Journal of Algebraic Combinatorics 2 (1993), 155–176

Alternating Sign Matrices and Some Deformations of Weyl's Denominator Formulas

SOICHI OKADA
Department of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-01, Japan

Received February 13, 1992; Revised October 30, 1992

Abstract. An alternating sign matrix is a square matrix whose entries are 1, 0, or −1, and which satisfies certain conditions. Permutation matrices are alternating sign matrices. In this paper, we use the (generalized) Littlewood's formulas to expand the products

$$\prod_{i=1}^{n}(1 - tx_i) \prod_{1\le i<j\le n} (1 - t^2 x_i x_j)(1 - t^2 x_i x_j^{-1}) \quad \text{and}$$

$$\prod_{i=1}^{n}(1 - tx_i)(1 + t^2 x_i) \prod_{1\le i<j\le n} (1 - t^2 x_i x_j)(1 - t^2 x_i x_j^{-1})$$

as sums indexed by sets of alternating sign matrices invariant under a 180° rotation. If we put $t = 1$, these expansion formulas reduce to the Weyl's denominator formulas for the root systems of type B_n and C_n. A similar deformation of the denominator formula for type D_n is also given.

Keywords: alternating sign matrix, monotone triangle, Weyl's denominator formula, Littlewood's formula

Introduction

An $n \times n$ matrix $A = (a_{ij})$ is called an *alternating sign matrix* if it satisfies the following four conditions:

(1) $a_{ij} \in \{1, 0, -1\}$.
(2) $\sum_{k=1}^{j} a_{ik} = 0$ or 1 for any i and j.
(3) $\sum_{k=1}^{i} a_{kj} = 0$ or 1 for any i and j.
(4) $\sum_{k=1}^{n} a_{kj} = \sum_{l=1}^{n} a_{il} = 1$ for any i and j.

Such matrices were introduced by W. Mills, D. Robbins and H. Rumsey, Jr. [3]. Their connection with descending plane partitions and self-complementary totally symmetric plane partitions was studied in [3] and [4].

If we denote by $\mathcal{A}_n$ the set of all $n \times n$ alternating sign matrices, then we have (see [6, 7])

$$\prod_{1\le i<j\le n}(1+tx_ix_j^{-1}) = \sum_{A\in\mathcal{A}_n} t^{i(A)}\left(1+\frac{1}{t}\right)^{s(A)} x^{\delta(A_{n-1})-A\delta(A_{n-1})}, \tag{1}$$

where $i(A) = \sum_{i<k,\,j>l} a_{ij}a_{kl}$ is the inversion number of A; $s(A)$ is the number of -1s in A; $\delta(A_{n-1}) = {}^t(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2})$; and $x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ for $\alpha = {}^t(\alpha_1, \ldots, \alpha_n)$. Alternating sign matrices with $s(A) = 0$ are the permutation matrices. So, substituting $t = -1$ in (1), we obtain the Weyl's denominator formula for the root system of type A_{n-1} (or $GL(n, \mathbb{C})$):

$$\prod_{1\le i<j\le n}(1-x_ix_j^{-1}) = \sum_{w\in S_n}(-1)^{l(w)}x^{\delta(A_{n-1})-w\delta(A_{n-1})},$$

where S_n is the symmetric group consisting of $n\times n$ permutation matrices and $l(w) = i(w)$ is the length of $w \in S_n$.

The aim of this article is to prove the following deformations of denominator formulas for the root systems of type B_n and C_n:

$$\prod_{i=1}^n(1-tx_i)\prod_{1\le i<j\le n}(1-t^2x_ix_j)(1-t^2x_ix_j^{-1})$$
$$= \sum_{A\in\mathcal{B}_n}(-1)^{i_1^+(A)+i_2(A)/2}t^{i(A)}\left(1-\frac{1}{t}\right)^{s(A)/2}x^{\delta(B_n)-A\delta(B_n)} \tag{2}$$

$$\prod_{i=1}^n(1-tx_i)(1+t^2x_i)\prod_{1\le i<j\le n}(1-t^2x_ix_j)(1-t^2x_ix_j^{-1})$$
$$= \sum_{A\in\mathcal{C}_n}(-1)^{i_1^+(A)+i_2(A)/2}t^{i(A)}\prod_{k=1}^{s(A)}\left(1+\frac{(-1)^k}{t}\right)x^{\delta(C_n)-A\delta(C_n)} \tag{3}$$

where $\mathcal{B}_n$ (resp. $\mathcal{C}_n$) is the set of all $2n\times 2n$ (resp. $(2n+1)\times(2n+1)$) alternating sign matrices which are invariant under a 180° rotation; $\delta(B_n) = {}^t(n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -(n-\frac{1}{2}))$; $\delta(C_n) = {}^t(n, n-1, \ldots, 1, 0, -1, \ldots, -n)$; and $x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ for $\alpha = {}^t(\alpha_1, \ldots, \alpha_n, (0), -\alpha_n, \ldots, -\alpha_1)$. (See Sections 2 and 3 for the definition of $i_1^+(A)$ and $i_2(A)$.) If we put $t = 1$ in (2) (resp. (3)), we can obtain the denominator formula for the root system of type B_n (resp. C_n). We also give a deformation corresponding to the root system of type D_n in Section 4.

It would be an interesting problem to give an intrinsic interpretation of alternating sign matrices in terms of root systems.

1. Alternating sign matrices and monotone triangles

In this article, we denote the set of integers by $\mathbb{Z}$. For nonnegative integers n and m, we put $[n] = \{1, 2, \ldots, n\}$ and $\sum_{n,m} = [n]\times[m]$.

We fix the notations concerning partitions (see [2]). A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers λ_i with finite sum $|\lambda| = \sum_{i\geq 1} \lambda_i$. The length $l(\lambda)$ of a partition λ is the number of nonzero terms of λ. We often identify a partition λ with its Young diagram $D(\lambda) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z}; 1 \leq j \leq \lambda_i, 1 \leq i \leq l(\lambda)\}$.

The *conjugate* partition of λ is the partition λ' whose Young diagram $D(\lambda')$ is obtained from $D(\lambda)$ by reflection with respect to the main diagonal. If $\lambda = \lambda'$, then we call λ a *self-conjugate* partition.

A partition λ is called *distinct* if $\lambda_1 > \lambda_2 > \cdots > \lambda_{l(\lambda)} > 0$. For example, $\delta_n = (n, n-1, \ldots, 2, 1)$ is a distinct partition.

Next we introduce the Frobenius notation. For a partition λ, we define

$$p = p(\lambda) = \#\{k \in \mathbb{Z} : \lambda_k \geq k\},$$
$$\alpha_k = \lambda_k - k, \quad \beta_k = \lambda'_k - k \quad (1 \leq k \leq p(\lambda)).$$

Then we write

$$\lambda = (\alpha_1, \ldots, \alpha_p \mid \beta_1, \ldots, \beta_p) = (\alpha \mid \beta).$$

The partition λ can be recovered from α and β by putting

$$\lambda_k = \alpha_k + k \quad \text{if } k \leq p \tag{4}$$
$$\lambda_k = \#\{j \in [p] : \beta_j + j \geq k\} \quad \text{if } k > p \tag{5}$$

1.1. Alternating sign matrices

A vector $a = (a_1, \ldots, a_n)$ is called *sign-alternating* if it satisfies

(1) $a_i \in \{1, 0, -1\}$.
(2) $\sum_{k=1}^{i} a_k = 0$ or 1 for $i = 1, \ldots, n$.

