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The Automorphism Group and the Convex Subgraphs of the Quadratic Forms Graph in Characteristic 2

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Abstract. We determine the automorphism group and the convex subgraphs of the quadratic forms graph $\mathrm{Quad}(n, q)$, q even.

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1. Introduction

The quadratic forms graph $\mathrm{Quad}(n, q)$ was introduced by Egawa [2]. It has as vertices all quadratic forms on an n-dimensional vector space over $GF(q)$. Two forms f and g are adjacent in $\mathrm{Quad}(n, q)$ whenever rank $(f - g) = 1$ or 2. If $n = 1$ or 2, then $\mathrm{Quad}(n, q)$ is a complete graph; the graph $\mathrm{Quad}(3, 2)$ is isomorphic to the distance-transitive graph $\mathrm{Alt}(4, 2)$ [2]. If $n \geq 3$ and $(n, q) \neq (3, 2)$ then $\mathrm{Quad}(n, q)$ is distance-regular, but not distance-transitive. In this paper we investigate the properties of the graph $\mathrm{Quad}(n, q)$ when q is even. We determine its automorphism group and describe all its convex (i.e., geodetically closed) subgraphs. In the case of odd q both problems have been solved earlier. The group $\mathrm{Aut}\,\mathrm{Quad}(n, q)$, q odd, was determined in [4]. The convex subgraphs of $\mathrm{Quad}(n, q)$, q odd, were determined in [5].

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Over a field of odd characteristic, the quadratic forms graph is the {1,2}-distance graph of the corresponding symmetric bilinear forms graph. In characteristic 2 this construction fails, and maybe it is the reason why $\mathrm{Aut}\,\mathrm{Quad}(n, q)$, q even, was not determined earlier. It is quite clear that the mappings $f \mapsto f + g$ ($g \in \mathrm{Quad}(n, q)$) and $f \mapsto \theta^{-1} \circ f \circ g$ ($g \in \Gamma L_n(q)$, θ is the field automorphism associated with g) are automorphisms of $\mathrm{Quad}(n, q)$. Together they generate a group $\mathrm{Aut}^0\,\mathrm{Quad}(n, q) \cong q^{n(n+1)/2} : \Gamma L_n(q)$, which was thought to be the full automorphism group of $\mathrm{Quad}(n, q)$. It is a little bit of a surprise that this natural conjecture fails, and $\mathrm{Quad}(n, q)$, q even, has more automorphisms.

THEOREM 1.1. *Assume* $n \geq 3$ *and* $(n, q) \neq (3, 2)$. *If* q *is a power of 2, then* $\mathrm{Aut}\,\mathrm{Quad}(n, q)$ *is the product of* $\mathrm{Aut}^0\,\mathrm{Quad}(n,q)$ *and the group of order* q^n *constituted by the following automorphisms:*

$$f \mapsto f + B_f(\cdot, v)^2 \quad (v \in V),$$

here B_f *is the alternating form associated with* f, *and* V *the basic* n*-dimensional vector space over* $GF(q)$.

It is easy to check that the permutations given in the theorem are indeed automorphisms of $\mathrm{Quad}(n, q)$ and that they form an elementary abelian group E of order q^n. Also, a nonidentity automorphism $e \in E$ preserves the rank of every form, but maps some (even rank) forms of plus type to forms of minus type, and vice versa. In particular, E has trivial intersection with $\mathrm{Aut}^0\,\mathrm{Quad}(n, q)$.

For a subspace $U \leq V$ and a form $f \in \mathrm{Quad}(n, q)$ let $Q(f, U)$ denote the subgraph in $\mathrm{Quad}(n, q)$ induced by the set of forms $\{f + g \mid \mathrm{Rad}\,(g) \geq U\}$. Then $Q(f, U)$ is naturally isomorphic to $\mathrm{Quad}(n - \dim U, q)$. Since the parameters c_i of $\mathrm{Quad}(n, q)$ do not depend on n, the subgraphs $Q(f, U)$ are convex.

