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Nonnegative Hall Polynomials

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Abstract. The number of subgroups of type μ and cotype ν in a finite abelian p-group of type λ is a polynomial $g^\lambda_{\mu\nu}(p)$ with integral coefficients. We prove $g^\lambda_{\mu\nu}(p)$ has nonnegative coefficients for all partitions μ and ν if and only if no two parts of λ differ by more than one. Necessity follows from a few simple facts about Hall-Littlewood symmetric functions; sufficiency relies on properties of certain order-preserving surjections φ that associate to each subgroup a vector dominated componentwise by λ. The nonzero components of $\varphi(H)$ are the parts of μ, the type of H; if no two parts of λ differ by more than one, the nonzero components of $\lambda - \varphi(H)$ are the parts of ν, the cotype of H. In fact, we provide an order-theoretic characterization of those isomorphism types of finite abelian p-groups all of whose Hall polynomials have nonnegative coefficients.

1. Introduction and statement of the main result

Hall polynomials were introduced by P. Hall [8], who was interested in the structure of lattices of subgroups of finite abelian groups. These polynomials play a central role in the work of J.A. Green [7] and A.O. Morris [12] and [13] on the representation theory of finite general linear groups. Hall polynomials are defined as follows: Any abelian group G of order p^n, where p is prime, is isomorphic to a direct product of cyclic groups

$$G \cong \mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_\ell}\mathbf{Z}$$

where $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. The partition λ is called the *type* of G. Hall studied the number $g^\lambda_{\mu\nu}(p)$ of subgroups H of a finite abelian p-group G of type λ, such that μ is the type of H and ν is the type of G/H (the *cotype* of H).

No satisfactory formula for $g^\lambda_{\mu\nu}(p)$ is known. In the 1950s Hall established that $g^\lambda_{\mu\nu}(p)$ is a polynomial in p with integral coefficients. In the 1960s T. Klein [9] discovered how to compute these polynomials. In the 1970s Macdonald provided, in his elegant text [10] on symmetric functions and Hall polynomials, a comprehensive treatment of the subject. Yet Macdonald readily admits that the method he gives for computing Hall polynomials is complicated.

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Example 1.1. The computation of Hall polynomials in [10] uses equations (4.9), (4.8), and (4.4) in Chapter II. The polynomial $g^{\lambda}_{\mu\nu}(p)$ is expressed as a sum of $c^{\lambda}_{\mu\nu}$ monic polynomials of the same degree. (The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is the number of ways of filling the Ferrers diagram of λ/μ, with ν_i letters i, so that the word read off from right to left in each row, beginning at the top row, is a lattice permutation. That is, in every initial factor of the word there must be at least as many i's as i's + 1.) Each of these monic polynomials is expressed as a product of signed sums of products of q-binomial coefficients in the variable $q = p^{-1}$ times powers of p. For example,

$$\begin{aligned} g^{221}_{11,21}(p) &= F_{(2,31,32)}(p) \cdot F_{(31,32,32)}(p) && \text{(by (4.9))} \\ &= (h_{211,211,221}h_{1,11,211} - h_{211,221,221}h_{11,11,211}) \\ &\quad (h_{221,221,221}h_{11,211,221}) && \text{(by (4.8))} \\ &= \left(1 \cdot (1 + p^{-1})p - 1 \cdot 1\right)\left(1 \cdot (1 + p^{-1})p\right) && \text{(by (4.4))} \\ &= ((p + 1) - 1)(p + 1) \end{aligned}$$

Contrast this computation with the one in Example 3.9.

A satisfactory formula for $g^{\lambda}_{\mu\nu}(p)$ not only should be simple, but also should make evident the symmetry $g^{\lambda}_{\mu\nu}(p) = g^{\lambda}_{\nu\mu}(p)$, which follows from Pontryagin duality for finite abelian groups. Moreover, in cases where $g^{\lambda}_{\mu\nu}(p)$ has nonnegative coefficients, this nonnegativity should be evident from the formula. We achieve both of these objectives for partitions λ whose parts differ by at most one.