Then the nonzero entries of a sign-alternating vector alternate in sign.

Definition. Let λ be a distinct partition with length n such that $\lambda_1 \leq m$. An $n \times m$ matrix $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ is a *λ-alternating sign matrix* if the following conditions hold:

(1) Every row and column is sign-alternating.
(2) $\sum_{j=1}^{m} a_{ij} = 1$ for any i.
(3) $\sum_{i=1}^{n} a_{ij} = 1$ if $j = \lambda_k$ for some k and 0 otherwise.

Let λ be a distinct partition with length n. It follows from the definition that, if A is an $n \times m$ λ-alternating sign matrix, then $a_{ij} = 0$ for all i and $j > \lambda_1$. So the number m of columns of a λ-alternating sign matrix is irrelevant so far as $m \geq \lambda_1$. We denote by $\mathcal{A}(\lambda)$ the set of all λ-alternating sign matrices. Then we have

$$\mathcal{A}(\delta_n) = \mathcal{A}_n,$$

the set of all $n \times n$ alternating sign matrices (defined in Introduction). For a λ-alternating sign matrix $A \in \mathcal{A}(\lambda)$, we define

$$i(A) = \sum_{i<k,\, j>l} a_{ij} a_{kl}, \tag{6}$$

called the *number of inversions* of A. And we denote by $s(A)$ the number of -1s in A (see [3]).

1.2. Monotone triangles

Definition. A triangular array

$$T = \begin{array}{ccccccccc} & & & & t_{11} & & & & \\ & & & t_{21} & & t_{22} & & & \\ & & t_{31} & & t_{32} & & t_{33} & & \\ & & & & \cdots & & & & \\ t_{n1} & & t_{n2} & & \cdots & & \cdots & & t_{nn} \end{array}$$

is a *monotone triangle* if it satisfies

(1) Each row is strictly increasing.
(2) $t_{i+1,j} \le t_{i,j} \le t_{i+1,j+1}$ for all $i = 1, \dots, n-1$ and $j = 1, \dots, i-1$.

For a distinct partition λ of length n, let $\mathcal{M}(\lambda)$ be the set of all monotone triangles with bottom row λ. For a monotone triangle $T = (t_{ij})$, we put

$$\begin{aligned} \max(T) &= \#\{(i, j) : t_{i+1,j} < t_{ij} = t_{i+1,j+1}\}, \\ \operatorname{sp}(T) &= \#\{(i, j) : t_{i+1,j} < t_{ij} < t_{i+1,j+1}\}, \\ x^T &= x_1^{s_1} x_2^{s_2 - s_1} \dots x_n^{s_n - s_{n-1}}, \end{aligned}$$

where s_i is the sum of the ith row of T.

To a λ-alternating sign matrix $A = (a_{ij}) \in \mathcal{A}(\lambda)$, we associate a matrix $B(A) = (b_{ij})$ by putting

$$b_{ij} = \sum_{k=1}^{i} a_{kj}. \tag{7}$$

Then we can define a triangular array $T = T(A)$ by the condition that the number j appears in the ith row of T if and only if $b_{ij} = 1$. For example, if

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

then we have

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad T = \begin{array}{ccccccc} & & & 3 & & & \\ & & 2 & & 3 & & \\ & 1 & & 2 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}.$$

PROPOSITION 1.1. *Let λ be a distinct partition with length n.*

(1) T gives a bijection from $\mathcal{A}(\lambda)$ to $\mathcal{M}(\lambda)$.
(2) For $A \in \mathcal{A}(\lambda)$, we have

$$i(A) = \max(T(A)) + \mathrm{sp}(T(A)),$$
$$s(A) = \mathrm{sp}(T(A)).$$

Proof. It is easy to see that T is a bijection and that $s(A) = \mathrm{sp}(T(A))$. From (6) and (7), we have

$$i(A) = \sum_{l<j} \sum_{i=1}^{n-1} b_{ij}(b_{i+1,l} - b_{i,l}).$$

It follows from the definition of $T(A) = (t_{ij})$ that

$$\sum_{l<j} b_{ij}(b_{i+1,l} - b_{i,l}) = \#\{k : t_{ik} > t_{i+1,k}\}.$$

Hence we have

$$i(A) = \#\{(i,j) : t_{ij} > t_{i+1,j}\} = \max(T) + \mathrm{sp}(T).$$ □

For a partition λ with length $\leq n$, we denote by $s_\lambda(x_1, \ldots, x_n)$ the Schur function corresponding to λ.

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}}{\det\left(x_i^{n-j}\right)_{1\leq i,j\leq n}}.$$

T. Tokuyama [7] proved the following formula by using the representation theory of general linear groups (see [5] for an alternate proof).

PROPOSITION 1.2. ([7, Theorem 2.1], [5, Theorem 4]) *Let λ be a partition with length $\leq n$. Then we have*

$$\prod_{1\leq i<j\leq n} (1 + t x_i x_j^{-1}) s_\lambda(x_1, \ldots, x_n)$$
$$= \sum_{T\in\mathcal{M}(\lambda+\delta_n)} t^{\max(T)+\mathrm{sp}(T)} \left(1 + \frac{1}{t}\right)^{\mathrm{sp}(T)} x^T x_1^{-1} x_2^{-2} \ldots x_n^{-n}.$$

We put $\lambda = 0$, the unique partition of 0, in Proposition 1.2. Then we can use Proposition 1.1. to obtain a deformation of the Weyl's denominator formula for the root system of type A_{n-1}.

COROLLARY 1.3.

$$\prod_{1\le i<j\le n}(1+tx_ix_j^{-1}) = \sum_{A\in\mathcal{A}_n} t^{i(A)}\left(1+\frac{1}{t}\right)^{s(A)} x^{\delta(A_{n-1})-A\delta(A_{n-1})}.$$

This corollary reduces to the denominator formula, if $t = -1$.

Let J_N be the $N \times N$ antidiagonal matrix given by

$$J_N = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}.$$

Then, for an $N \times N$ matrix $A = (a_{ij})$, $J_N A J_N$ is the matrix obtained by a 180° rotation, i.e., if $J_N A J_N = (a'_{ij})$, then $a'_{ij} = a_{N+1-i,N+1-j}$. Here we quote a lemma from [2].