THEOREM 1.2. *If* Σ *is a noncomplete convex subgraph of* $\mathrm{Quad}(n, q)$, *q even, then* $\Sigma = Q(f, U)$ *for some* $f \in \mathrm{Quad}(n, q)$ *and a subspace* $U < V$, $\dim U \leq n - 3$.

The maximal cliques of $\mathrm{Quad}(n, q)$ were determined in [3].

2. Preliminaries

Throughout the paper Γ denotes the graph $\mathrm{Quad}(n, q)$ and V denotes the n-dimensional vector space over $GF(q)$ on which the quadratic forms from Γ (and associated alternating forms) are defined, q is a power of 2. Here and below we use the same letter to denote a graph and its set of vertices.

For a given $f \in \Gamma$, we denote by $R(f)$ the set $\{f + h \mid \mathrm{rank}\,(h) = 1\}$. Clearly, $R(f)$ is a clique of size q^n; after [1, section 9.6], we call such cliques R_1-*cliques*. If B_f denotes the alternating form associated with f (i.e., $B_f(x, y) =$

$f(x + y) - f(x) - f(y)$ then $B_f = B_g$ if and only if $R(f) = R(g)$. It will be convenient to identify R_1-cliques and corresponding alternating form. Under this identification we will use notation like Rad $(X - Y)$, rank (X), where X and Y are R_1-cliques. Two R_1-cliques $R(f)$ and $R(g)$ are at distance 1 from each other whenever rank $(B_f - B_g) = 2$. It means that the identification of $R(f)$ and B_f reveals the structure of the alternating forms graph Alt(n, q) on the set Δ of R_1-cliques.

Every R_1-clique X naturally carries the structure of an n-dimensional affine space (denote it by Aff X). The i-dimensional subspaces of Aff X are all sets of the form $S(f, U) = \{f + h | h \in R(0), \text{Rad } (h) \geq U\}$, where $f \in X$ and U is an $(n - i)$-dimensional subspace of V. When U is fixed and f runs through X, the sets $S(f, U)$ form a parallel class of i-spaces in Aff X. Notice also that the intersection of subspaces $S(f, U_1)$ and $S(f, U_2)$ is the subspace $S(f, \langle U_1, U_2\rangle)$.

LEMMA 2.1. *Suppose R_1-cliques X and Y are at distance s from each other. Then for every $f \in X$ one has that $\Gamma_s(f) \cap Y = S(g, \text{Rad } (X - Y))$ for some $g \in Y$, i.e., $\Gamma_s(f) \cap Y$ is a subspace of* Aff Y *of dimension $2s$.*

For an alternating form $B \in$ Alt(n, q) and a subspace $U \leq V$ let $A(B, U) = \{B + B' | \text{ Rad } (B') \geq U\}$. The subgraph $A(B, U)$ of Alt(n, q) is naturally isomorphic to Alt$(n-$ dim $U, q)$. The following result was proved in [5, Proposition (5.26)].

LEMMA 2.2. *Every noncomplete convex subgraph of $\Delta =$ Alt(n, q) coincides with $A(B, U)$ for some B and $U < V$,* dim$U \leq n - 4$.

LEMMA 2.3. *If $n \geq 3$ and $(n, q) \neq (3, 2)$ then Γ does not contain a subgraph $\Sigma \cong$ Alt(n, q) such that Σ intersects every R_1-clique of Γ in exactly one vertex.*

Proof. Suppose that there exists such a subgraph Σ. Without loss of generality we may assume that 0 is a vertex of Σ. First consider the case $q > 2$ and $n \geq 3$.

It is known [1, proof of 9.5.6] that the singular lines of Alt(n, q), containing the zero alternating form, are the sets $L(B) = \{\alpha B | \alpha \in GF(q)\}$, where B is a rank 2 alternating form (in case $n = 3$ Alt$(3, q)$ is complete and still $L(B)$ is a part of the (trivial) singular line). Let $L = \{0, f, g, \ldots\}$ be the preimage of $L(B)$ in Σ. Then, clearly, Rad $(f) =$ Rad (g). Let $h \in \{0, f\}^{\perp}$ (in Γ) and let $t \in R(h) \cap \Sigma$. Then t is adjacent to 0, f and g. Since h also is adjacent to f, Rad $(h - t) >$ Rad(f). Therefore, the equality Rad $(f) =$ Rad (g) implies that h is adjacent to g. It means that L is contained in a singular line of Γ. On the other hand, it is known [2, Lemma 3.6] that, for an f of rank 2, the singular line $\{0, f\}^{\perp\perp}$ of Γ has size 2. Since $|L| = q > 2$, we obtain a contradiction.

