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Fredman's reciprocity, invariants of abelian groups, and the permanent of the Cayley table

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Abstract Let $\mathcal{R}$ be the regular representation of a finite abelian group G and let $\mathcal{C}_n$ denote the cyclic group of order n. For $G = \mathcal{C}_n$, we compute the Poincaré series of all $\mathcal{C}_n$-isotypic components in $\mathcal{S}^{\cdot}\mathcal{R} \otimes \wedge^{\cdot}\mathcal{R}$ (the symmetric tensor exterior algebra of $\mathcal{R}$). From this we derive a general reciprocity and some number-theoretic identities. This generalises results of Fredman and Elashvili–Jibladze. Then we consider the Cayley table, $\mathcal{M}_G$, of G and some generalisations of it. In particular, we prove that the number of formally different terms in the permanent of $\mathcal{M}_G$ equals $(\mathcal{S}^n\mathcal{R})^G$, where n is the order of G.

Keywords Molien formula · Poincaré series · Permanent · Ramanujan's sum

1 Introduction

In the beginning of the 1970s, Fredman [7] considered the problem of computing the number of vectors $(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ with non-negative integer components that satisfy

$$\lambda_0 + \cdots + \lambda_{n-1} = m \quad \text{and} \quad \sum_{j=0}^{n-1} j\lambda_j \equiv i \bmod n. \tag{1.1}$$

He denoted this number by $S(n, m, i)$. Using generating functions, Fredman obtained an explicit formula for $S(n, m, i)$, which immediately showed that $S(n, m, i) = S(m, n, i)$. The latter is said to be *Fredman's reciprocity*. Using a necklace interpretation, he also constructed a bijection between the vectors enumerated by $S(n, m, i)$

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and those enumerated by $S(m,n,i)$. However, these results did not attract attention and remained unnoticed.

Later, Elashvili and Jibladze [4, 5] (partly with Pataraia [6]) rediscovered these results using Invariant Theory. Let $\mathcal{C}_n \simeq \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n and $\mathcal{R}$ the space of its regular representation over $\mathbb{C}$. Choose a basis $(v_0, v_1, \dots, v_{n-1})$ for $\mathcal{R}$ consisting of $\mathcal{C}_n$-eigenvectors. More precisely, if $\gamma \in \mathcal{C}_n$ is a generator and $\zeta = \sqrt[n]{1}$ a fixed primitive root of unity, then $\gamma \cdot v_i = \zeta^i v_i$. Write χ_i for the linear character $\mathcal{C}_n \to \mathbb{C}^\times$ that takes γ to ζ^i. The monomial $v_0^{\lambda_1} \cdots v_{n-1}^{\lambda_{n-1}}$ has degree m and weight χ_i if and only if $(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$ satisfies (1.1). Thus, $S(n,m,i)$ is the dimension of the space of $\mathcal{C}_n$-semi-invariants of weight χ_i in the mth symmetric power $\mathcal{S}^m\mathcal{R}$. This space can also be understood as the $\mathcal{C}_n$-isotypic component of type χ_i in $\mathcal{S}^m\mathcal{R}$, denoted by $(\mathcal{S}^m\mathcal{R})_{\mathcal{C}_n,\chi_i}$. To stress the connection with cyclic groups, we will write $a_i(\mathcal{C}_n, m)$ in place of $S(n,m,i)$. The celebrated Molien formula provides a closed expression for the generating function (Poincaré series)

$$\mathcal{F}\big((\mathcal{S}^\cdot\mathcal{R})_{\mathcal{C}_n,\chi_i}; t\big) = \sum_{m=0}^{\infty} a_i(\mathcal{C}_n, m) t^m,$$

where $(\mathcal{S}^\cdot\mathcal{R})_{\mathcal{C}_n,\chi_i} = \bigoplus_{m\ge 0} (\mathcal{S}^m\mathcal{R})_{\mathcal{C}_n,\chi_i}$ is the $(\mathcal{C}_n, \chi_i)$-isotypic component in $\mathcal{S}^\cdot\mathcal{R}$. Then extracting the coefficient of t^m yields a formula for $a_i(\mathcal{C}_n, m)$, see (2.2). It is worth stressing that Molien's formula is a very efficient tool that provides a uniform approach to various combinatorial problems and paves the way for further generalisations; see, e.g. [12].

