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Parallelogram-free distance-regular graphs having completely regular strongly regular subgraphs

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Abstract Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d. A *parallelogram* of length i is a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, z) = i$, and $\partial(x, w) = \partial(y, w) = \partial(y, z) = i - 1$. A subset Y of X is said to be a *completely regular code* if the numbers

$$\pi_{i,j} = |\Gamma_j(x) \cap Y| \quad (i, j \in \{0, 1, \dots, d\})$$

depend only on $i = \partial(x, Y)$ and j. A subset Y of X is said to be *strongly closed* if

$$\{x \mid \partial(u, x) \le \partial(u, v), \partial(v, x) = 1\} \subset Y, \text{ whenever } u, v \in Y.$$

Hamming graphs and dual polar graphs have strongly closed completely regular codes. In this paper, we study parallelogram-free distance-regular graphs having strongly closed completely regular codes. Let Γ be a parallelogram-free distance-regular graph of diameter $d \ge 4$ such that every strongly closed subgraph of diameter two is completely regular. We show that Γ has a strongly closed subgraph of diameter $d - 1$ isomorphic to a Hamming graph or a dual polar graph. Moreover if the covering radius of the strongly closed subgraph of diameter two is $d - 2$, Γ itself is isomorphic to a Hamming graph or a dual polar graph. We also give an algebraic characterization of the case when the covering radius is $d - 2$.

Keywords Distance-regular graph · Association scheme · Homogeneity · Completely regular code

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1 Introduction

The study of completely regular codes in a distance-regular graph has a long history [3, 5]. Most of the completely regular codes studied are those with large minimum distance because of the requirements to apply the theory to error-correcting codes. Recently Brouwer et al. [2] studied a special class of completely regular codes in a Q-polynomial distance-regular graph satisfying extremal conditions from a different point of view. Let us call these codes extremal. These extremal codes afford induced structure of a Q-polynomial distance-regular graph and hence they are necessarily connected as a graph or minimum distance one. Independently, we studied the Terwilliger algebra with respect to a subset in [9]. The thin condition of the principal module of this Terwilliger algebra is equivalent to the complete regularity of the base subset. We also gave a sufficient condition, called tight, that the module generated by an end-point-zero vector is thin. In the case of the principal module, if the subset is extremal, then it is tight.

In a recent paper [10], H. Tanaka classified all extremal completely regular codes in certain classical association schemes. For example if the underlying graph is a dual polar graph, then extremal codes are strongly closed. In the literature, one also finds weak-geodesically closed used in place of strongly closed.

In this paper, we study a converse, i.e., we classify parallelogram-free distance-regular graphs having strongly closed completely regular codes. To state our results, we make a few definitions. For notation, terminology and the general theory of distance-regular graphs, we refer the reader to [1].

Let $\Gamma = (X, R)$ be a connected graph of diameter d with vertex set X and edge set R. For vertices x and y, $\partial(x, y)$ denotes the distance between x and y, i.e., the length of a shortest path connecting x and y. More generally, for each $x \in X$ and a subset $S \subset X$ we write $\partial(x, S) = \min\{\partial(x, s) \mid s \in S\}$.

For a vertex $u \in X$ and $j \in \{0, 1, \ldots, d\}$, let

$$\Gamma_j(u) = \{x \in X \mid \partial(u, x) = j\} \text{ and } \Gamma(u) = \Gamma_1(u).$$

A subset Y of X is said to be *completely regular*, or a *completely regular code*, if the following numbers

$$\pi_{i,j} = |\Gamma_j(x) \cap Y| \quad (i, j \in \{0, 1, \ldots, d\})$$

depend only on $i = \partial(x, Y)$ and j. We write $\gamma_i = \pi_{i,i}$. For $Y \subset X$, the number $t(Y) = \max\{\partial(x, Y) \mid x \in X\}$ is called the *covering radius* of Y, and $w(Y) = \max\{\partial(x, y) \mid x, y \in Y\}$ is called the *width* of Y.

For two vertices u and $v \in X$ with $\partial(u, v) = j$, let

$$\begin{aligned} C(u, v) = C_j(u, v) &= \Gamma_{j-1}(u) \cap \Gamma(v), \\ A(u, v) = A_j(u, v) &= \Gamma_j(u) \cap \Gamma(v), \text{ and} \\ B(u, v) = B_j(u, v) &= \Gamma_{j+1}(u) \cap \Gamma(v). \end{aligned}$$

A connected graph Γ is said to be *distance-regular* or a *distance-regular graph* if the cardinalities $c_j = |C(u,v)|$, $a_j = |A(u,v)|$ and $b_j = |B(u,v)|$ depend only on $j = \partial(u,v)$ for all $j \in \{0, 1, \ldots, d\}$. These numbers c_j's, a_j's and b_j's are called the *intersection numbers* of Γ.

A subset Y of the vertex set X is often called a *code*, but in this paper, it is also regarded as the induced subgraph on Y. A nonempty subset Y of X is said to be *strongly closed* if

$$C(u,v) \cup A(u,v) \subset Y \text{ for all } u, v \in Y.$$

In this case Y is also called a *strongly closed subgraph*. For two vertices x and y, $\ll x, y \gg$ denotes the smallest strongly closed subgraph containing x and y. Note that since the intersection of two strongly closed subgraphs is strongly closed and Γ itself is a strongly closed subgraph containing x and y, $\ll x, y \gg$ always exists.