Now let $q = 2$ and $n \geq 4$. By [7, Proposition 3.1], the local graph $\Gamma(0)$ is isomorphic to the Grassmann graph $\left[{W \atop 2}\right]$ for an $(n + 1)$-dimensional vector space

W over $GF(2)$. Let us identify $\Gamma(0)$ with $[{W \atop 2}]$.

Choose $f \in \Sigma_2(0)$. Then by Lemma 2.2 the convex closure of 0 and f (in Σ) is a subgraph $\Theta \cong \mathrm{Alt}(4, 2)$. The local subgraph $\Theta(0)$ is isomorphic to the Grassmann graph $[{U \atop 2}]$ for a 4-dimensional vector space U over $GF(2)$. By [5, Proposition (5.14)], the isomorphism between $\Theta(0)$ and $[{U \atop 2}]$ can be established by choosing as U certain 4-dimensional subspace of W (i.e., $\Theta(0) = [{U \atop 2}]$ for that subspace U). On the other hand, f must be rank 4 as a quadratic form (otherwise, $\Gamma(f)$ contains a rank 1 form). It was proved in [7, Lemma 5.7 and the remark after it] that, for a rank 4 form f, $\Gamma(0)\cap\Gamma(f)$ is not contained in $[{U \atop 2}]$ for any 4-dimensional subspace $U < W$; a contradiction since $\Gamma(0)\cap\Gamma(f) = \Theta(0)\cap\Theta(f)$. □

3. The automorphism group

In this section we prove Theorem 1.1. Throughout the section we assume that $n \geq 3$. If $(n, q) \neq (3, 2)$, it was shown in [2] that $G = \mathrm{Aut}\,\mathrm{Quad}(n, q)$does not mix the edges (f, g) with rank $(f-g) = 1$ and the edges with rank $(f-g) = 2$. It implies that, under the above restriction, G leaves the set of R_1-cliques invariant. It defines a homomorphism of G into the group $\mathrm{Aut}\,\mathrm{Alt}(n, q)$. The proof of Theorem 1.1 will be given in two steps. First we show that G and the group $G_0 = E \cdot \mathrm{Aut}^0\,\mathrm{Quad}(n, q)$ from the introduction have the same kernel under this homomorphism. The second step is to show that G and G_0 have the same image in $\mathrm{Aut}\,\mathrm{Alt}(n, q)$.

Let K be the subgroup of G, consisting of all the automorphisms of Γ, stabilizing every R_1-clique. Notice that until Proposition 3.2 we do not exclude the case $(n, q) = (3, 2)$.

LEMMA 3.1. *For every R_1-clique X, K induces on X the group of translations of* Aff X.

Proof. It suffices to prove that every $a \in K$ induces a translation of Aff X. It follows from Lemma 2.1 that a preserves every parallel class of subspaces in Aff X, and hence a is either a translation or a collineation. Suppose a is a nontrivial collineation with a center $x \in X$ (in particular, $q > 2$). Take an R_1-clique Y at distance 1 from X. By Lemma 2.1, $S = \Gamma(x) \cap Y$ is a plane of Aff Y, and $T = \Gamma(y) \cap X$ (y is an arbitrary element of S) is a plane in Aff X. Clearly, a stabilizes T and does not stabilize every plane parallel to T. By Lemma 2.1, adjacency relation establishes a bijection between the parallel classes of S and T (in Aff Y and Aff X, respectively). Consequently, a stabilizes S and does not stabilize every plane of Aff Y, parallel to S. It follows that a cannot induce a translation of Aff Y, hence it induces a nontrivial collineation.