In Section 3 we show that if no two parts of λ differ by more than one, then the Hall polynomials $g^{\lambda}_{\mu\nu}(p)$ have nonnegative coefficients. The proof is elementary and yields a satisfactory formula for these Hall polynomials. In Section 2 we show that if two parts of λ differ by more than one, then some Hall polynomial $g^{\lambda}_{\mu\nu}(p)$ has a negative coefficient. The proof employs machinery from the theory of Hall-Littlewood symmetric functions. Together these lemmas yield our main result.

THEOREM 1.2. *The Hall polynomial $g^{\lambda}_{\mu\nu}(p)$ has nonnegative coefficients for all μ and ν if and only if no two parts of λ differ by more than one.*

Proof. Necessity follows from Lemma 2.5 below. Sufficiency follows from Lemma 3.7 below. □

Finally, in Section 4 we provide an order-theoretic characterization of those isomorphism types of finite abelian p-groups all of whose Hall polynomials have nonnegative coefficients.

2. Proof of necessity

Definition 2.3. For any partitions μ, ν, and λ let $g^{\lambda}_{\mu\nu}$ be the number of elements α in the chain product $[0, \lambda_1] \times \cdots \times [0, \lambda_\ell]$ such that the nonzero components of

α are the parts of μ and the nonzero components of $\lambda - \alpha$ are the parts of ν.

PROPOSITION 2.4. *The Hall polynomial $g^\lambda_{\mu\nu}(p)$ evaluates, at $p = 1$, to $g^\lambda_{\mu\nu}$.*

Proof. Hall-Littlewood symmetric functions (see [8] and [10]) have the property that

$$t^{n(\mu)}P_\mu(x;t)t^{n(\nu)}P_\nu(x;t) = \sum_\lambda g^\lambda_{\mu\nu}(t^{-1})t^{n(\lambda)}P_\lambda(x;t)$$

where $n(\lambda) = \sum(i-1)\lambda_i$. Hall-Littlewood symmetric functions $P_\lambda(x;t)$ reduce to monomial symmetric functions $m_\lambda(x)$ when $t = 1$. So, evaluating the above identity at $t = 1$, we find

$$m_\mu(x)m_\nu(x) = \sum_\lambda g^\lambda_{\mu\nu}(1)m_\lambda(x).$$

Since monomial symmetric functions are a basis for the ring of symmetric functions, it suffices to check

$$m_\mu(x)m_\nu(x) = \sum_\lambda g^\lambda_{\mu\nu}m_\lambda(x).$$

Define the *type* of a vector $\alpha = (\alpha_1, \alpha_2, \ldots)$ with nonnegative components only finitely many of which are nonzero to be the partition whose parts are the nonzero components of α.

$$\begin{aligned} m_\mu(x)m_\nu(x) &= \left(\sum_{\text{type}\,\alpha=\mu} x^\alpha\right)\left(\sum_{\text{type}\,\beta=\nu} x^\beta\right) \\ &= \sum_\lambda \sum_{\text{type}\,\gamma=\lambda} \#\{(\alpha,\beta)|\text{type}\,\alpha = \mu,\ \text{type}\,\beta = \nu,\ \text{and}\,\alpha+\beta = \gamma\}\,x^\gamma \end{aligned}$$

Since $\#\{(\alpha,\beta)|\text{type}\,\alpha = \mu,\ \text{type}\,\beta = \nu,\ \text{and}\,\alpha+\beta = \gamma\}$ depends only on the type of γ,

$$m_\mu(x)m_\nu(x) = \sum_\lambda \#\{(\alpha,\beta)|\text{type}\,\alpha = \mu,\ \text{type}\,\beta = \nu,\ \text{and}\,\alpha+\beta = \gamma\}m_\lambda(x).$$

□

LEMMA 2.5. *If $\lambda_i > \lambda_{i+1}$, $\lambda_j > \lambda_{j+1}$, and $\lambda_i - 1 > \lambda_j$, then $g^\lambda_{\mu\nu}(p)$ has a negative coefficient, where $\mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots)$ and $\nu = (2)$.*