LEMMA 1.4. (**[2, I.(1.7)]**). *For a partition $\nu \subset (m^n)$, we have*

$$\{\nu_k + n + 1 - k : k \in [n]\} \cup \{n + l - \nu'_l : l \in [m]\} = [n+m]$$

LEMMA 1.5. *Let A be an $N \times N$ alternating sign matrix and $A' = J_N A J_N$. For $i = 1, \ldots, N$, let $\lambda^{(i)}$ (resp. μ^i) be the partition such that $\lambda^{(i)} + \delta_i$ (resp. $\mu^{(i)} + \delta_i$) is the ith row of $T(A)$ (resp. $T(A')$). Then $\lambda^{(i)}$ is the conjugate partition of $\mu^{(N-i)}$.*

Proof. Let $\lambda = \lambda^{(i)}$ and $\mu = \mu^{(N-i)}$. If we put $B(A) = (b_{ij})$ and $B(JAJ) = (b'_{ij})$, then we have $b_{ij} = 1 - b'_{N-i,N+1-j}$. Hence, the number j appears in the ith row of $T(A)$ if and only if $N+1-j$ does not appear in the $(N-i)$-th row of $T(A')$. That is,

$$\{\lambda_k + i + 1 - k : k \in [i]\} \cup \{N + 1 - (\mu_l + N - i + 1 - l) : l \in [N-i]\} = [N].$$

On the other hand, by applying Lemma 1.4 to $\lambda \subset ((N-i)^i)$, we have

$$\{\lambda_k + i + 1 - k : k \in [i]\} \cup \{i + l - \lambda'_l : l \in [N-i]\} = [N].$$

Hence, we see that

$$i + l - \lambda'_l = N + 1 - (\mu_l + N - i + 1 - l) \qquad (l = 1, \ldots, N-i)$$

This gives $\lambda'_l = \mu_l$. □

2. Deformation for B_n type

In this section we give a deformation of the Weyl's denominator formula for the root system of type B_n.

Let $\mathcal{B}_n$ be the set of all $2n \times 2n$ alternating sign matrices invariant under a 180° rotation, i.e.,

$$\mathcal{B}_n = \{A \in \mathcal{A}_{2n} : J_{2n} A J_{2n} = A\}.$$

Definition. Let $L = \{(i,j;k,l) \in \Sigma_{2n,2n} \times \Sigma_{2n,2n} : i < k, j > l\}$ and define subsets L_1, L_2, L_+, and $L_\pm$ of L as follows:

$$\begin{aligned} L_1 &= \{(i, j; k, l) \in L : i + k = 2n + 1, j + l = 2n + 1\}, \\ L_2 &= L - L_1, \\ L_+ &= \{(i, j; k, l) \in L : i \le n, k \le n\}, \\ L_\pm &= \{(i, j; k, l) \in L : i \le n, k \ge n + 1\}. \end{aligned}$$

For each subset L_*, $* = 1, 2, +, \pm$, and $A \in \mathcal{B}_n$, we put

$$i_*(A) = \sum_{(i,j;k,l)\in L_*} a_{ij} a_{kl}.$$

Moreover we put

$$\begin{aligned} i_1^+(A) &= \#\{(i, j) : 1 \le i \le n, n + 1 \le j \le 2n, a_{ij} = 1\}, \\ i_1^-(A) &= \#\{(i, j) : 1 \le i \le n, n + 1 \le j \le 2n, a_{ij} = -1\}. \end{aligned}$$

From the definition, we have, for $A \in \mathcal{B}_n$,

$$\begin{aligned} i(A) &= i_1(A) + i_2(A) \qquad (8) \\ &= 2i_+(A) + i_\pm(A), \\ i_1(A) &= i_1^+(A) + i_1^-(A). \qquad (9) \end{aligned}$$

The main result of this section is

THEOREM 2.1.

$$\prod_{i=1}^{n}(1 - tx_i) \prod_{1\le i<j\le n} (1 - t^2 x_i x_j)(1 - t^2 x_i x_j^{-1}) = \sum_{A\in\mathcal{B}_n} (-1)^{i_1^+(A)+i_2(A)/2} t^{i(A)} \left(1 - \frac{1}{t^2}\right)^{s(A)/2} x^{\delta(B_n)-A\delta(B_n)},$$

where $\delta(B_n) = {}^t(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -(n - \frac{1}{2}))$ *and* $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *for* $\alpha = {}^t(\alpha_1, \ldots, \alpha_n, -\alpha_n, \ldots, -\alpha_1)$.

In order to prove this theorem, first we note the following.

PROPOSITION 2.2. *For $A \in \mathcal{B}_n$, let $T^+(A)$ be the monotone triangle consisting of the first n rows of $T(A)$. Then the correspondence T^+ gives a bijection*

$$T^+ : \mathcal{B}_n \to \coprod_{\lambda} \mathcal{M}(\lambda + \delta_n),$$

where λ runs over all self-conjugate partitions λ such that $\lambda \subset (n^n)$.

Proof. follows from Proposition 1.1 and Lemmas 1.4 and 1.5.

LEMMA 2.3. *Let $A \in \mathcal{B}_n$ and $T = T^+(A) \in \mathcal{M}(\lambda + \delta_n)$. Then we have*

(1) $i_+(A) = \max(T) + \mathrm{sp}(T)$.
(2) $s(A) = 2\mathrm{sp}(T)$.
(3) $x^{\delta(B_n) - A\delta(B_n)} = x^T x_1^{-1} x_2^{-2} \cdots x_n^{-n}$.
(4) $i_\pm(A) = |\lambda|$.
(5) $i_1^+(A) - i_1^-(A) = p(\lambda)$.

Proof. (1) and (2) follows from Proposition 1.1.(2).

(3) Since $\sum_{j=1}^{2n} a_{ij} = 1$, the ith component of $\delta(B_n) - A\delta(B_n)$ is equal to $\frac{1}{2}\left\{(2n - 2i + 1) - \sum_{j=1}^{2n} a_{ij}(2n - 2j + 1)\right\} = \sum_{j=1}^{2n} j a_{ij} - i$, which is the sum of the ith row of T.

(4) The definition says that

$$i_\pm(A) = \sum_{1 \le l < j \le 2n} \left(\sum_{i=1}^{n} a_{ij}\right)\left(\sum_{k=n+1}^{2n} a_{kl}\right).$$

By the symmetry $J_{2n} A J_{2n} = A$, we have

$$\sum_{i=1}^{n} a_{ij} = \begin{cases} 1 & \text{if } j = \lambda_f + n + 1 - f \text{ for some } f \in [n] \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$\sum_{k=n+1}^{2n} a_{kl} = \begin{cases} 1 & \text{if } 2n + 1 - l = \lambda_g + n + 1 - g \text{ for some } g \in [n] \\ 0 & \text{otherwise} \end{cases}$$

Hence, we have

$$i_\pm(A) = \#\{(f, g) \in [n] \times [n] : \lambda_f + \lambda_g \ge f + g\}.$$

Since λ is self-conjugate, we see that

$$D(\lambda) = \{(f, g) \in [n] \times [n] : \lambda_f + \lambda_g \ge f + g\}.$$

Therefore we obtain $i_\pm(A) = |\lambda|$.

(5) By definition and (10), we have

$$\begin{aligned} i_1^+(A) - i_1^-(A) &= \sum_{j=n+1}^{2n} \sum_{i=1}^{n} a_{ij} \\ &= \#\{k \in [n] : \lambda_k + n + 1 - k \geq n + 1\} \\ &= p(\lambda). \end{aligned}$$

□

The following formula due to D.E. Littlewood is the key to our proof of Theorem 2.1.

LEMMA 2.4. **([1, p. 238], [2, I. Ex. 5.9])**

$$\prod_{i=1}^{n}(1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j) = \sum_{\lambda} (-1)^{(|\lambda|+p(\lambda))/2} s_\lambda(x_1, \ldots, x_n)$$

where λ runs over all self-conjugate partitions such that $\lambda \subset (n^n)$.