By connectivity, a induces a nontrivial collineation on every R_1-clique Y. In particular, a fixes exactly one form t_Y in every Y. Moreover, the above argument shows that t_{Y_1} and t_{Y_2} are neighbors whenever Y_1 and Y_2 are such. It means that the subgraph Σ induced by all t_Y's is naturally isomorphic to Alt(n, q). This contradicts Lemma 2.3. □

LEMMA 3.2. *Let $A \leq K$ and let Σ be the subgraph induced by all the vertices of Γ, fixed by A. Then*

(1) for every $f \in \Sigma$, one has $R(f) \subseteq \Sigma$;
(2) the image of Σ in $\Delta =$ Alt(n, q) is a convex subgraph.

Proof. The part (1) follows from Lemma 3.1. For (2), let $X, Y \subseteq \Sigma$ be two R_1-cliques at distances $s > 1$ from each other. Suppose that an R_1-clique T lies on a shortest path from X to Y, say, rank $(X - Y) = 2i$ and rank $(T - Y) = 2j = \text{rank}(X - Y) - \text{rank}(X - T) = 2s - 2i$. By Lemma 2.1, for $x \in X$ and $y \in Y$, $S_1 = \Gamma_i(x) \cap T$, is a $2i$-dimensional subspace of Aff T, and $S_2 = \Gamma_j(y) \cap T$ is a $2j$-dimensional subspace of Aff T. Clearly, we may choose x and y in such a way that $S_1 \cap S_2 \neq \emptyset$. According to the remark before Lemma 2.1, $S_1 \cap S_2 = S(f, \langle \text{Rad}\ (X - T), \text{Rad}\ (T - Y)\rangle)$ for an $f \in S_1 \cap S_2$. But the equality rank $(X - Y) = \text{rank}(X - T) + \text{rank}(T - Y)$ implies that $\langle \text{Rad}\ (X - T), \text{Rad}\ (T - Y)\rangle = V$ [1, Lemma 9.5.5 (i)]. It means that $S_1 \cap S_2$ consists of a unique form, and hence A fixes T elementwise by Lemma 3.1. □

PROPOSITION 3.1. *K is contained in G_0.*

Proof. It is easy to see that $|K \cap G_0| = q^{2n}$. Hence it suffices to prove that $|K| \leq q^{2n}$.

First we consider the case of even n (recall that $n \geq 3$). Let f be a form of rank n. By Lemma 2.2, the geodetic closure of $\{R(0), R(f)\}$ coincides with the whole of Δ. Hence Lemma 3.2 implies that the stabilizer in K of both 0 and f is trivial. Therefore, $|K| \leq |R(0)| \cdot |R(f)| = q^{2n}$.

Now let us separately consider the case $n = 3$. Choose a rank 2 form f and let $X = R(f)$. Let Y be an R_1-clique such that Rad $(Y) \neq$ Rad (X). This condition and Lemma 2.1 imply that $\Gamma(0) \cap Y$ and $\Gamma(f) \cap Y$ are nonparallel planes of Aff Y. Hence $L = \Gamma(0) \cap \Gamma(f) \cap Y$ is a line of Aff Y. Pick a form $g \in L$. If T is an R_1-clique such that Rad $(T) \not\leq \langle \text{Rad}\ (X)\ \text{Rad}\ (Y)\rangle$, then the line $\Gamma(0) \cap \Gamma(f) \cap T$ is not parallel to the plane $\Gamma(g) \cap T$ and hence $\Gamma(0) \cap \Gamma(f) \cap \Gamma(g) \cap T$ has cardinality 1. It means that an element of K fixes T whenever it fixes 0, f and g. Let T' be an arbitrary R_1-clique different from X, Y and T. Then Rad (T') is not contained in at least one of $\langle$Rad (X), Rad $(Y)\rangle$, $\langle$Rad (X), Rad $(T)\rangle$, and $\langle$Rad (Y), Rad $(T)\rangle$. We can repeat the above argument for appropriate R_1-cliques to obtain that an element K fixes T' whenever it fixes 0, f, and g. Thus we have shown that the subgroup of K fixing 0, f and g is trivial.