Proof. The Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$, which is the leading coefficient of $g^\lambda_{\mu\nu}(p)$, is one. We show that the sum of the coefficients of $g^\lambda_{\mu\nu}(p)$ is zero. By Proposition 2.4, this sum is $g^\lambda_{\mu\nu}$. By Definition 2.3, $g^\lambda_{\mu\nu}$ counts placements of the rows of μ into distinct rows of λ that leave vacant exactly two squares in the same row of λ. The number of parts of μ greater than or equal to $\lambda_i - 1$ equals the number of parts of λ greater than or equal to $\lambda_i - 1$. So, if $r = \lambda_i - 1$, the first λ'_r rows of μ must be placed into the first λ'_r rows of λ, leaving exactly one square vacant so far. Since the two squares left vacant are never in the same row of λ, $g^{\lambda}_{\mu\nu}$ is zero. □

3. Proof of sufficiency

Our proof of sufficiency relies on properties of certain order-preserving surjections [4] that associate to each subgroup a vector dominated componentwise by λ. Thereby, for each p, we define a map φ from the lattice $L_\lambda(p)$ of subgroups of $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_\ell}\mathbf{Z}$ onto the chain product $[0, \lambda]$. This map is illustrated, for $p = 2$ and $\lambda = 221$, in Figure 1.

Definition 3.6. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition. Let H be a subgroup of type $\mu = (\mu_1, \ldots, \mu_k)$ in $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_\ell}\mathbf{Z}$. An ordered set $\{h^{(1)}, \ldots, h^{(k)}\}$ of elements of H is called a set of *Hall generators* for H if it satisfies the following four conditions.

(1) $H = \langle h^{(1)}, \ldots, h^{(k)} \rangle$.

(2) The order of $h^{(i)} = (h^{(i)}_1, \ldots, h^{(i)}_\ell)$ is p^{μ_i}.

Define a map $i \mapsto I$ so that I is largest with $p^{\mu_i} = \operatorname{order}(h^{(i)}_I)$.

(3) If $j > i$, then $h^{(j)}_I = 0$.

(4) If $j > i$ and $\mu_j = \mu_i$, then $J < I$.

Define $\varphi : L_\lambda(p) \to [0, \lambda_1] \times \cdots \times [0, \lambda_\ell]$ by $\varphi(H) = \vee_i \mu_i e_I$, where e_I has a 1 in the I^{th} component.

For example, the elements $h^{(1)} = (2, 0, 1)$ and $h^{(2)} = (2, 2, 0)$ are Hall generators for a subgroup H of type $\mu = (1, 1)$ in $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. The rightmost component of $h^{(1)}$ that has order 2 is the third component and the rightmost component of $h^{(2)}$ that has order 2 is the second component, so $\varphi(H) = (0, 0, 1) \vee (0, 1, 0) = (0, 1, 1)$.

Since the unit vectors e_I are distinct, the nonzero components of $\varphi(H)$ are the parts of μ. Using ideas in [2], it is easy to see that every subgroup H has a set of Hall generators and that $\varphi(H)$ does not depend on the choice of Hall generators for H. In [3, Chapter 2, Section 7], we provide a simple way of calculating the cardinality of $\varphi^{-1}(\alpha)$. The surjections are compatible for different values of p in that $\#\varphi^{-1}(\alpha) = p^{\text{inv}}$, where inv is calculated from α and λ as described in Definition 3.8.

Finally, in [4], we show that φ is order-preserving, a fact we exploit in the proof of Theorem 4.11. From Lemma 3.7 below, it follows that if no two parts

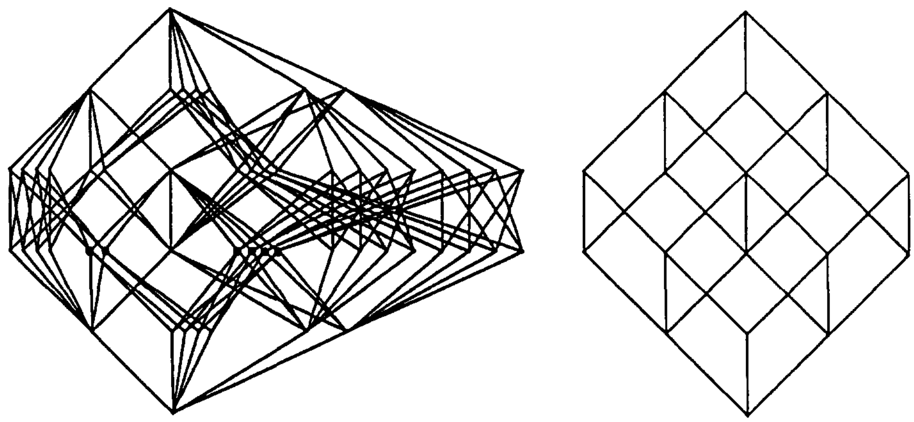


Figure 1. The order-preserving surjection $\varphi : L_{221}(2) \to [0, 221]$.