Proof of Theorem 2.1. Replacing x_i by tx_i in Lemma 2.4 and using Proposition 1.2, we obtain

$$\begin{aligned} &\prod_{i=1}^{n}(1 - tx_i) \prod_{1 \leq i < j \leq n} (1 - t^2 x_i x_j)(1 - t^2 x_i x_j^{-1}) \\ &\quad = \sum_{\lambda \subset (n^n), \lambda' = \lambda} \ \sum_{T \in \mathcal{M}(\lambda + \delta_n)} (-1)^{(|\lambda|+p(\lambda))/2 + \max(T) + \operatorname{sp}(T)} \\ &\qquad \times t^{|\lambda| + 2\max(T) + 2\operatorname{sp}(T)} x^T x_1^{-1} x_2^{-2} \cdots x_n^{-n} \end{aligned}$$

Then the proof follows from Proposition 2.2 and Lemma 2.3 together with (8) and (9). □

If we put $W(B_n) = \{A \in \mathcal{B}_n : s(A) = 0\}$, then $W(B_n)$ is the Weyl group of the root system

$$\Delta(B_n) = \{\pm\varepsilon_i : i \in [n]\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : i, j \in [n], i < j\},$$

where $\varepsilon_i = {}^t(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0, \overset{2n+1-i}{-1}, 0 \ldots, 0)$. By substituting $t = 1$ in Theorem 2.1, we obtain the denominator formula

COROLLARY 2.5. (to Theorem 2.1).

$$\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)(1-x_ix_j^{-1}) = \sum_{A\in W(B_n)}(-1)^{l(A)}x^{\delta(B_n)-A\delta(B_n)},$$

where $l(A) = i_1(A) + i_2(A)/2$ *is the length of* $A \in W(B_n)$.

3. Deformation for C_n type

Next we consider a deformation of the denominator formula for the root system of type C_n.

Let $\mathcal{C}_n$ be the set of all $(2n+1)\times(2n+1)$ alternating sign matrices invariant under a 180° rotation, i.e.,

$$\mathcal{C}_n = \{A \in \mathcal{A}_{2n+1} : J_{2n+1}AJ_{2n+1} = A\}$$

Definition. Let $L = \{(i, j; k, l) \in \Sigma_{2n+1,2n+1} \times \Sigma_{2n+1,2n+1} : i < k, j > l\}$ and define subsets L_0, L_1, L_2, L_+, and $L_\pm$ of L as follows:

$$\begin{aligned}
L_0 &= \{(i, j; k, l) \in L : i = n+1 \text{ or } k = n+1\},\\
L_1 &= \{(i, j; k, l) \in L : i + k = 2n+2,\ j + l = 2n+2\},\\
L_2 &= L - (L_0 \cup L_1),\\
L_+ &= \{(i, j; k, l) \in L : i \le n,\ k \le n\},\\
L_\pm &= \{(i, j; k, l) \in L : i \le n,\ k \ge n+2\}.
\end{aligned}$$

For each subset L_*, $* = 0, 1, 2, +, \pm$, and $A \in \mathcal{C}_n$, we put

$$i_*(A) = \sum_{(i,j;k,l)\in L_*} a_{ij}a_{kl}$$

Moreover we put

$$\begin{aligned}
i_1^+(A) &= \#\{(i, j) : 1 \le i \le n,\ n+2 \le j \le 2n+1,\ a_{ij} = 1\},\\
i_1^-(A) &= \#\{(i, j) : 1 \le i \le n,\ n+2 \le j \le 2n+1,\ a_{ij} = -1\}.
\end{aligned}$$

Then we have

$$i(A) = i_0(A) + i_1(A) + i_2(A), \tag{11}$$

$$i_1(A) + i_2(A) = 2i_+(A) + i_\pm(A), \tag{12}$$

$$i_1(A) = i_1^+(A) + i_1^-(A). \tag{13}$$

The main result of this section is

THEOREM 3.1.

$$\prod_{i=1}^{n}(1-tx_i)(1+t^2x_i)\prod_{1\le i<j\le n}(1-t^2x_ix_j)(1-t^2x_ix_j^{-1})$$

$$=\sum_{A\in C_n}(-1)^{i_1^+(A)+i_2(A)/2}t^{i(A)}\prod_{k=1}^{s(A)}\left(1+\frac{(-1)^k}{t}\right)x^{\delta(C_n)-A\delta(C_n)},$$

where $\delta(C_n)={}^t(n,n-1,\dots,1,0,-1,\dots,-n)$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ *if* $\alpha={}^t(\alpha_1,\dots,\alpha_n,0,-\alpha_n,\dots,-\alpha_1)$.

For simplicity, we write

$$\left(1\mp\frac{1}{t}\right)^s=\prod_{k=1}^{s}\left(1+\frac{(-1)^k}{t}\right)$$

$$=\begin{cases}(1-\frac{1}{t^2})^{s/2} & \text{if } s \text{ is even}\\ (1-\frac{1}{t})(1-\frac{1}{t^2})^{(s-1)/2} & \text{if } s \text{ is odd}\end{cases}$$

Now we consider a partition $\lambda=(\alpha_1,\dots,\alpha_p|\beta_1,\dots,\beta_p)$ satisfying

$$n\ge\beta_1+1\ge\alpha_1\ge\beta_2+1\ge\alpha_2\ge\ldots\ge\beta_p+1\ge\alpha_p. \tag{14}$$

For such a partition λ, we put

$$q(\lambda)=\#\{k\in[n]:\lambda_k>k\}$$
$$r(\lambda)=\#\{(k,l)\in[n]\times[n]:\lambda_k+\lambda_l>k+l\}$$
$$u(\lambda)=\#\{(k,l)\in[n]\times[n]:\lambda_k+\lambda_l=k+l\}$$

Then these quantities can be expressed in terms of α and β.

LEMMA 3.2. *Let* $\lambda=(\alpha|\beta)$ $(\alpha=(\alpha_1,\dots,\alpha_p),\beta=(\beta_1,\dots,\beta_p),p=p(\lambda))$ *be a partition satisfying (14). Then we have*

(1) $q(\lambda)=\#\{k\in[p]:\alpha_k>0\}$.
(2) Let $\nu=(\gamma|\gamma)$ *be the self-conjugate partition defined by*

$$\gamma_k=\begin{cases}\max(\alpha_k-1,\beta_{k+1}+1) & \text{if } k\le p-1\\ \alpha_p-1 & \text{if } \alpha_p>0 \text{ and } k=p\end{cases}$$

Then

$$D(\nu)=\{(k,l)\in[n]\times[n]:\lambda_k+\lambda_l>k+l\}.$$

In particular, we have $r(\lambda)=|\nu|$.
(3) $u(\lambda)$ *is given by*

$$u(\lambda) = 2\#\{k \in [p] : \beta_k + 1 > \alpha_k > \beta_{k+1} + 1\} + \begin{cases} 1 & \textit{if } \alpha_p = 0 \\ 0 & \textit{if } \alpha_p > 0 \end{cases}$$

where $\beta_{p+1} + 1 = 0$.

Proof.

(1) This follows from (4).

(2) First we show that, if $(k, m) \in D(\nu)$ and $k \leq m$, then $\lambda_k + \lambda_m > k + m$. In this case, we note that $k \leq p$ and $m \leq \gamma_k + k$.

   (i) If $k < m \leq p$, then $\lambda_k > k$ and $\lambda_m \geq m$, so we have $\lambda_k + \lambda_m > k + m$.