Now we can estimate the order of K. The stabilizer of 0 moves f within the plane $\Gamma(0) \cap X$. The stabilizer of 0 and f moves g within the line L. Therefore, $|K| \leq q^{3+2+1} = q^6$.

Finally, let us consider the case $n > 3$ is odd. Choose a form f of the maximal even rank $2s = n - 1$. Let g be a rank 2 form such that Rad $(f) \not\leq$ Rad(g). Then Lemma 2.2 implies that the geodetic closure in Δ of $R(0)$, $R(f)$, and $R(g)$ coincides with the whole of Δ. By Lemma 3.2, the stabilizer in K of 0, f and g is trivial. By Lemma 2.1, $H = \Gamma_s(0) \cap R(f)$ is a hyperplane of Aff $R(f)$. Furthermore, $S_1 = \Gamma_s(f) \cap R(g)$ is a hyperplane of Aff $R(g)$, $S_2 = \Gamma(0) \cap R(g)$ is a plane of Aff $R(g)$ and S_2 is not parallel to S_1. Hence $|K| \leq |R(0)| \cdot |H| \cdot |S_1 \cap S_2| = q^{n+(n-1)+1} = q^{2n}$. □

In case $(n, q) \neq (3, 2)$, K coincides with the kernel of G acting on Δ. Hence, in order to prove Theorem 1.1 it remains to establish the following.

PROPOSITION 3.2. *Assume* $(n, q) \neq (3, 2)$. *Then the images of* G *and* G_0 *in* Aut Δ *coincide.*

Proof. In what follows, a bar over an element, or a subgroup of G means taking image in the action on Δ.

If $n \geq 5$, then Aut Δ is known to be $\bar{G}_0$ [6], so the assertion is trivial. In case $n = 4$ the automorphism group of Δ is twice larger (i.e., Aut $\Delta = \bar{G}_0.2$), since it contains an element interchanging the two classes $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of maximal cliques in Δ. These two classes are represented, respectively, by the cliques

$$Q_1 = \{B \in \Delta(0) | \text{Rad } (B) \leq U_1\} \cup \{0\}$$

and

$$Q_2 = \{B \in \Delta | \text{Rad } (B) \geq U_2\},$$

where U_1 and U_2 are fixed subspaces of V of dimensions 3 and 1, respectively. Let $f \in \Gamma$ be a rank 1 form, such that Rad $f = U_1$. Then 0 and f have the same set of neighbors in the preimage of Q_1. On the other hand, the preimage of Q_2, naturally isomorphic to Quad(3, q), does not contain a pair of vertices with this property. It follows that $\bar{G}$ leaves $\mathcal{Q}_1$ and $\mathcal{Q}_2$ invariant and, hence, $\bar{G} = \bar{G}_0$.

It remains to consider the case $n = 3$, $q > 2$. In this case Δ is a complete graph with q^3 vertices. For $f \in \Gamma$ and a subspace $U < V$, dim $U = n - 2$, the subgraph $Q(f, U) \cong$ Quad(2, q) (see Introduction) is a clique of size q^3. Such cliques will be called *cubic*. Consider the plane $S = S(x, U)$ of Aff $R(x)$. Then $S^\perp$ coincides with the union of $R(x)$ and the cubic clique $Q = Q(x, U)$. Moreover, no element from $Q \setminus S$ is adjacent to an element from $R(x) \setminus S$. Since the set of planes of Aff X, where X runs through all R_1-cliques, is invariant under G, this implies that the set of cubic cliques is also invariant under the action of G. Now the vertices of Δ and the images of the cubic cliques (each image consists of q elements) form a 3-dimensional affine space over $GF(q)$.

Since $\bar{G}_0$ is isomorphic to the full automorphism group of that affine space, we obtain $\bar{G} = \bar{G}_0$. □

4. Convex subgraphs

In this section we prove Theorem 1.2. Clearly, it suffices to consider the case $n \geq 3$; we start with $n = 3$.