A *parallelogram* of length i is a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x,y) = \partial(z,w) = 1$, $\partial(x,z) = i$, and $\partial(x,w) = \partial(y,w) = \partial(y,z) = i - 1$.

A parallelogram of length 2 is isomorphic to $K_{2,1,1}$. If a distance-regular graph Γ does not have a parallelogram of length 2, then it is said to have *order* (s,t) for some positive integers s and t, as every edge is contained in a maximal clique of constant size $s+1$, and every vertex is contained in exactly $t+1$ maximal cliques. In particular, the valency $k = s(t+1)$ and the neighborhood $\Gamma(x)$ of each vertex x is isomorphic to a disjoint union of $t+1$ cliques of size s. If $c_2 = 1$, then Γ is of order (s,t) for some positive integers s and t. If $a_1 = 0$ then Γ is of order $(1, k-1)$.

A distance-regular graph $\Gamma = (X, R)$ of diameter d is said to be a *regular near polygon* if it is of order (s,t) for some integers s and t, and for every maximal clique L and a vertex $x \in X$ with $\partial(x, L) = i < d$, $|\Gamma_i(x) \cap L| = 1$. A regular near polygon having the property that no maximal clique is contained in $\Gamma_d(x)$ for any $x \in X$ is called a *regular near* $2d$*-gon*. A regular near 4-gon is called a *generalized quadrangle*. A regular near polygon is often defined as an incidence structure, and in that case our regular near polygon is called the collinearity graph of a regular near polygon, or the point graph of it. See [1, Section 6.4].

If a graph does not contain parallelograms of any length, it is called *parallelogram free*. A regular near polygon is parallelogram free, and the parallelogram-free condition is closely related to the existence of strongly closed subgraphs. See Theorem 2.2.

Throughout this paper by strongly regular graphs we mean distance-regular graphs of diameter two, hence connected.

Now we state our main results.

Theorem 1.1 *Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph of diameter $d \geq 4$ such that $b_1 > b_2$ and $a_2 \neq 0$. Suppose every strongly closed subgraph C of diameter 2 is completely regular. Then the following hold.*

(i) *Γ is a regular near polygon with $c_2 > 1$, and for every pair of vertices x, y at distance $d-1$, Γ has a strongly closed subgraph Y of diameter $d-1$ containing x and y.*
(ii) *The covering radius $t(C)$ of each strongly closed subgraph C of diameter 2 is at least $d-2$, and*

(a) *If* $t(C) = d - 2$, *then* Γ *is isomorphic to a Hamming graph or a dual polar graph.*
(b) *If* $t(C) \geq d - 1$, *then every strongly closed subgraph* Y *of diameter* $d - 1$ *is isomorphic to a Hamming graph or a dual polar graph.*

When $q = 1$ and $d \geq 4$, we can prove that Γ itself is isomorphic to a Hamming graph without assuming that the covering radius is $d - 2$ by [6, 11]. See the last section.

We have the following characterization of the case that a strongly regular subgraph is completely regular with covering radius $d - 2$.

Theorem 1.2 *Let* $\Gamma = (X, R)$ *be a parallelogram-free distance-regular graph of order* (s, t) *and diameter* $d \geq 4$. *Suppose* $b_1 > b_2$ *and* $a_2 \neq 0$. *Let* $q = c_2 - 1$. *Then the following are equivalent.*

(i) *There is a completely regular code* C *of covering radius* $d - 2$ *such that the induced subgraph on* C *is strongly regular.*
(ii) *There is a strongly closed completely regular code* C *of width* 2 *and covering radius* $d - 2$.
(iii) *Every strongly closed subgraph of diameter* 2 *is completely regular with covering radius* $d - 2$.
(iv) *Every strongly closed subgraph of diameter* 2 *is completely regular with covering radius* $d - 2$ *and that it is a generalized quadrangle.*
(v) $q \neq 0$ *and* Γ *has eigenvalues* $-t - 1$ *and* $s - t/q$.
(vi) Γ *is isomorphic to a Hamming graph or a dual polar graph.*

2 Preliminaries

Lemma 2.1 ([1, Remark on page 86], [8, Lemma 2.6]) *Let* Γ *be a strongly regular graph with* $a_2 \neq 0$, *and let* u *be a vertex of* Γ. *Then the induced subgraph* Δ *on* $\Gamma_2(u)$ *is connected of diameter at most three.*

Theorem 2.2 ([12, Proposition 6.7], [8, Theorem 1.1]) *Let* $\Gamma = (X, R)$ *be a distance-regular graph of diameter* d, *and let* m *be a positive integer such that* $2 \leq m \leq d$. *Assume that* Γ *contains no parallelogram of length* i *for any* $i = 2, \ldots, m + 1$ *and that* $b_1 > b_2$. *In addition assume one of the following*:

(i) $m = 2$, $c_2 > 1$ *and* $a_2 \neq 0$,
(ii) $c_2 > 1$ *and* $a_1 \neq 0$,
(iii) $m = 2$ *and* $c_2 = 1$,
(iv) $c_2 = 1$ *and* $a_1 \neq 0$, *or*
(v) $c_{m+1} = 1$.