   (ii) If $k = m = p$, then it follows from $(k, m) \in D(\nu)$ that $\alpha_p > 0$, so $\lambda_p > p$. Hence, we have $\lambda_k + \lambda_m = 2\lambda_p > 2p = k + m$.

   (iii) If $k \leq p < m$ and $\alpha_k > \beta_{k+1} + 1$, then $\gamma_k = \alpha_k - 1$ and $\beta_k + k \geq \alpha_k - 1 + k = \gamma_k + k \geq m$. So we have $\lambda_k = \alpha_k + k = \gamma_k + 1 + k > m$ and $\lambda_m \geq k$, hence, $\lambda_k + \lambda_m > k + m$.

   (iv) If $k \leq p < m$ and $\alpha_k = \beta_{k+1} + 1$, then $\gamma_k = \alpha_k$ and $\beta_{k+1} + k + 1 = \gamma_k + k \geq m$. So we have $\lambda_k = \alpha_k + k = \gamma_k + k \geq m$ and $\lambda_m > k$, hence $\lambda_k + \lambda_m > k + m$.

   Therefore we obtain $\lambda_k + \lambda_m > k + m$ for $(k, m) \in D(\nu)$. Similarly we can show that $\lambda_k + \lambda_m \leq k + m$ for $(k, m) \notin D(\nu)$. Hence, we have

$$D(\nu) = \{(k, m) : \lambda_k + \lambda_m > k + m\}.$$

(3) Here we note that, if $k = m$ and $\lambda_k + \lambda_m = k + m$, then $k = m = p$ and $\alpha_p = 0$. Now we suppose that $\lambda_k + \lambda_m = k + m$ and $k < m$.
First we show that $\lambda_k = m$ and $\lambda_m = k$. If $\lambda_k < m$, then we can see that $k \leq p < m$. Hence, we have

$$\alpha_k + k < m, \quad \beta_{k+1} + k + 1 \geq m,$$

so that $\beta_{k+1} + 1 \geq m - k > \alpha_k$. This contradicts (14). We have a similar contradiction if $\lambda_k > m$. Therefore we have $\lambda_k = m$ and $\lambda_m = k$.
Next we show that $\beta_k + 1 > \alpha_k > \beta_{k+1} + 1$. Then we can check that $k \leq p < m$. From (4), we have $\lambda_k = \alpha_k + k = m$. On the other hand, from (5), we have $\beta_k + k \geq m$ and $\beta_{k+1} + k + 1 < m$. Hence, we see $\beta_k + 1 > \alpha_k > \beta_{k+1} + 1$.
Therefore we obtain

$$\begin{aligned}\{(k, m) : \lambda_k + \lambda_m = k + m\} = &\{(k, \lambda_k) : k \in [p], \beta_k + 1 > \alpha_k > \beta_{k+1} + 1\} \\ &\cup \{(\lambda_k, k) : k \in [p], \beta_k + 1 > \alpha_k > \beta_{k+1} + 1\} \\ &\cup \{(p, p) : \alpha_p = 0\}.\end{aligned}$$

So, considering the cardinalities of both sides completes the proof. □

PROPOSITION 3.3. *For $A \in \mathcal{C}_n$, let $T^+(A)$ be the monotone triangle consisting of the first n rows of $T(A)$. Then the correspondence T^+ gives a bijection*

$$T^+ : \mathcal{C}_n \to \coprod_{\lambda} \mathcal{M}(\lambda + \delta_n)$$

where λ runs over all partitions λ satisfying (14).

Proof. For a partition $\lambda = (\alpha|\beta) \subset ((n+1)^n)$, we put

$$\begin{aligned} t_i &= \lambda_i + n + 1 - i, \quad i \in [n] \\ t'_j &= \lambda'_j + n + 2 - j, \quad j \in [n+1] \end{aligned}$$

Then, by considering the shifted diagrams of $\lambda + \delta_n$ and $\lambda' + \delta_{n+1}$, we can see that λ satisfies (14) if and only if

$$t'_1 \geq t_1 \geq t'_2 \geq t_2 \geq \ldots \geq t'_n \geq t_n \geq t'_{n+1}.$$

Now, if $A \in \mathcal{C}_n$ and $T^+(A)$ has the bottom row $\lambda + \delta_n$, then the $(n+1)$-th row of $T(A)$ is $\lambda' + \delta_{n+1}$ by Lemma 1.5, so that λ satisfies (14). Conversely, given $T \in \mathcal{M}(\lambda + \delta_n)$, where λ satisfies (14), we can define a monotone triangle $\widetilde{T}$ by adjoining $\lambda' + \delta_{n+1}$ to T. If $V = (v_{ij})_{1 \leq i \leq n+1, 1 \leq j \leq 2n+1}$ corresponds to $\widetilde{T}$ under the bijection in Proposition 1.1, then if follows from Lemma 1.4 that the matrix $A = (a_{ij})_{1 \leq i,j \leq 2n+1}$ defined by

$$a_{ij} = \begin{cases} v_{ij} & \text{if } i \leq n+1 \\ v_{2n+2-i, 2n+2-j} & \text{if } i \geq n+2 \end{cases}$$

is an alternating sign matrix belonging to $\mathcal{C}_n$ and $T^+(A) = T$. □

LEMMA 3.4 *Let $A \in \mathcal{C}_n$ and $T = T^+(A) \in \mathcal{M}(\lambda + \delta_n)$. Then we have*

(1) $i_+(A) = \max(T) + \mathrm{sp}(T)$.
(2) $x^{\delta(\mathcal{C}_n) - A\delta(\mathcal{C}_n)} = x^T x_1^{-1} x_2^{-2} \ldots x_n^{-n}$.
(3) $i_\pm(A) = r(\lambda)$.
(4) $i_1^+(A) - i_1^-(A) = q(\lambda)$
(5) $i_0(A) = 2(|\lambda| - r(\lambda))$
(6) The number of -1s in the $(n+1)$-th row of A is equal to $u(\lambda)$. Hence

$$s(A) = 2\mathrm{sp}(T) + u(\lambda).$$

Proof. We prove only (5) and (6). The other statements can be proved in the same way as in the proof of Lemma 2.3. If we put $B(A) = (b_{ij})$, then it follows from Lemma 1.6 that

$$b_{n,l} = \begin{cases} 1 \text{ if } l = \lambda_k + n + 1 - k \text{ for some } k \\ 0 \text{ if } l = n + k - \lambda'_k \text{ some } k \end{cases} \tag{15}$$

$$b_{n+1,l} = \begin{cases} 1 \text{ if } l = \lambda'_k + n + 2 - k \text{ for some } k \\ 0 \text{ if } l = n + k + 1 - \lambda_k \text{ some } k \end{cases} \tag{16}$$

(3) By the symmetry $J_{2n+1}AJ_{2n+1} = A$ and (15), we have

$$\begin{aligned} \frac{1}{2} i_0(A) &= \sum_{1 \le l < j \le n} \left(\sum_{i=1}^{n} a_{ij} \right) a_{n+1,l} \\ &= \sum_{f=1}^{n} \sum_{l < \lambda_f + n + 1 - f} (b_{n+1,l} - b_{n,l}) \end{aligned}$$

From (15), we see

$$\sum_{l < \lambda_f + n + 1 - f} b_{n,l} = f - 1$$

and, from (16),

$$\begin{aligned} \sum_{l < \lambda_f + n + 1 - f} b_{n+1,l} &= \lambda_f + n + 1 - f - 1 - \#\{l < \lambda_f + n + 1 - f : b_{n+1,l} = 0\} \\ &= \lambda_f + n + 1 - f - 1 \\ &\quad - \#\{g \in [n] : n + 1 + g - \lambda_g < \lambda_f + n + 1 - f\} \\ &= \lambda_f + n - f - \#\{g \in [n] : \lambda_f + \lambda_g > f + g\} \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{1}{2} i_0(A) &= \sum_{f=1}^{n} \{\lambda_f + n - f - (f - 1)\} - \#\{(f, g) \in [n] \times [n] : \lambda_f + \lambda_g > f + g\} \\ &= |\lambda| - r(\lambda). \end{aligned}$$

(4) The number of -1s in the $(n + 1)$-th row of A is equal to

$$\#\{l \in [n] : b_{n,l} = 0, b_{n+1,l} = 1\},$$

which is $u(\lambda)$ by (15) and (16). □

Our proof of Theorem 3.1 needs the following generalization of the Littlewood's formula.

