_1$. From $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{pq}$ it follows that $(\ell^2 + 1)(\ell + 1) \equiv 0 \pmod{pq}$. Since $\ell^2 - 1 \ne 0 \pmod{pq}$, one of the following holds:

$$(1)\ \ell^2 + 1 \equiv 0 \pmod{pq},$$
$$(2)\ \ell^2 + 1 \equiv 0 \pmod{p}, \qquad \ell + 1 \equiv 0 \pmod{q},$$
$$(3)\ \ell + 1 \equiv 0 \pmod{p}, \qquad \ell^2 + 1 \equiv 0 \pmod{q}.$$

Suppose (1) holds. Then $a^{\beta^2} = a^{-1}$ and $b^{\beta^2} = a^{\ell+1}b$. For any $x \in G$, one may compute $x^{\beta^2 R(a)\beta^2} = (x^{\beta^2}a)^{\beta^2} = x^{\beta^4}a^{-1} = xa^{-1} = x^{R(a^{-1})}$, that is, $R(a)^{\beta^2} = R(a^{-1})$. Since $R(a)^{R(b)} = R(bab) = R(a^{-1})$, one has $R(a)^{\beta^2 R(b)} = R(a^{-1})^{R(b)} = R(a)$. This implies that $\beta^2 R(b) \in C_A(M)$, contrary to the fact that $M = C_A(M)$.

Thus, (2) and (3) hold and by Example 3.5, one has $X \cong \mathcal{CD}^0_{2pq}$ or $\mathcal{CD}^1_{2pq}$, which correspond to the ninth and tenth rows in Table 1. □

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