Then for any vertices $x, y \in X$ *with* $\partial(x, y) \leq m$, *the diameter of the strongly closed subgraph* $\ll x, y \gg$ *is* $\partial(x, y)$. *In particular, if* $a_2 \neq 0$, *then for any vertices* $x, y \in X$ *with* $\partial(x, y) = 2$, *there is a strongly closed subgraph of diameter* 2 *containing* x *and* y.

Lemma 2.3 ([12, Lemma 6.9], [8, Lemma 4.1]) *Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $d \geq 3$. Suppose Γ contains no parallelogram of any length. Let x be a vertex and Y a strongly closed subgraph of diameter 2. Suppose $u \in \Gamma_i(x) \cap Y$ and $\Gamma_{i+2}(x) \cap Y \neq \emptyset$ with $i + 2 \leq d$. Then for all $y \in Y$, we have $\partial(x, y) = i + \partial(u, y)$.*

3 Terwilliger algebras and completely regular codes

Let $\Gamma = (X, R)$ be a connected graph of diameter d and C a subset of X with width $w = w(C)$ and covering radius $t = t(C)$. Let $C_i = \{x \in X \mid \partial(x, C) = i\}$ for $i \in \{0, 1, \dots, t\}$.

Let $V = \boldsymbol{C}^X = \mathrm{Span}(\hat{x} \mid x \in X)$ be a vector space over the complex number field consisting of the set of column vectors with rows indexed by the elements of X, and $\hat{x}$ denotes the unit vector whose x-entry is 1 and 0 otherwise.

For each $i = 0, 1, \dots, d$, let $A_i \in \mathrm{Mat}_X(\boldsymbol{C})$ be the *i-th adjacency matrix* defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \partial(x, y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

We call $A = A_1$ the *adjacency matrix* of Γ.

For $i \in \{0, 1, \dots, t\}$, $E_i^* = E_i^*(C) \in \mathrm{Mat}_X(\boldsymbol{C})$ are defined as follows.

$$(E_i^*)_{x,y} = \begin{cases} 1 & \text{if } x = y \text{ and } x \in C_i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix E_i^* induces the projection onto the subspace $E_i^* V = \mathrm{Span}(\hat{x} \mid x \in C_i)$.

Definition 3.1 The *Terwilliger algebra* $\mathcal{T} = \mathcal{T}(C)$ of a connected graph $\Gamma = (X, R)$ associated with a subset C of X is a matrix subalgebra over $\boldsymbol{C}$ of $\mathrm{Mat}_X(\boldsymbol{C})$ generated by A together with $E_0^*, E_1^*, \dots, E_t^*$, where $t = t(C)$. A *$\mathcal{T}$-module* W is a $\mathcal{T}$-invariant linear subspace of V. A nonzero $\mathcal{T}$-module W is said to be *irreducible* if W does not contain proper nonzero $\mathcal{T}$-modules. An irreducible $\mathcal{T}$-module W is said to be *thin* if $\dim E_i^* W \leq 1$ for every $i = 0, 1, \dots, t$.

Definition 3.2 Let $\Gamma = (X, R)$ be a connected graph, and C a nonempty subset of X. Let $\mathbf{1}_C = \sum_{x \in C} \hat{x} \in V = \boldsymbol{C}^X$. Then C is said to be a *completely regular code* if $\mathcal{T}(C)\mathbf{1}_C$ is a thin irreducible $\mathcal{T}(C)$-module.

Note that if Γ is a distance-regular graph, the definition of complete regularity in the introduction coincides with the one given above. The proof is straightforward. See [9, Proposition 7.2] and [5].

Let $\Gamma = (X, R)$ be a connected graph. Then it is immediate that Γ is distance-regular if and only if it is regular and every singleton $\{x\}$ with $x \in X$ is completely regular. It is not difficult to show that if Γ is distance-regular of diameter d, then every edge $\{x, y\}$ with $x, y \in X$ is completely regular if and only if $a_1 = a_2 = \cdots = a_{d-1} = 0$, i.e., Γ is almost bipartite or bipartite.

Thin Irreducible Modules. Let $\Gamma = (X, R)$ be a distance-regular graph of valency k and diameter d. Let A_i be the i-th adjacency matrix and $A = A_1$. Then there is a polynomial $v_i(\lambda) \in \boldsymbol{C}[\lambda]$ of degree exactly i such that $v_i(A) = A_i$. Let $k_i = v_i(k)$. Then $k_i = |\Gamma_i(x)|$ for every $x \in X$. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ be distinct eigenvalues of A and let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of $\boldsymbol{C}[A]$ corresponding to each of the distinct eigenvalues. Then each column of E_i is an eigenvector of the same eigenvalue θ_i of A, and $AE_i = \theta_i E_i$. Let $m(\theta_i) = \mathrm{tr}(E_i)$. Then $m(\theta_i)$ is the multiplicity of θ_i as an eigenvalue of A. Set $\Theta = \{\theta_0, \theta_1, \ldots, \theta_d\}$.