LEMMA 3.5.

$$\prod_{i=1}^{n}(1-x_i)(1+tx_i)\prod_{1\le i<j\le n}(1-x_ix_j)$$
$$=\sum_{\lambda}(-1)^{(q(\lambda)+r(\lambda))/2}t^{|\lambda|-r(\lambda)}\left(1\mp\frac{1}{t}\right)^{u(\lambda)}s_\lambda(x_1,\dots,x_n),$$

summed over all partitions λ satisfying (14).

Remark. If $t=0$ and 1, then the above Lemma reduces to the known Littlewood's formulas (see[1, p. 238], [2, I. Ex. 5.9.]):

$$\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)=\sum_{\tau}(-1)^{(|\tau|+p(\tau))/2}s_\tau(x_1,\dots,x_n)$$
$$\prod_{i=1}^{n}(1-x_i^2)\prod_{1\le i<j\le n}(1-x_ix_j)=\sum_{\rho}(-1)^{|\rho|/2}s_\rho(x_1,\dots,x_n)$$

where τ (resp. ρ) runs over all partitions of the form $\tau=(\alpha_1,\dots,\alpha_p|\alpha_1,\dots,\alpha_p)$ (resp. $\rho=(\alpha_1+1,\dots,\alpha_p+1|\alpha_1,\dots,\alpha_p)$) such that $\alpha_1\le n-1$.

To prove Lemma 3.5, we fix a partition $\lambda=(\alpha|\beta)$ satisfying (14). We put $\varepsilon_k=\min(\alpha_k,\beta_k)$ $(k=1,\dots,p)$ and $\sigma=(\varepsilon|\varepsilon)$. Then

$$\varepsilon=\begin{cases}\alpha_k & \text{if } \beta_k+1>\alpha_k\\ \alpha_k-1 & \text{if } \beta_k+1=\alpha_k\end{cases}$$

And we put

$$S=\{k\in[p]:\beta_k+1>\alpha_k>\beta_{k+1}+1\}.$$

LEMMA 3.6.

$$|\sigma|=r(\lambda)+u(\lambda).$$

Proof. By Lemma 3.2, if $\alpha_p>0$, then

$$u(\lambda)=2\#S,\quad r(\lambda)=p+2\sum_{k=1}^{p}\gamma_k,\quad |\sigma|=p+2\sum_{k=1}^{p}\varepsilon_k,$$

and, if $\alpha_p=0$, then

$$u(\lambda) = 2\#S + 1, \quad r(\lambda) = p - 1 + 2\sum_{k=1}^{p-1} \gamma_k, \quad |\sigma| = p + 2\sum_{k=1}^{p-1} \varepsilon_k,$$

So it is enough to show

$$\varepsilon_k - \gamma_k = \begin{cases} 1 & \text{if } k \in S \\ 0 & \text{if } k \notin S \end{cases} \tag{17}$$

If $\beta_k + 1 > \alpha_k > \beta_{k+1} + 1$, then we have $\gamma_k = \alpha_k - 1$ and $\varepsilon_k = \alpha_k$; hence, $\varepsilon_k - \gamma_k = 1$. If $\beta_k + 1 = \alpha_k > \beta_{k+1} + 1$, then we have $\gamma_k = \varepsilon_k = \alpha_k - 1$. If $\beta_k + 1 > \alpha_k = \beta_{k+1} + 1$, then we have $\gamma_k = \varepsilon_k = \alpha_k$. Therefore, we obtain (17). □

For a subset J of S, we put

$$\varepsilon(J)_k = \begin{cases} \varepsilon_k - 1 & \text{if } k \in J \\ \varepsilon_k & \text{if } k \notin J \end{cases}$$

It is checked that $\varepsilon(J)_1 > \varepsilon(J)_2 > \ldots > \varepsilon(J)_p \geq 0$. So we can define a partition $\sigma(J) = (\varepsilon(J)|\varepsilon(J))$. Similarly, if $\alpha_p = 0$, then we can define a self-conjugate partition $\overline{\sigma(J)} = (\overline{\varepsilon(J)}|\overline{\varepsilon(J)})$, where

$$\overline{\varepsilon(J)} = (\varepsilon(J)_1, \ldots, \varepsilon(J)_{p-1})$$

LEMMA 3.7.

(1) $\lambda/\sigma(J)$ *is a vertical strip, i.e.,* $0 \leq \lambda_i - \sigma(J)_i \leq 1$ *for all* i.
(2) $\lambda/\overline{\sigma(J)}$ *is a vertical strip.*
(3) If π *is a self-conjugate partition such that* $p(\pi) = p$ *and* λ/π *is a vertical strip, then there exists a subset* J *and* S *such that* $\pi = \sigma(J)$.
(4) If π *is a self-conjugate partition such that* $p(\pi) = p - 1$ *and* λ/π *is a vertical strip, then there exists a subset* J *of* S *such that* $\pi = \overline{\sigma(J)}$.

Proof. (1) If $k \leq p$, then $\varepsilon(J)_k = \alpha_k$ or $\alpha_k - 1$, hence

$$\lambda_k - \sigma(J)_k = \alpha_k + k - (\varepsilon(J)_k + k) \leq 1$$

Let $k > p$ and suppose that $\lambda_k - \sigma(J)_k \geq 2$. Then, by (5), there exists an integer i such that

$$\beta_i + i \geq k > \varepsilon(J)_i + i, \quad \beta_{i+1} + i + 1 \geq k > \varepsilon(J)_{i+1} + i + 1$$

so that $\beta_{i+1} > \varepsilon(J)_i$.

(i) If $i \in J$, then it follows from $\beta_i + 1 > \alpha_i$ (resp. $\beta_i + 1 = \alpha_i$) that $\varepsilon_i = \alpha_i \geq \beta_{i+1} + 1$ (resp. $\varepsilon_i = \alpha_i - 1 \geq \beta_{i+1} + 1$), which contradicts $\beta_{i+1} > \varepsilon(J)_i$.

(ii) If $i \notin J$, then $\varepsilon(J)_i = \alpha_i - 1$, so we have $\beta_{i+1} + 1 \geq \alpha_i$, which contradicts $\alpha_i > \beta_{i+1} + 1$.

Therefore we have $\lambda_k - \sigma(J)_k \leq 1$ for any k.

(2) This follows from (1).