of λ differ by more than one, then $g^{\lambda}_{\mu\nu}(p)$ is the sum of $\#\varphi^{-1}(\alpha)$ over certain elements α of the chain product, namely those such that α is a rearrangement of μ and $\lambda - \alpha$ is a rearrangement of ν.

LEMMA 3.7. *Let H be a subgroup of $\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_\ell}\mathbf{Z}$. If no two parts of the partition λ differ by more than one, the nonzero components of $\lambda - \varphi(H)$ are the parts of the cotype of H.*

Proof. Let $G = \mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_\ell}\mathbf{Z}$. Fix a sequence $h^{(1)}, \ldots, h^{(k)}$ of Hall generators of a subgroup H of type μ. It suffices to construct a direct sum decomposition $\langle g^{(1)} \rangle \oplus \cdots \oplus \langle g^{(\ell)} \rangle$ of G such that for $i \leq k$,

(1) The generator $h^{(i)}$ is a multiple of $g^{(i)}$; and
(2) The order of $g^{(i)}$ is p^{λ_I}, where I is largest with $p^{\mu_i} = \operatorname{order}(h^{(i)}_I)$.

For $i \leq k$, we construct the element $g^{(i)}$ from $h^{(i)}$ as follows. Suppose

$$h^{(i)} = (a_1 p^{d_1}, \ldots, a_I p^{d_I}, \ldots, a_\ell p^{d_\ell}),$$

where no a_r is divisible by p. (However, for a 0 in the r^{th} component we write $0p^{\lambda_r}$.) We have $d_r \geq \lambda_r - \mu_i$ for $r = 1, \ldots, \ell$, with equality for $r = I$ and for no larger r. Since no two parts of λ differ by more than one, $d_I = \min\{d_1, \ldots, d_\ell\}$. (This is the key point; it depends on the I^{th} component of $h^{(i)}$ being the *rightmost* of order p^{μ_i}.) Define

$$g^{(i)} = (a_1 p^{d_1 - d_I}, \ldots, a_I p^0, \ldots, a_\ell p^{d_\ell - d_I}).$$

We have $h^{(i)} = p^{d_I} g^{(i)}$, and the order of $g^{(i)}$ is $p^{d_I+\mu_i} = p^{\lambda_I}$. Property 3 of Hall generators guarantees that the sum of the cyclic subgroups $\langle g^{(1)}\rangle, \dots, \langle g^{(k)}\rangle$ is direct. To complete the direct sum decomposition of G, define the generators $g^{(k+1)}, \dots, g^{(\ell)}$ to be the vectors e_r where r lies outside the image of the map $i \mapsto I$. $\square$

Since $\varphi^{-1}(\alpha)$ has cardinality a power of p, Lemma 3.7 shows that $g^{\lambda}_{\mu\nu}(p)$ has nonnegative coefficients if no two parts of λ differ by more than one. Below we use a combinatorial formula for $\#\varphi^{-1}(\alpha)$ to find an expression for each of these Hall polynomials as a product of p-binomial coefficients times a power of p. The p-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ gives the number of subgroups of order p^k in the elementary abelian p-group $(\mathbf{Z}/p\mathbf{Z})^n$. It may be computed as $[n]!/([k]![n-k]!)$, where $[n] = 1 + p + \cdots + p^{n-1}$ and $[n]! = [n][n-1]\cdots[1]$. It is a polynomial in p with nonnegative coefficients. Hence, the formula in Corollary 3.10 makes evident the fact that $g^{\lambda}_{\mu\nu}(p)$ has nonnegative coefficients if no two parts of λ differ by more than one. This nonnegativity is not evident from the method of computing Hall polynomials given in [9] or [10].