Let C be a nonempty subset of X and $\mathcal{T} = \mathcal{T}(C)$. We consider an irreducible $\mathcal{T}$-module W such that $E_0^* W \neq 0$, which is called a module of *endpoint* 0.

We review some facts proved in [9].

Let $\boldsymbol{v} = E_0^* \boldsymbol{v}$ be a nonzero vector. Set

$$\rho_{\boldsymbol{v}}(\lambda) = \frac{1}{|X|} \sum_{i=0}^{d} \frac{{}^t\overline{\boldsymbol{v}} A_i \boldsymbol{v}}{\|\boldsymbol{v}\|^2} \frac{v_i(\lambda)}{k_i} \in \boldsymbol{R}[\lambda].$$

The following is called the inner distribution of the vector $\boldsymbol{v}$.

$$a(\boldsymbol{v}) = \left(\frac{{}^t\overline{\boldsymbol{v}} A_0 \boldsymbol{v}}{\|\boldsymbol{v}\|^2}, \ldots, \frac{{}^t\overline{\boldsymbol{v}} A_i \boldsymbol{v}}{\|\boldsymbol{v}\|^2}, \ldots, \frac{{}^t\overline{\boldsymbol{v}} A_d \boldsymbol{v}}{\|\boldsymbol{v}\|^2} \right).$$

By definition, if $w = w(C)$ is the width of C, then the degree of $\rho_{\boldsymbol{v}}(\lambda)$ is at most w. On the other hand by direct computation we have

$$\frac{\|E_i \boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = \rho_{\boldsymbol{v}}(\theta_i) m(\theta_i).$$

Since $\boldsymbol{C}[A]\boldsymbol{v} = \mathrm{Span}(E_0\boldsymbol{v}, E_1\boldsymbol{v}, \ldots, E_d\boldsymbol{v})$, we have

$$\dim \boldsymbol{C}[A]\boldsymbol{v} \geq d + 1 - (\#\text{ of roots of } \rho_{\boldsymbol{v}}(\lambda) \text{ in } \Theta) \geq d + 1 - w(C).$$

Set $r = r(\boldsymbol{v}) = \dim \boldsymbol{C}[A]\boldsymbol{v} - 1$. The number $r(\boldsymbol{v})$ is called the *dual degree* of $\boldsymbol{v}$. If $\mathbf{1}_C$ is the characteristic vector of C, we write $r(C)$ for $r(\mathbf{1}_C)$ and call the dual degree of C. Now we have the following.

Theorem 3.1 ([9, Theorem 1.1]) *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d, and C a nonempty subset of X. Let $E_0^* = E_0^*(C)$ and $\boldsymbol{v} = E_0^*\boldsymbol{v}$ a nonzero vector. Then the following hold.*

(i) $\dim \boldsymbol{C}[A]\boldsymbol{v} + w(C) \geq d + 1$.
(ii) *If* $\dim \boldsymbol{C}[A]\boldsymbol{v} + w(C) = d + 1$, *then $\mathcal{T}(C)\boldsymbol{v}$ is a thin irreducible $\mathcal{T}(C)$-module.*

A nonzero vector $\boldsymbol{v} \in E_0^* V$ satisfying the condition in Theorem 3.1 (ii) is called a tight vector. When $E_0^* V$ is spanned by tight vectors, we call C a tight code.

The case that $\boldsymbol{v}$ is the characteristic vector $\mathbf{1}_C$ of C is also studied in [2]. See also [4].

Corollary 3.2 ([2, Theorem 1]) *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d, and C a nonempty subset of X with dual degree $r = r(C)$. If $r + w(C) = d$, then C is a completely regular code. Moreover, we have $t = r$ in this case.*

Note that the condition in the corollary can be checked if we have $a(\mathbf{1}_C)$ together with the set of eigenvalues of A. In the literature, the inner distribution $a(\mathbf{1}_C)$ is also called the inner distribution of the code C and denoted $a(C)$.

4 Completely regular subgraphs

Proposition 4.1 *Let $\Gamma = (X, R)$ be a distance-regular graph of valency k and diameter d. Let C be a subset of X contained in a proper strongly closed subgraph Y of Γ. In addition assume that $|\Gamma_i(z) \cap C|$ depends only on i whenever $\partial(z, C) = 1$. Then C is strongly closed.*

Proof First note that the maximal valency of Y is not k. Suppose not, and let m be the diameter of Y. Then $c_m + a_m = k$ and $b_m = 0$. This implies $m = d$ and Y is not regular. This contradicts Theorem 1.1 in [7].