PROPOSITION 4.1. *Let $n = 3$ and let $f \in \Gamma$ be a form of rank 3. Then the convex closure of 0 and f coincides with the whole of Γ.*

Proof. Let C denote the convex closure of 0 and f. Let X be an R_1-clique such that Rad $(X) \neq$ Rad $(R(f))$. Then the line $L_1 = \Gamma(0) \cap \Gamma(f) \cap X$ of Aff X is contained in C. Choose now an R_1-clique Y, such that Rad $(Y) \not\leq \langle \text{Rad}(R(f)), \text{Rad}(X)\rangle$. Then the line $L_2 = \Gamma(0) \cap \Gamma(f) \cap Y$ of Aff Y also is contained in C. Moreover, for $x \in L_1$ and $y \in L_2$, one has that L_1 is not parallel to the plane $\Gamma(y) \cap X$ and L_2 is not parallel to the plane $\Gamma(x) \cap Y$. Now every $a \in X$ is adjacent to every x on L_1 and to some y on L_2. Clearly, we can choose x to be nonadjacent to that y. Therefore, a belongs to C. It follows that X (and, symmetrically, Y) is contained in C. Notice now that the sets of lines $A(X, U)$, where $X \in \Delta$ and $U < V$ has dimension 1, makes Δ an affine 3-space. In terms of this affine space the above conclusion can be rephrased as follows: if P, Q, $R \in \Delta$ are noncollinear and $P \cap C \neq \emptyset \neq Q \cap C$, then $R \subseteq C$. Now for every R_1-clique T we can choose two out of three R_1-cliques $R(f)$, X and $R(0)$ to form together with T a noncollinear triple. It means that $T \subseteq C$ and hence $C = \Gamma$. □

Recall that $\Delta \cong \text{Alt}(n, q)$ is the graph of R_1-cliques.

LEMMA 4.1. *If Σ is a convex subgraph of Γ then the image of Σ in Δ is convex.*

Proof. Let X and Y be two R_1-cliques at distance $s > 1$ from each other, such that $X \cap \Sigma \neq \emptyset \neq Y \cap \Sigma$. Let T be an R_1-clique such that T is at distance i from X and at distance j from Y with $s = i + j$. It suffices to prove that $T \cap \Sigma$ is nonempty.

Let $x \in X \cap \Sigma$ and $y \in Y \cap \Sigma$. If rank $(x - y) = 2s + 1$ then $\Gamma_s(y) \cap X$ is a nonempty set contained in Σ. Replacing x by any element from $\Gamma_s(y) \cap X$, we may assume without loss of generality that rank $(x - y) = 2s$. By Lemma 2.1, both x and y belong to the subgraph $\Gamma_0 = Q(x, \text{Rad}(X - Y)) \cong \text{Quad}(2s, q)$. Since Rad $(X - Y) =$ Rad $(X - T) \cap$ Rad $(T - Y)$ [1, Lemma 9.5.5 (i)], we have that $T \cap \Gamma_0 \neq \emptyset$. Also Γ_0 is a convex subgraph and hence $\Sigma_0 = \Gamma_0 \cap \Sigma$ is convex. It means that we can substitute Γ_0 and Σ_0 in place of Γ and Σ. In other words we may assume that $n = 2s$.

Finally, $V = \langle \text{Rad } (X - T), \text{Rad } (T - Y) \rangle$ implies that the $2i$-dimensional subspace $\Gamma_i(x) \cap T$ and the $2j$-dimensional subspace $\Gamma_j(y) \cap T$ of Aff T must intersect each other in exactly one point. □

According to this lemma and Lemma 2.2, the image of Σ must be a clique, or a natural subgraph Alt(k, q), $k \leq n$.

PROPOSITION 4.2. *Suppose $\Sigma \subseteq \Gamma$ is convex and its image in Δ is a clique. Then either Σ is a clique, or $\Sigma = Q(f, U) \cong$ Quad$(3, q)$ for an $f \in \Sigma$ and some subspace $U < V$,* dim $U = n - 3$.