Definition 3.8. A *tabloid* of shape λ is a filling of the Ferrers diagram of λ such that entries weakly increase along rows. The *inversion number* of a tabloid is the sum, over entries x in the tabloid, of the number of smaller entries below x in the same column or above x one column farther right. We write $\mathrm{inv}(T)$ for the inversion number of a tabloid T.

In [3] we show that $\#\varphi^{-1}(\alpha)$ is $p^{\mathrm{inv}(T)}$, where T is the tabloid that has α_i 1's and $\lambda_i - \alpha_i$ 2's in row i.

Example 3.9. Since

$$\mathrm{inv}\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 1 & \end{pmatrix} = 1 \qquad \text{and} \qquad \mathrm{inv}\begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 1 & \end{pmatrix} = 2$$

$g^{221}_{11,21}(p) = p + p^2$. The p subgroups in $\varphi^{-1}(101)$ and the p^2 subgroups in $\varphi^{-1}(011)$ are heavily marked in Figure 1. The subgroups in $\varphi^{-1}(110)$ have cotype 111, so they are not counted in $g^{221}_{11,21}(p)$.

In [3], we show how, from the combinatorial method of computing $\#\varphi^{-1}(\alpha)$, to easily recover the following well-known expression (see [6] and [11]) for the number of subgroups of type μ in a finite abelian p-group of type λ.

$$\prod_{i \geq 1} p^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{bmatrix}. \tag{1}$$

If $\lambda = n^s$, every subgroup of type μ has cotype $\nu = (n - \mu_s, \dots, n - \mu_1)$; so the above gives a product formula for Hall polynomials when the Ferrers diagram of λ is a rectangle. It is only a little harder to obtain a product formula for Hall polynomials when λ is almost a rectangle. In particular, since p-binomial coefficients are symmetric and unimodal (see, e.g., [1]), the corollary below shows that each of these Hall polynomials not only is nonnegative, but also is symmetric and unimodal.

COROLLARY 3.10. *A finite abelian p-group of type $\lambda = n^s(n-1)^t$ has a subgroup of type μ and cotype ν if and only if $g^{\lambda}_{\mu\nu}$ is nonzero. In that case, the number of subgroups is*

$$\left(\prod_{i=1}^{n} p^{\sigma'_{i+1}(s-\sigma'_i)} \begin{bmatrix} s-\sigma'_{i+1} \\ \sigma'_i - \sigma'_{i+1} \end{bmatrix}\right) \left(\prod_{i=1}^{n-1} p^{\tau'_{i+1}(t-\tau'_i)} \begin{bmatrix} t-\tau'_{i+1} \\ \tau'_i - \tau'_{i+1} \end{bmatrix}\right) \prod_{i=1}^{n-1} p^{(s-\sigma'_i)\tau'_i} \prod_{i=1}^{n-1} p^{(t-\tau'_i)\sigma'_{i+1}}$$

where the partitions σ and τ are obtained as follows: By Definition 2.3, there is a vector α with type $\alpha = \mu$ and type $(\lambda - \alpha) = \nu$. Let $\sigma = \text{type}(\alpha_1, \ldots, \alpha_s)$ and $\tau = \text{type}(\alpha_{s+1}, \ldots, \alpha_{s+t})$.

Proof. A combinatorial formula for the number of subgroups of type μ and cotype ν in a finite abelian p-group of type $\lambda = n^s(n-1)^t$ is obtained by p-counting certain tabloids of shape λ and entries 1 and 2. For a tabloid with α_i 1's in row i to contribute a monomial to the Hall polynomial, the nonzero components of α must be the parts of μ, and, by Lemma 3.7, the nonzero components of $\lambda - \alpha$ must be the parts of ν. Since λ has at most two parts, we observe that contributing tabloids are unique up to permutation of the top s rows and permutation of the bottom t rows. (So the partitions σ and τ do not depend on the choice of α in the statement of this corollary.) Observe further that the inversion number of such a tabloid is easily calculated from the tabloid S composed of the top s rows and the tabloid T composed of the bottom t rows. It is inv(S) plus inv(T) plus, for each 2 in S, the number of 1's in T in the same column, plus, for each 2 in T, the number of 1's in S one column farther right. The product formula above is easily deduced from these two observations. □

In the above formula, the first product is the number of subgroups of type σ in a finite abelian p-group of rectangular type n^s, and the second product is the number of subgroups of type τ in a finite abelian p-group of rectangular type $(n-1)^t$.