(3) Let $\pi = (\eta|\eta)$ be a self-conjugate partition such that $p(\pi) = p$ and λ/π is a vertical strip. Then, putting $J = \{k \in [p] : \varepsilon_k - \eta_k = 1\}$, we show that $J \subset S$. Suppose that $\varepsilon_k - \eta_k = 1$. If $\beta_k + 1 = \alpha_k$, then $\varepsilon_k = \alpha_k - 1$, hence, $\lambda_k - \pi_k = 2$, which contradicts the assumption that λ/π is a vertical strip. If $\alpha_k = \beta_{k+1} + 1$, then $\varepsilon_k = \alpha_k$ and it follows from $\pi = \pi'$ that $\lambda_m - \pi_m \geq 2$ where $m = k + \lambda_k$, which also contradicts the assumption. Hence we see that $J \subset S$ and $\pi = \sigma(J)$.

(4) This also follows from (3). □

The following lemma is well known.

LEMMA 3.8.

(1)

$$s_{(1^r)}s_\tau = \sum_\mu s_\lambda$$

summed over all partitions μ such that μ/τ is a vertical r-strip

(2) If τ is a self-conjugate partition and μ/τ is a vertical strip, then μ satisfies (14).

(3)

$$\sum_{r=0}^{n} s_{(1^r)}(x_1, \ldots, x_n)t^r = \prod_{i=1}^{n}(1 + x_i t)$$

Proof. (1) See [2, I.(5.17)]; (2) is easy; (3) see [2, I.(2.2)]. □

Now we are in position to prove Lemma 3.5.

Proof of Lemma 3.5. If $u(\lambda)$ is even, then the coefficient of s_λ on the right-hand side of Lemma 3.5 is equal to

$$\sum_{J\subset S}(-1)^{(q(\lambda)+r(\lambda))/2+(\#S-\#J)}t^{|\lambda|-r(\lambda)-2(\#S-\#J)}.$$

By Lemma 3.6 and the definition of $\sigma(J)$, it is equal to

$$\sum_{J\subset S}(-1)^{(|\sigma(J)|+p(\sigma(J)))/2}t^{|\lambda|-|\sigma(J)|}.$$

Similarly, if $u(\lambda)$ is odd, then the coefficient of s_λ on the right-hand side of Lemma 3.5 is equal to

$$\sum_{J\subset S}(-1)^{(|\sigma(J)|+p(\sigma(J)))/2)}t^{|\lambda|-|\sigma(J)|}+\sum_{J\subset S}(-1)^{(|\overline{\sigma(J)}|+p(\overline{\sigma(J)}))/2}t^{|\lambda|-|\overline{\sigma(J)}|}$$

Hence, it follows from Lemma 3.7 that the right-hand side of Lemma 3.5 is

$$\sum_{\tau}(-1)^{(|\tau|+p(\tau))/2}\sum_{\lambda}t^{|\lambda|-|\tau|}s_\lambda,$$

where, in the above summations, τ runs over all self-conjugate partitions such that $\tau\subset(n^n)$, and λ runs over all partitions such that λ satisfies (14) and λ/τ is a vertical strip. But, for fixed τ, Lemma 3.8 implies

$$\sum_{\lambda}t^{|\lambda|-|\tau|}s_\lambda=\prod_{i=1}^{n}(1+tx_i)s_\tau.$$

Therefore Lemma 2.4 completes the proof of Lemma 3.5. □

Remark. By similar argument, we can prove

$$\prod_{i=1}^{n}(1+tx_i)\prod_{1\le i<j\le n}(1+x_ix_j)$$
$$=\sum_{\lambda}t^{|\lambda|-2t(\lambda)}\left(1+\frac{1}{t^2}\right)^{v(\lambda)}s_\lambda(x_1,\ldots,x_n),$$

where $\lambda=(\alpha|\beta)$ runs over all partitions satisfying

$$n-1\ge\beta_1\ge\alpha_1\ge\beta_2\ge\alpha_2\ge\ldots\ge\beta_{p(\lambda)}\ge\alpha_{p(\lambda)}$$

and

$$t(\lambda)=\#\{(k,m):k\le m,\lambda_k+\lambda_{m+1}>k+m\}$$
$$v(\lambda)=\#\{(k,m):k\le m,\ \lambda_k+\lambda_{m+1}=k+m\}$$

It would be interesting to give a bijective proof to this identity.

Proof of Theorem 3.1. By substituting tx_i for x_i in Lemma 3.5 and using Proposition 1.2, we obtain

$$\prod_{i=1}^{n}(1-tx_i)(1+t^2x_i)\prod_{1\le i<j\le n}(1-t^2x_ix_j)(1-t^2x_ix_j^{-1})$$

$$= \sum_{\lambda:(3.4)}\sum_{T\in\mathcal{M}(\lambda+\delta_n)}(-1)^{\max(T)+\mathrm{sp}(T)+(q(\lambda)+r(\lambda))/2}$$

$$\times t^{2\max(T)+2\mathrm{sp}(T)+2|\lambda|-r(\lambda)}\left(1\mp\frac{1}{t}\right)^{2\mathrm{sp}(T)+u(\lambda)} x^T x_1^{-1}\ldots x_n^{-n}.$$

Now by using (11)–(13), Proposition 3.2 and Lemma 3.3, we have

$$\prod_{i=1}^{n}(1-tx_i)(1+t^2x_i)\prod_{1\le i<j\le n}(1-t^2x_ix_j)(1-t^2x_ix_j^{-1})$$

$$= \sum_{A\in\mathcal{C}_n}(-1)^{i_1^+(A)+i_2(A)/2}t^{i(A)}\left(1\mp\frac{1}{t}\right)^{s(A)} x^{\delta(C_n)-A\delta(C_n)}.$$

□

If we put $W(C_n)=\{A\in\mathcal{C}_n : s(A)=0\}$, then $W(C_n)$ is the Weyl group of the root system

$$\Delta(C_n)=\{\pm 2\varepsilon_i : i\in[n]\}\cup\{\pm\varepsilon_i\pm\varepsilon_j : i,j\in[n], i<j\},$$

where $\varepsilon_i = {}^t(0,\ldots,0,\overset{i}{1},0,\ldots,0,\overset{2n+2-i}{-1},0,\ldots,0)$. By substituting $t=1$ in Theorem 3.1, we obtain the denominator formula.

COROLLARY 3.9. (to Theorem 3.1).

$$\prod_{i=1}^{n}(1-x_i^2)\prod_{1\le i<j\le n}(1-x_ix_j)(1-x_ix_j^{-1}) = \sum_{A\in W(C_n)}(-1)^{l(A)}x^{\delta(C_n)-A\delta(C_n)},$$

where $l(A)=i_1(A)+i_2(A)/2$ *is the length of* $A\in W(C_n)$.

4. Deformation for D_n Type

Finally we give a deformation for the root system of type D_n.

Definition. Let $\mathcal{D}_n$ be the set of all $2n\times(2n-1)$ matrices $A=(a_{ij})_{1\le i\le 2n, 1\le j\le 2n-1}$ satisfying the following conditions:

(1) Every row is sign-alternating.
(2) Every column, except for the nth column, is sign-alternating.
(3) $a_{ij}=a_{2n+1-i,2n-j}$.