In this note, we elaborate on two topics. *First*, generalising results of Fredman and Elashvili–Jibladze, we compute the Poincaré series for each $\mathcal{C}_n$-isotypic component in the bi-graded algebra $\mathcal{S}^\cdot\mathcal{R} \otimes \wedge^\cdot\mathcal{R}$ and then $\dim(\mathcal{S}^p\mathcal{R} \otimes \wedge^m\mathcal{R})_{\mathcal{C}_n,\chi_i}$ for all p, m, i (Theorem 3.2). From this we derive a more general reciprocity, see (3.5). As a by-product of these computations, we obtain some interesting identities, e.g.

$$\exp\left(\frac{z}{1-z^2}\right) = \prod_{d=1}^{\infty} \big(1 + z^d\big)^{\varphi(d)/d},$$

where φ is Euler's totient function. In Sect. 4, several identities related to isotypic components in $\wedge^\cdot\mathcal{R}$ are given; some of them are valid for an arbitrary finite abelian group G, see Theorem 4.4. *Second*, in Sect. 5, we study some properties of the Cayley table, $\mathcal{M}_G$, of G. If $G = \{x_0, x_1, \dots, x_{n-1}\}$, then $\mathcal{M}_G$ can be regarded as n by n matrix with entries in $\mathbb{C}[x_0, \dots, x_{n-1}] \simeq \mathcal{S}^\cdot\mathcal{R}$. For $G = \mathcal{C}_n$, $\mathcal{M}_G$ is nothing but a generic *circulant matrix*. The permanent of $\mathcal{M}_G$, $\operatorname{per}(\mathcal{M}_G)$, is a sum of monomials in x_i's of degree n. Using [8], we prove that the number of different monomials occurring in this sum equals $\dim(\mathcal{S}^n\mathcal{R})^G$. Then we introduce the extended Cayley table, $\widetilde{\mathcal{M}}_G$ (which is a matrix of order $n+1$), and characterise the monomials occurring in $\operatorname{per}(\widetilde{\mathcal{M}}_G)$ (Theorem 5.8). This characterisation implies that the number of different monomials in $\operatorname{per}(\widetilde{\mathcal{M}}_G)$ equals $\dim(\mathcal{S}^{n+1}\mathcal{R})^G$. Both $\operatorname{per}(\mathcal{M}_G)$ and $\det(\mathcal{M}_G)$ belong to $\mathcal{S}^n\mathcal{R}$, and we prove that $\operatorname{per}(\mathcal{M}_G)$ is G-invariant, whereas $\det(\mathcal{M}_G)$ is a semi-invariant whose weight is the sum of all elements of the dual group $\hat{G}$. The latter means that in many cases $\det(\mathcal{M}_G)$ is invariant, too. In Sect. 6, we discuss some open problems related to $(\mathcal{S}^\cdot\mathcal{R})^G$ and $\operatorname{per}(\mathcal{M}_G)$.

Notation $\#(M)$ is the cardinality of a finite set M; (n,m) is the greatest common divisor of $n,m\in\mathbb{N}$; G is always a finite group.

2 Preliminaries

2.1 Ramanujan's sums

Two important number-theoretic functions are *Euler's totient function* φ and the *Möbius function* μ. Recall that $\varphi(n)$ is the number of all primitive roots of unity of order n. Given $i,n\in\mathbb{N}$, $n\ge 1$, the *Ramanujan's sum*, $c_n(i)$, is the sum of ith powers of the primitive roots of unity of order n. In particular, $c_n(0)=\varphi(n)$. There are two useful expressions for Ramanujan's sums:

$$c_n(i)=\sum_{d|(n,i)}\mu\left(\frac{n}{d}\right)d,\qquad c_n(i)=\frac{\varphi(n)}{\varphi(\frac{n}{(n,i)})}\cdot\mu\left(\frac{n}{(n,i)}\right),$$

see [9, Theorems 271 and 272]. These formulae also show that $c_n(1)=\mu(n)$, $c_n(i)=c_n(n-i)$, and $c_n(i)$ is always a rational integer.