Let $x, y \in C$ such that $\partial(x, y) = \ell$. Since the maximal valency of Y is less than k, there is a vertex $u \in X \setminus Y$ adjacent to x. Let $v \in C$. Since $C \subset Y$ and Y is strongly closed $\partial(u, v) = \partial(x, v) + 1$. Let $z \in \Gamma(x)$ such that $\partial(z, y) \le \ell$. We claim that $z \in C$. Suppose not. Then $\partial(z, C) = 1$ and the following hold.

$$\sum_{v \in C} \partial(x, v) + |C| = \sum_{v \in C} \partial(u, v) = \sum_{v \in C} \partial(z, v).$$

Since $\partial(z, v) \le \partial(x, v) + 1$. The above holds only if $\partial(z, v) = \partial(x, v) + 1$ holds for all $v \in C$. Since $y \in C$ and $\partial(z, y) \le \ell = \partial(x, y)$, this is absurd. Thus we proved the claim. Hence C is strongly closed. □

An induced subgraph on Y of a graph $\Gamma = (X, R)$ is called *weakly closed* if the distance in the subgraph is equal to the distance in Γ.

Corollary 4.2 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d. Let C be a weakly closed distance-regular subgraph in Γ of diameter ℓ, and $u, v \in C$ with $\partial(u, v) = \ell$. In addition assume that $|\Gamma_i(z) \cap C|$ depends only on i whenever $\partial(z, C) = 1$. If both u and v are contained in a proper strongly closed subgraph Y of Γ, then $C \subset Y$ and C is strongly closed.*

Proof By Proposition 4.1, it suffices to show that $C \subset Y$. Since C is connected for each $w \in C$, there is a path $u = u_0 \sim u_1 \sim \cdots \sim u_m = w$ in C. Since the diameter of C is ℓ, C is weakly closed and Y is strongly closed, $C \cap \Gamma(u) \subset Y$ and $C \cap \Gamma(v) \subset Y$. Since C is distance-regular and weakly closed, there is a vertex $v_1 \in (\Gamma(v) \cup \{v\}) \cap C$ such that $\partial(u_1, v_1) = \ell$. Since $v_1 \in Y$, we can proceed by induction to show $w \in Y$. □

Lemma 4.3 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d. Let $1 \le m \le d-1$ be an integer. Suppose for $u, v \in X$ with $\partial(u, v) = m$, there is a strongly closed subgraph C of diameter m containing u and v and C is completely regular. Then the parameters $\pi_{i,j}$ of C are determined by m and the parameters of Γ.*

Proof Since C is strongly closed in Γ, the parameters of C and hence the inner distribution of C is determined by the parameters of Γ and m. Now the assertion follows from [9, Corollary 10.3]. □

5 Completely regular strongly regular subgraphs

In this section, we study parallelogram-free distance-regular graphs having completely regular strongly regular subgraphs. The goal is to establish the following result.

Theorem 5.1 *Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph of diameter $d \ge 4$ such that $b_1 > b_2$ and $a_2 \ne 0$. Suppose every strongly closed subgraph C of diameter 2 is completely regular. Let $c_2 = q + 1$. Then Γ is a regular near polygon, $q \ge 1$ and $c_i = \left[{i \atop 1}\right]_q$ for $i \in \{1, 2, \ldots, d-1\}$. Moreover if the covering radius of C is $d-2$, then $c_d = \left[{d \atop 1}\right]_q$ and Γ is a regular near $2d$-gon.*

We first remark that under the hypothesis of Theorem 5.1, for two vertices x, y with $\partial(x, y) = 2$, there is a strongly closed subgraph $\ll x, y \gg$ of diameter 2 containing x and y by Theorem 2.2.

Hypothesis 5.1 *Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph of diameter $d \ge 4$ such that $b_1 > b_2$ and $a_2 \ne 0$. Every strongly closed subgraph C of diameter 2 is completely regular.*

Let $s = a_1 + 1$ and $t = b_1/s$. Then Γ in Hypothesis 5.1 is of order (s, t).

Lemma 5.2 *Under Hypothesis 5.1, for every $i \le d-2$ and $u \in X$ with $\partial(u, C) = i$, $\gamma_i = \gamma_i(u) = |C \cap \Gamma_i(u)| = 1$, $\alpha_i = \alpha_i(u) = |C \cap \Gamma_{i+1}(u)| = \kappa = a_2 + c_2$. In particular, the covering radius of C is at least $d-2$ and the parameters γ_i and α_i of C as a completely regular code do not depend on the choice of strongly closed subgraphs of diameter 2 up to $i \le d-2$.*

Proof Let $x, y \in C$ with $\partial(x, y) = 2$. Since $i \le d-2$, there is a vertex $u \in \Gamma_i(x) \cap \Gamma_{i+2}(y)$. Then by Lemma 2.3, we have the desired conclusion. Since C is completely regular, this is the case for all $u \in X$ with $\partial(u, C) = i$. □

Lemma 5.3 *Under Hypothesis 5.1, C is a generalized quadrangle. In particular $c_2 > 1$.*

Proof Since Γ is parallelogram free and C is strongly closed, C is of order (s,τ) for some integer τ. Let $u \in C$. Suppose that there are adjacent vertices $v, w \in \Gamma_2(u) \cap C$ such that $A(v,w) \subset \Gamma_2(u)$. Let $x \in B(u,w)$. Since C is strongly closed, $\partial(v,x) = 2$. Let $C' = \ll v, x \gg$. Since $\gamma_2 = 1$ and $v, w \in C \cap \Gamma_2(u)$, $\partial(u, C') = 1$. Let $\{y\} = \Gamma(u) \cap C'$. Then v, w and all vertices in $C' \cap \Gamma_2(u)$ are in $\Gamma(y)$, which is absurd as $\{v, w\} \cup A(v,w)$ is a maximal clique. Hence C is a generalized quadrangle. □