From the combinatorial version of the formula in Corollary 3.10, the symmetry $g^{\lambda}_{\mu\nu}(p) = g^{\lambda}_{\nu\mu}(p)$ is evident, for $\lambda = n^s(n-1)^t$. Suppose the tabloid T contributes a term to $g^{\lambda}_{\mu\nu}(p)$. (So T has α_i entries equal to 1 and $\lambda_i - \alpha_1$ entries equal to 2 in row i. The nonzero components of α are the parts of μ, and the nonzero components of $\lambda - \alpha$ are the parts of ν.) Construct a tabloid T' that contributes a term to $g^{\lambda}_{\nu\mu}(p)$ as follows: For $1 \le i \le s$, let row $s+1-i$ of T' have $\lambda_i - \alpha_i$ entries equal to 1 and α_i entries equal to 2. For $1 \le i \le t$, let row $s+t+1-i$ of

T' have $\lambda_{s+i} - \alpha_{s+i}$ entries equal to 1 and α_{s+i} entries equal to 2. Since $T \mapsto T'$ is a bijection and $\operatorname{inv}(T) = \operatorname{inv}(T')$, it is evident that $g^{\lambda}_{\mu\nu}(p) = g^{\lambda}_{\nu\mu}(p)$.

4. An order-theoretic characterization

The argument used to prove Lemma 3.7 may be strengthened to prove the following theorem.

THEOREM 4.11. *If no two parts of λ differ by more than one, then there are antiautomorphisms, F and f, of $L_\lambda(p)$ and $[0, \lambda]$, respectively, such that $\varphi \circ F = f \circ \varphi$, where $\varphi : L_\lambda(p) \to [0, \lambda]$ is the order-preserving surjection in Definition 3.6.*

Proof. Let $\lambda = k^s(k-1)^t$. (So $\lambda_r = k$ for $1 \leq r \leq s$, and $\lambda_r = k-1$ for $s < r \leq s+t$.) For such a partition we modify well-known antiautomorphisms of $L_\lambda(p)$ and $[0, \lambda]$, so that $\varphi \circ F = f \circ \varphi$.

First we define F and f. The classical inner product on $(\mathbf{Z}/p^k\mathbf{Z})^s \times (\mathbf{Z}/p^{k-1}\mathbf{Z})^t$

$$h \cdot x = \sum_{r \leq s} h_r x_r + p \sum_{r > s} h_r x_r \qquad (\operatorname{mod} p^k)$$

yields the Pontryagin antiautomorphism $H \mapsto H^{\perp} = \{x | h \cdot x = 0 \ (\text{mod } p^k) \text{ for all } h \in H\}$. However $\varphi(H^{\perp}) \neq \lambda - \varphi(H)$. Study of the example $L_{221}(2)$ suggests that we define $f(\alpha) = (\lambda - \alpha)^{\text{rev}}$ where $\beta^{\text{rev}} = (\beta_s, \ldots, \beta_1, \beta_{s+t}, \ldots, \beta_{s+1})$, and that we use this operation of reversing within blocks of equal parts of λ to modify the Pontryagin antiautomorphism. That is, define $F(H) = \{x^{\text{rev}} | x \in H^{\perp}\}$.

To show that $\varphi(F(H)) = f(\varphi(H))$, we need only show that $\varphi(\langle y \rangle) \leq f(\varphi(H))$ for any $0 \neq y \in F(H)$. Suppose $y = x^{\text{rev}}$ and that x_I is the rightmost component of y whose order is the order of y. We must show that the order of x_I is less than or equal to $p^{\lambda_I - \alpha_I}$, where $\alpha = \varphi(H)$. As in Lemma 3.7, write

$$x = (b_1 p^{f_1}, \ldots, b_I p^{f_I}, \ldots, b_{s+t} p^{f_{s+t}}).$$

We must show $f_I \geq \alpha_I$. Without loss of generality, $\alpha_I > 0$. Hence, by Definition 3.6, there is a Hall generator of H