(4) The vector $(a_{1n}, \ldots, a_{nn})$ is sign-alternating and $\sum_{i=1}^{n} a_{in} = 1$.

Let $L = \{(i, j; k, l) \in \Sigma_{2n,2n-1} \times \Sigma_{2n,2n-1} : i < k, j > l\}$ and define subsets L_1, L_2, L_+, and $L_\pm$ of L as follows:

$$\begin{aligned} L_1 &= \{(i, j; k, l) \in L : i + k = 2n + 1, j + l = 2n\}, \\ L_2 &= L - L_1, \\ L_+ &= \{(i, j; k, l) \in L : i \leq n, k \leq n\}, \\ L_\pm &= \{(i, j; k, l) \in L : i \leq n, k \geq n + 1\}. \end{aligned}$$

For each subset L_*, $* = 1, 2, +, \pm$ and $A \in \mathcal{D}_n$, we put

$$i_*(A) = \sum_{(i,j;k,l) \in L_*} a_{ij} a_{kl}.$$

Moreover, for $A \in D_n$, we put

$$\begin{aligned} i_1^+(A) &= \#\{(i, j) : 1 \leq i \leq n, n \leq j \leq 2n - 1, a_{ij} = 1\}, \\ i_1^-(A) &= \#\{(i, j) : 1 \leq i \leq n, n \leq j \leq 2n - 1, a_{ij} = -1\}. \end{aligned}$$

THEOREM 4.1.

$$\prod_{1 \leq i < j \leq n} (1 + t x_i x_j)(1 + t x_i x_j^{-1}) = \sum_{A \in \mathcal{D}_n} t^{i_1^-(A) + i_2(A)/2} \left(1 + \frac{1}{t}\right)^{s(A)/2} x^{\delta(D_n) - A\delta'(D_n)},$$

where $\delta(D_n) = {}^t(n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -(n-1))$ *and* $\delta'(D_n) = {}^t(n-1, n-2, \ldots, 1, 0, -1, \ldots, -(n-1))$.

We can prove this theorem in a way similar to that of Theorem 2.1, so we omit the proof.

Let $W(D_n)$ be the subgroup of $W(B_n)$ consisting of matrices A such that $i_1(A)$ is even. Then $W(D_n)$ is the Weyl group of

$$\Delta(D_n) = \{\pm\varepsilon_i \pm \varepsilon_j : i, j \in [n], i < j\},$$

where $\varepsilon_i = {}^t(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0, \overset{2n+1-i}{-1}, 0, \ldots, 0)$. The subset $\overline{W}(D_n) = \{A \in \mathcal{D}_n : s(A) = 0\}$ of $\mathcal{D}_n$ can be identified with the Weyl group $W(D_n)$ as follows.

PROPOSITION 4.2. *For* $A \in W(D_n)$, *let* $\overline{A} = (\overline{a_{ij}})_{1 \leq i \leq 2n, 1 \leq j \leq 2n-1}$ *be the matrix defined by*

$$\overline{a_{ij}} = \begin{cases} a_{ij} & \text{if } j < n \\ a_{i,n} + a_{i,n+1} & \text{if } j = n \\ a_{i,j+1} & \text{if } j > n \end{cases}$$

Then this correspondence $A \mapsto \overline{A}$ is a bijection between $W(D_n)$ and $\overline{W}(D_n)$. Moreover the length of A is given by

$$l(A) = i_2(\overline{A})/2.$$

Therefore, substituting -1 for t in Theorem 4.1, we obtain the Weyl's denominator formula.

COROLLARY 4.3. (to Theorem 4.1).

$$\prod_{1\le i<j\le n} (1 - x_i x_j)(1 - x_i x_j^{-1}) = \sum_{A\in W(D_n)} (-1)^{l(A)} x^{\delta(D_n)-A\delta(D_n)}$$

By considering $2n \times (2n + 1)$ matrices, we can give another deformation for the root system of type C_n.

Definition. Let C'_n be the set of all $2n \times (2n+1)$ matrices $A = (a_{ij})_{1\le i\le 2n, 1\le j\le 2n+1}$ satisfying the following conditions:

(1) Every row is sign-alternating.
(2) Every column, except for the $(n + 1)$-th column, is sign-alternating.
(3) $a_{ij} = a_{2n+1-i, 2n+2-j}$.
(4) The vector $(a_{1,n+1}, \ldots, a_{n,n+1})$ is a sign-alternating vector and $\sum_{i=1}^{n} a_{i,n+1} = 0$.

Let $L = \{(i, j; k, l) \in \Sigma_{2n,2n+1} \times \Sigma_{2n,2n+1} : i < k, j > l\}$ and define subset L_1, L_2, L_+ and $L_\pm$ of L as follows:

$$\begin{aligned} L_1 &= \{(i, j; k, l) \in L : i + k = 2n + 1, j + l = 2n + 2\}, \\ L_2 &= L - L_1, \\ L_+ &= \{(i, j; k, l) \in L : i \le n, k \le n\}, \\ L_\pm &= \{(i, j; k, l) \in L : i \le n, k \ge n + 1\}. \end{aligned}$$

For each subset L_*, $* = 1, 2, +, \pm$ and $A \in C'_n$, we put

$$i_*(A) = \sum_{(i,j;k,l)\in L_*} a_{ij} a_{kl}$$

Moreover, for $A \in C'_n$, we put

$$\begin{aligned} i_1^+(A) &= \#\{(i, j) : 1 \le i \le n, n + 2 \le j \le 2n + 1, a_{ij} = 1\}, \\ i_1^-(A) &= \#\{(i, j) : 1 \le i \le n, n + 2 \le j \le 2n + 1, a_{ij} = -1\}. \end{aligned}$$

THEOREM 4.4.

$$\prod_{i=1}^{n}(1 - tx_i^2) \prod_{1\le i<j\le n} (1 + tx_i x_j)(1 + tx_i x_j^{-1})$$

$$= \sum_{A \in C_n'} t^{i_1^+(A)+i_2(A)/2} \left(1 + \frac{1}{t}\right)^{s(A)/2} x^{\delta'(C_n) - A\delta(C_n)}$$

where $\delta(C_n) = {}^t(n, n-1, \ldots, 1, 0, -1, \ldots, -n)$ *and* $\delta'(C_n) = {}^t(n, n-1, \ldots, 1, -1, \ldots, -n)$.

If $t = -1$, then this theorem reduces to the denominator formula.

References

1. D.E. Littlewood, *The Theory of Group Characters*, Oxford University Press, London, 2nd edition, 1950.
2. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, Oxford, 1979.
3. W.H. Mills, D.P. Robbins, and H. Rumsey, Jr, "Alternating sign matrices and descending plane partitions," *J. Combin. Theory Ser. A* **34** (1983) 340–359.
4. W.H. Mills, D.P. Robbins, and H. Rumsey, Jr, "Self-complementary totally symmetric plane partitions," *J. Combin. Theory Ser. A* **42** (1986) 277–292.
5. S. Okada, "Partially strict shifted plane partitions," *J. Combin. Theory Ser. A* **53** (1990) 143–156.
6. D.P. Robbins and H. Rumsey, Jr, "Determinants and alternating sign matrices," *Adv. Math.* **62** (1986) 169–184.
7. T. Tokuyama, "A generating function of strict Gelfand patterns and some formulas on characters of general linear group," *J. Math. Soc. Japan* **40** (1988) 671–685.