Lemma 5.4 *Under Hypothesis 5.1, Γ is a regular near polygon. Moreover if the covering radius of C is $d-2$, then Γ is a regular near $2d$-gon.*

Proof Let L be a maximal clique and $\partial(u, L) = i \le d-1$ for some vertex u. We will show that $|\Gamma_i(u) \cap L| = 1$. We may assume that $i \ge 2$ as L is a maximal clique. By way of contradiction assume that two vertices v and w are in $\Gamma_i(u) \cap L$.

First assume that $\Gamma_{i+1}(u) \cap L = \emptyset$. Let $x \in C(u,v)$. Then $\partial(x,w) = 2$. Let $C = \ll x, w \gg$. Then either $\partial(u,C) = i-1$ or $\partial(u,C) = i-2$. The first case does not occur as otherwise $\partial(x,w) = 1$ by Lemma 5.2. Suppose $\partial(u,C) = i-2$. By Lemma 5.3, we have a contradiction as we assumed that $\Gamma_{i+1}(u) \cap L = \emptyset$. This part also proves that if the covering radius of C is $d-2$, there is no maximal clique L such that $\partial(u,L) = d$.

Next assume that $\Gamma_{i+1}(u) \cap L \neq \emptyset$. Let $x \in \Gamma_{i+1}(u) \cap L$ and $y \in C(u,v)$. Then $\partial(x,y) = 2$. Let $C = \ll x, y \gg$. Since $v, w \in C$, this contradicts Lemma 5.2. □

Lemma 5.5 *Let $q = c_2 - 1$. Under Hypothesis 5.1 the following hold.*

$$c_{i+1} - 1 = (c_2 - 1)c_i, \text{ and } c_{i+1} = 1 + q + \cdots + q^i = \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q \text{ for all } i \le d-2. \quad (1)$$

Moreover, if every strongly closed subgraph C of diameter 2 is of covering radius $d-2$, then (1) holds for $i = d-1$ as well.

Proof Let $u, v, w \in X$ with $\partial(u,v) = i+1 \le d$ and $w \in C(u,v)$. We count the number of pairs in the following set.

$$N = \{(x,y) \mid x \in C(u,w), y \in C(x,v) \setminus \{w\}\}.$$

First there are c_i choices of x and then for each $x \in C(u,w)$, there are $c_2 - 1$ choices of y. Hence we have $|N| = (c_2 - 1)c_i$.

Next let $y \in C(u,v) \setminus \{w\}$. Since Γ is a regular near polygon by Lemma 5.4, $\partial(y,w) = 2$. Let Y be the strongly closed subgraph of diameter 2, containing y and w. Since $\{y, w\} \subset \Gamma_i(u)$ and $v \in Y \cap \Gamma_{i+1}(u)$, $\partial(u,Y) = i-1$ if $i \le d-2$ or $i = d-1$ and every strongly closed subgraph C of diameter 2 is of covering radius $d-2$. By Lemma 5.2, there exists a vertex x such that $\Gamma_{i-1}(u) \cap Y = \{x\}$ and that $y, w \in \Gamma(x)$. Therefore x is the unique vertex in $C(y,w) \cap \Gamma_{i-1}(u)$. Hence $(x,y) \in N$ and $|N| = c_{i+1} - 1$.

Since $q = c_2 - 1$ and $c_{i+1} = qc_i + 1$, we have the formula for c_{i+1} by induction. □

Proof of Theorem 5.1 Since C is a generalized quadrangle with $a_2 \neq 0$ by Lemma 5.3, $c_2 \geq 2$ and $q \geq 1$. Now we have the assertions by Lemma 5.4 and Lemma 5.5.

Proof of Theorem 1.1 By Theorem 5.1 $c_2 > 1$ and Γ is a regular near polygon. Since $a_2 \neq 0, a_1 \neq 0$. Hence for every pair of vertices x, y at distance $d-1$, Γ has a strongly closed subgraph Y of diameter $d-1$ containing x and y by Theorem 2.2. Let Y be a strongly closed subgraph of diameter $d-1$ in Γ. Then Y is a regular near $2(d-1)$-gon with $c_i = \left[{i \atop 1}\right]_q$. Hence it is with classical parameters $(d-1, q, 0, a_1+1)$. Now Y is isomorphic to a Hamming graph or a dual polar graph if $d \geq 4$ by Theorem 9.4.4 in [1]. The covering radius of C is at least $d-2$ by Lemma 5.2 and the result for the case the covering radius is $d-2$ follows similarly using the characterization in [1, Theorem 9.4.4]. □

6 Tight completely regular codes of small width

In this section, we consider the case that a subset C of small width $w \leq 2$ becomes a completely regular code with smallest covering radius $d-w$ or $\mathbf{1}_C$ is tight that satisfies the condition in Corollary 3.2.