$$h^{(i)} = (a_1 p^{d_1}, \ldots, a_I p^{d_I}, \ldots, a_{s+t} p^{d_{s+t}}).$$

such that $a_I p^{d_I}$ is the rightmost component of $h^{(i)}$ whose order is the order of $h^{(i)}$. (That is, $d_I = \lambda_I - \alpha_I$.) The claim below shows that $h^{(i)} \cdot x = 0 \ (\operatorname{mod} p^k)$ implies $f_I \geq \alpha_I$.

CLAIM. *The power of p in the I^{th} term of $h^{(i)} \cdot x$ is strictly less than the power of p in any other term of $h^{(i)} \cdot x$.*

The proof below uses the following fact: Suppose $a_I p^{d_I}$ is the rightmost component whose order is the order of $(a_1 p^{d_1}, \ldots, a_{s+t} p^{d_{s+t}}) \in (\mathbf{Z}/p^k\mathbf{Z})^s \times (\mathbf{Z}/p^{k-1}\mathbf{Z})^t$. Then, not only is d_I the minimum of the exponents d_r (as noted in the proof of Lemma 3.7) but also equality holds only if

(a) The I^{th} and r^{th} components are in the same block ($\lambda_I = \lambda_r = k$ or $\lambda_I = \lambda_r = k-1$) and the I^{th} component is to the right of the r^{th} component, or
(b) The I^{th} component is in the first block ($\lambda_I = k$) and the r^{th} component is in the second block ($\lambda_r = k-1$).

Proof of Claim. Consider the r^{th} term of $h^{(i)} \cdot x$, where $r \neq I$.

Suppose $I \leq s$ and $r \leq s$. The power of p in the I^{th} term is $p^{\lambda_I - \alpha_I + f_I}$; the power of p in the r^{th} term is $p^{d_r + f_r}$. Since the I^{th} component of $h^{(i)}$ is the rightmost whose order is the order of $h^{(i)}$, we have $\lambda_I - \alpha_I \leq d_r$ with equality only if $r \leq I$. Since the component $b_I p^{f_I}$ of x^{rev} is the rightmost whose order is the order of x^{rev}, we have $f_I \leq f_r$ with equality only if $r \geq I$. Hence, $\lambda_I - \alpha_I + f_I < d_r + f_r$ for $r \neq I$.

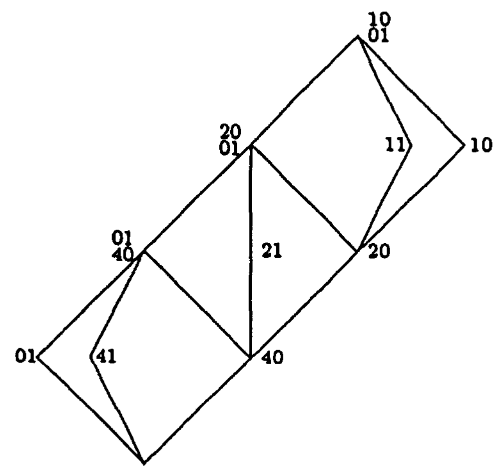
Suppose $I > s$ and $r > s$. The power of p in the I^{th} term is $p^{\lambda_I - \alpha_I + f_I}$; the power of p in the r^{th} term is $p^{1 + d_r + f_r}$. As above, we have $\lambda_I - \alpha_I \leq d_r$ with equality only if $r \leq I$, and $f_I \leq f_r$ with equality only if $r \geq I$. Hence, $1 + \lambda_I - \alpha_I + f_I < 1 + d_r + f_r$ for $r \neq I$.

Suppose $I \leq s$ and $r > s$. The power of p in the I^{th} term is $p^{\lambda_I - \alpha_I + f_I}$; the power of p in the r^{th} term is $p^{1 + d_r + f_r}$. Since the I^{th} component of $h^{(i)}$ is the rightmost whose order is the order of $h^{(i)}$, we have $\lambda_I - \alpha_I \leq d_r$. Since the component $b_I p^{f_I}$ of x^{rev} is the rightmost whose order is the order of x^{rev}, we have $f_I \leq f_r$. Hence, $\lambda_I - \alpha_I + f_I < 1 + d_r + f_r$.