Proof. Suppose Σ is not a clique. Then there exist $f, g \in \Sigma$ such that rank $(f - g) = 3$. By Proposition 4.1, the closure Σ_0 of f and g coincides with $Q(f, \text{Rad } (f - g)) \cong$ Quad$(3, q)$. Notice that the image of Σ_0 in Δ is a maximal clique. It follows that the images of Σ and Σ_0 coincide. Suppose $\Sigma \neq \Sigma_0$ and $x \in \Sigma \setminus \Sigma_0$. Let $X = R(x)$ and let $Y \neq X$ be another R_1-clique such that $Y \cap \Sigma \neq \emptyset$. Then also $T \cap \Sigma_0 \neq \emptyset$. Choose $y \in Y \cap \Sigma_0$. Then the plane $\Gamma(y) \cap X$ is contained in Σ_0 and hence rank $(x - y) = 3$. Moreover, Rad $(x - y) \neq$ Rad $(f - g)$. One more application of Proposition 4.1 gives that the image of Σ is bigger than the image of Σ_0; a contradiction. □

PROPOSITION 4.3. *Suppose that Σ is a convex subgraph of Γ such that the image of Σ in Δ coincides with $A(B, U)$ for some alternating form B and subspace $U < V$,* dim $U \leq n - 4$. *Then $\Sigma = Q(f, U)$ for an $f \in \Sigma$.*

Proof. Let $f \in \Sigma$ and $\Sigma_0 = Q(f, U)$. We first prove that $\Sigma \subseteq \Sigma_0$. Indeed, let an R_1-clique $Y \in A(B, U)$ be a neighbor of $R(f)$. Suppose there is $g \in Y \cap \Sigma$ such that $g \notin Q(f, U)$. Then rank $(g - f) = 3$ and Rad $(g - f) \not\geq U$. By Proposition 4.1 it follows that $Q(f, \text{Rad } (g - f)) \subseteq \Sigma$. But $A(B_f \text{ Rad } (g - f)) \not\subseteq A(B, U) = A(B_f, U)$, a contradiction. Hence $Y \cap \Sigma \subseteq \Sigma_0$. By connectivity of $A(B, U)$, we obtain that $\Sigma \subseteq \Sigma_0$. It means that without loss of generality we may assume that Σ covers the whole of Δ (i.e., $U = 0$).

Now we claim that there is an R_1-clique X such that $|X \cap \Sigma| > 1$. Indeed, suppose $|X \cap \Sigma| = 1$ for every R_1-clique X. Let $\{t_X\} = X \cap \Sigma$. If two R_1-cliques X and Y are adjacent then t_X and t_Y are adjacent, otherwise Proposition 4.1 forces a contradiction. Therefore, Σ is isomorphic to Δ; a contradiction with Lemma 2.3, since $n \geq 4$.

Let X be an R_1-clique such that $x_1 \neq x_2 \in X \cap \Sigma$. Let L be the line of Aff X containing x_1 and x_2. Let Y be an R_1-clique at distance 1 from X, such that for $y \in Y$ the plane $\Gamma(y) \cap X$ is not parallel to L. Notice that $\Gamma(x_1) \cap Y \cap \Sigma$ is nonempty. Indeed, if $y \in Y \cap \Sigma$ is not adjacent to x_1 then the whole plane $\Gamma(x_1) \cap Y$ is contained in Σ. Let us choose $y \in \Gamma(x_1) \cap Y \cap \Sigma$. Since L is not parallel to $\Gamma(y) \cap X$, y is not adjacent to x_2. By convexity of Σ, $\Gamma(y) \cap X \subseteq \Sigma$. It follows that all planes of Aff X, which contain x_1 and do not contain x_2, are contained in Σ. Substituting now other elements of X in place of x_2 we establish that $X \subseteq \Sigma$.

Finally, for an R_1-clique Y at distance 1 from X, and for $y \in Y \cap \Sigma$ we can find an $x \in X$ which is not adjacent to y. Therefore, $Y \cap \Sigma \supseteq \Gamma(x) \cap Y$ has cardinality at least q^2. Applying the argument from the previous paragraph to the R_1-clique Y, we obtain $Y \subseteq \Sigma$. Repeating this argument and using the connectivity of Δ we eventually establish that $\Gamma = \Sigma$. □

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