Lemma 6.1 *Let C be a subset of a distance-regular graph $\Gamma = (X, R)$ of diameter $d \geq 2$. Let $\boldsymbol{v}$ be a non-zero vector such that* $\mathrm{supp}(\boldsymbol{v}) \subset C$. *Let*

$$\rho_{\boldsymbol{v}}(\lambda) = \frac{1}{|X|}\sum_{i=0}^{d} \eta_i \frac{v_i(\lambda)}{k_i} \in \boldsymbol{R}[\lambda], \text{ where } \eta_i = \eta_i(\boldsymbol{v}) = \frac{{}^t\overline{\boldsymbol{v}} A_i \boldsymbol{v}}{\|\boldsymbol{v}\|^2}.$$

Then the following hold.

(i) *If $w(C) = 1$, then*

$$\rho_{\boldsymbol{v}}(\lambda) = \frac{1}{|X| b_0}(b_0 + \eta_1 \lambda).$$

(ii) *If $w(C) = 2$, then*

$$\rho_{\boldsymbol{v}}(\lambda) = \frac{1}{|X| b_0 b_1}(b_0(b_1 - \eta_2) + (\eta_1 b_1 - \eta_2 a_1)\lambda + \eta_2 \lambda^2).$$

Proof Since

$$v_0(\lambda) = 1, \; v_1(\lambda) = \lambda, \text{ and } c_2 \cdot v_2(\lambda) = \lambda^2 - a_1\lambda - b_0,$$

the formulas above follow by direct computation using the fact that $\eta_0 = 1$ and $\eta_i = 0$ for all $i > w(C)$. □

Corollary 6.2 *Let C be a subset of a distance-regular graph $\Gamma = (X, R)$ of order (s, t) of diameter $d \geq 2$. Let $\mathbf{1}_C$ be the characteristic vector of C. Then the following hold.*

(i) *Suppose C is a maximal clique of size $s+1$. Then*

$$\rho_{\mathbf{1}_C}(\lambda) = \frac{1}{|X|(t+1)}(t+1+\lambda).$$

(ii) *Suppose C is strongly regular and strongly closed. In addition assume that $c_2 + a_2 = (q+1)s$ with $q = c_2 - 1$, i.e., C is a generalized quadrangle. Then*

$$\rho_{\mathbf{1}_C}(\lambda) = \frac{1}{|X|(t+1)t}(q\lambda + t - qs)(\lambda + t + 1).$$

Proof (i) is immediate. For (ii), $\eta_0 = 1$, $\eta_1 = (q+1)s$ and $\eta_2 = qs^2$. Hence the formula is immediate. □

Proposition 6.3 *Let $\Gamma = (X, R)$ be a distance-regular graph of order (s,t) of diameter $d \geq 3$. Suppose C is a strongly closed generalized quadrangle in Γ. Then the following are equivalent.*

(i) *Γ has eigenvalues $-t-1$ and $s - t/q$, where $q = c_2 - 1$.*
(ii) *C is completely regular with covering radius $d-2$.*

Moreover if (i), (ii) *hold, then every maximal clique C_1 is completely regular with covering radius $d-1$.*

Proof This is a direct consequence of Theorem 3.1, Corollary 3.2 and Corollary 6.2. □

Proof of Theorem 1.2 (iv)⇒(iii)⇒(ii)⇒(i) is clear, and (vi)⇒(v) is well-known. See [1, p. 261, p. 276].

(i)⇒(ii): Since the diameter of C is two, it is weakly closed. Hence by Corollary 4.2, C is strongly closed.

(ii)⇒(iii): Let $\rho(\lambda) = \rho_{\mathbf{1}_C}(\lambda)$. Since $\rho(\lambda)$ is determined by $\kappa_1 = |\Gamma(x) \cap C| = c_2 + a_2$ and $\kappa_2 = |\Gamma_2(x) \cap C| = (c_2 + a_2)(c_2 - a_2 - s)/c_2$, ρ does not depend on the choice of strongly closed subgraph. Moreover by (ii), two distinct eigenvalues of Γ are the roots of ρ. Therefore, every strongly closed subgraph of diameter 2 is completely regular with covering radius $d-2$.

(iii)⇒(iv): We need to show that the induced subgraph on C is a generalized quadrangle. This follows from Lemma 5.3.

(v)⇒(iv): Since Γ has an eigenvalue $-t-1$, every maximal clique of size $s+1$ is a completely regular code with covering radius $d-1$. Let C be a strongly closed subgraph of diameter 2. Then every maximal clique of size $s+1$ contained in C is completely regular with covering radius 1. Since C is of order (s, τ) with a suitable choice of an integer τ, C is a generalized quadrangle. In particular $q \neq 0$ as $a_2 \neq 0$. Note that if Y is a maximal clique and $x \in C \setminus Y$, $|\Gamma(x) \cap Y| = 1$ as Y is maximal.

Hence ρ_C is as in Corollary 6.2 and ρ_C has two eigenvalues $-t-1$ and $s-t/q$ as roots. Therefore, C is completely regular with covering radius $d-2$.