Suppose $I > s$ and $r \leq s$. The power of p in the I^{th} term is $p^{1 + \lambda_I - \alpha_I + f_I}$; the power of p in the r^{th} term is $p^{d_r + f_r}$. Since the order of the I^{th} component of $h^{(i)}$ is the order of $h^{(i)}$, we have $\lambda_I - \alpha_I < d_r$. Since the order of the component $b_I p^{f_I}$ of x^{rev} is the order of x^{rev}, we have $f_I < f_r$. Hence, $1 + \lambda_I - \alpha_I + f_I < d_r + f_r$. □

Together with Lemma 2.5, this theorem permits us to view the existence of negative coefficients in a Hall polynomial $g^{\lambda}_{\mu\nu}(p)$ as obstructing the existence of certain antiautomorphisms of the lattice $L_\lambda(p)$. After all, the type of H is determined by the partial order on the subgroups contained in H. (A subgroup is cyclic if its subgroups form a chain. H is the direct sum of certain cyclic subgroups if their join is H and each meets the join of the rest in the identity.) Similarly, the cotype of H is determined by the partial order on the subgroups containing H.

Example 4.12. For $\lambda = 31$, there is no way of clustering the $1 + p$ subgroups of order p^2 into two clusters, of cardinality 1 and p, so that the subgroups within each cluster have the same type and cotype. The lattice of subgroups of $\mathbf{Z}/8\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ is shown below.



The subgroup $\langle 21 \rangle$ has the same type as $\langle 20 \rangle$ but the same cotype as $\langle 01, 40 \rangle$.

COROLLARY 4.13. *Let λ be a partition. The Hall polynomial $g^{\lambda}_{\mu\nu}(p)$ has nonnegative coefficients, for all μ and ν, if and only if, for all p, there is a triple (φ, F, f), where φ is an order-preserving surjection from $L_\lambda(p)$ onto $[0, \lambda]$, F is an antiautomorphism of $L_\lambda(p)$, and f is an antiautomorphism of $[0, \lambda]$, satisfying the following conditions.*

(1) The type of H is the partition whose parts are the nonzero components of $\varphi(H)$.
(2) The cardinality of $\varphi^{-1}(\alpha)$, as a function of p, is a power of p.
(3) $\varphi \circ F = f \circ \varphi$.

Proof. If such triples exist, then the subgroups in $\varphi^{-1}(\alpha)$ not only have the same type, they also have the same cotype. (Since F is an antiautomorphism, the cotype of H is the type of $F(H)$.) In fact, by conditions (1) and (3), the nonzero components of $f(\varphi(H))$ are the parts of the cotype of H. Hence, if we define the *type* of $\alpha \in [0, \lambda]$ to be the partition whose parts are the nonzero components of α, then

$$g^{\lambda}_{\mu\nu}(p) = \sum_{\substack{\text{type}\,\alpha = \mu \\ \text{type}\, f(\alpha) = \nu}} \#\varphi^{-1}(\alpha).$$

Condition (2) now implies that $g^{\lambda}_{\mu\nu}(p)$ is a polynomial in p with nonnegative coefficients.

If $g^{\lambda}_{\mu\nu}(p)$ has nonnegative coefficients, for all μ and ν, then Lemma 2.5

guarantees that no two parts of λ differ by more than one. This corollary is thereby a consequence of Theorem 4.11. $\square$

We say informally that there are antiautomorphisms of $L_\lambda(p)$ and $[0, \lambda]$ that commute with the collapse φ of the subgroup lattice onto the chain product if and only if all Hall polynomials $g^\lambda_{\mu\nu}(p)$ have nonnegative coefficients.

In a subsequent paper [5] we show that finite abelian p-groups whose types are almost rectangles are precisely those which admit "homogeneous flags," i.e., maximal chains of subgroups in which any two quotients of the same size are isomorphic. This result is used to study the question of whether the subgroup lattice $L_\mu(p)$ embeds as a poset in $L_\lambda(p)$, where μ and λ are partitions of the same integer.

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