(iv)⇒(vi): This is a direct consequence of Proposition 6.3 and Theorem 1.1. □

7 Remarks

For the case $q=1$, the following two propositions cover most of our results. We only sketch their proofs.

Theorem 7.1 *Let Γ be a parallelogram-free distance-regular graph of order (s,t) with $c_2=2$, $a_2=2(s-1)$ and $c_3=3$ with $s>1$. If the diameter $d\geq 3$, then Γ is isomorphic to the Hamming graph $H(d,s+1)$.*

Proof We proceed by induction on d. If $d=3$, then by [6, Corollary], Γ is isomorphic to $H(3,s+1)$. Note that we do not need the assumption $s\neq 3$ as Γ is parallelogram free. Suppose the assertion holds for $d-1$. By Theorem 2.2, there is a strongly closed subgraph Δ of diameter $d-1$ in Γ. By induction hypothesis, Δ is isomorphic to $H(d-1,s+1)$ with $d\geq 4$. Now by [6, Theorem 1], there is a $(d-1)$-error correcting completely regular code of covering radius d in a $H(n,s+1)$ with $s+1\geq 3$. These are uniformly packed codes classified by H. van Tilborg [11] and the only possibility for Γ is $H(d,s+1)$. □

Corollary 7.2 *Let Γ be a parallelogram-free distance-regular graph of order (s,t), diameter $d\geq 4$ with $c_2=2$. Suppose Γ contains a strongly regular (vertex induced) subgraph with parameters (κ,λ,μ). If $\kappa\neq\mu$ and $\pi_{i,j}=|\Gamma_j(x)\cap C|$ depends only on $i=\partial(x,C)$ and j whenever $(i,j)=(1,1),(1,2)$ or $(2,2)$. Then Γ is isomorphic to the Hamming graph $H(d,s+1)$.*

Proof By our assumption, $c_2>1$ and $a_2\neq 0$. By Theorem 2.2, for each pair of distance two there is a strongly closed subgraph of diameter two containing the pair. Hence by Corollary 4.2, C is strongly closed. Now by Lemma 2.3, $\pi_{1,1}=1$, $\pi_{1,2}=\kappa=c_2+a_2$ and $\pi_{2,2}=1$. Hence by the proof of Lemma 5.3, C is a generalized quadrangle and $a_2=2(s-1)$. By mimicking the proof of Lemma 5.5, $c_3=3$. We are now ready to apply Theorem 7.1 to conclude that Γ is isomorphic to $H(d,s+1)$. □

The consideration of the case $q=1$ above suggests us to classify distance-regular graphs of order (s,t) of diameter $d\geq 4$ with the following parameters:

$$c_i=1+q+\cdots+q^{i-1},\ a_i=c_i(s-1) \text{ for all } i\in\{1,2,\ldots,d-1\}$$

with $q\geq 2$ and $s>1$.

The results in this paper also suggest problems to characterize distance-regular graphs by a given completely regular subgraph. Since $\Gamma_d(x)$ is always completely regular, this problem is connected to the problem to characterize distance-regular graphs by the structure of $\Gamma_d(x)$. We close this paper by giving a possible improvement of the result of this paper.

> Replace the hypothesis 'parallelogram-free' in Theorem 5.1 and Corollary 5.1 by the following:
> Γ is of order (s, t) and every maximal clique is completely regular.

References

1. Brouwer, A.E., Cohen, A.M., Neumaier, A.: Distance-Regular Graphs. Springer, Berlin (1989)
2. Brouwer, A.E., Godsil, C.D., Koolen, J.H., Martin, W.J.: Width and dual width of subsets in polynomial association schemes. J. Comb. Theory A **102**, 255–271 (2003)
3. Delsarte, P.: An algebraic approach to the association schemes of coding theory. Philips Research Reports Supplements 1973, No.10
4. Hosoya, R., Suzuki, H.: Tight distance-regular graphs with respect to subsets of width two. Eur. J. Comb. **28**, 61–74 (2007)
5. Neumaier, A.: Completely regular codes. Discrete Math. **106–107**, 353–360 (1992)
6. Nomura, K.: Distance-regular graphs of Hamming type. J. Comb. Theory B **50**, 160–167 (1990)
7. Suzuki, H.: On strongly closed subgraphs of highly regular graphs. Eur. J. Comb. **16**, 197–220 (1995)
8. Suzuki, H.: Strongly closed subgraphs of a distance-regular graph with geometric girth five. Kyushu J. Math. **50**, 371–384 (1996)
9. Suzuki, H.: The Terwilliger algebra associated with a set of vertices in a distance-regular graph. J. Algebr. Comb. **22**, 5–38 (2005)
10. Tanaka, H.: Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs. J. Comb. Theory A **113**, 903–910 (2006)
11. van Tilborg, H.C.A.: Uniformly packed codes. Ph.D. thesis, Eindhoven (1976). http://alexandria.tue.nl/extra1/PRF2B/7602641.pdf
12. Weng, C.: Weak-geodesically closed subgraphs in distance-regular graphs. Graphs Comb. **14**, 275–304 (1998)