2.2 Molien's formula for the symmetric algebra

Let G be a finite group and V a finite-dimensional G-module. The original Molien formula computes the Poincaré series of the graded algebra of invariants $(\mathcal{S}^{\cdot}V)^G=\bigoplus_{m\ge 0}(\mathcal{S}^mV)^G$. More generally, there is a similar formula for the Poincaré series of any G-isotypic component in $\mathcal{S}^{\cdot}V$. Let χ be an irreducible representation of G and $(\mathcal{S}^{\cdot}V)_{G,\chi}$ the isotypic component of type χ in $\mathcal{S}^{\cdot}V$. By definition, the Poincaré series of $(\mathcal{S}^{\cdot}V)_{G,\chi}$ is the power series $\mathcal{F}((\mathcal{S}^{\cdot}V)_{G,\chi};t):=\sum_{m\ge 0}\dim((\mathcal{S}^mV)_{G,\chi})t^m$. Then

$$\mathcal{F}\big((\mathcal{S}^{\cdot}V)_{G,\chi};t\big)=\frac{\deg(\chi)}{\#(G)}\sum_{\gamma\in G}\frac{\operatorname{tr}(\chi(\gamma^{-1}))}{\det_V(\mathbb{1}-t\gamma)},$$

see, e.g. [12, Theorem 2.1]. Here $\mathbb{1}$ is the identity matrix in $GL(V)$. (The algebra of invariants corresponds to the trivial one-dimensional representation, i.e., if $\deg(\chi)=1$ and $\chi(\gamma)=1$ for all $\gamma\in G$.)

Let $\mathcal{R}$ be the space of the regular representation of G. For the G-module $\mathcal{R}$, Molien's formula can be presented in a somewhat simpler form.

Proposition 2.1 [2, V.1.8] *Let $\varphi_G(d)$ be the number of elements of order d in G. Then*

$$\mathcal{F}\big((\mathcal{S}^{\cdot}\mathcal{R})^G;t\big)=\frac{1}{\#(G)}\sum_{d\ge 1}\frac{\varphi_G(d)}{(1-t^d)^{(\#G)/d}}.$$

This can easily be extended to an arbitrary χ. If $\mathsf{ord}(\gamma)$ is the order of $\gamma\in G$, then

$$\mathcal{F}\big((\mathcal{S}^{\cdot}\mathcal{R})_{G,\chi};t\big)=\frac{\deg(\chi)}{\#(G)}\sum_{d|\#G}\frac{\sum_{\gamma:\,\mathsf{ord}(\gamma)=d}\operatorname{tr}(\chi(\gamma^{-1}))}{(1-t^d)^{(\#G)/d}}.\tag{2.1}$$

In fact, we prove below a more general formula (Lemma 3.1).

2.3 Formulae of Fredman and Elashvili–Jibladze

Recall that $a_i(\mathcal{C}_n, m) = \dim \mathcal{S}^m(\mathcal{R})_{\mathcal{C}_n,\chi_i}$ or, equivalently, it is the number of vectors satisfying (1.1). In particular, $a_0(\mathcal{C}_n, m) = \dim \mathcal{S}^m(\mathcal{R})^{\mathcal{C}_n}$. If the elements of $\mathcal{C}_n$ are regarded as the roots of unity of order n, then χ_i is the character $\xi \mapsto \xi^i$, $\xi \in \mathcal{C}_n$. Here $\varphi_{\mathcal{C}_n}(d)$ is almost Euler's totient function. That is, $\varphi_{\mathcal{C}_n}(d) = \varphi(d)$, if $d|n$; and $\varphi_{\mathcal{C}_n}(d) = 0$ otherwise. Using (2.1) with $G = \mathcal{C}_n$ and $\chi = \chi_i$, we see that $\deg(\chi_i) = 1$ and $\sum_{\gamma:\,\mathrm{ord}(\gamma)=d} \chi_i(\gamma^{-1}) = c_d(d-i)$. Then extracting the coefficient of t^m yields a nice-looking formula (Fredman [7], Elashvili–Jibladze [5])

$$a_i(\mathcal{C}_n, m) = \frac{1}{n+m} \sum_{d|(n,m)} c_d(i) \binom{n/d + m/d}{n/d}. \tag{2.2}$$

Remark 2.2 Both Fredman's approach, see (1.1), and cyclic group interpretation presuppose that $a_i(\mathcal{C}_n, m)$ is defined for $n \ge 1$ and $m \ge 0$. But (2.2) shows that $a_i(\mathcal{C}_n, m)$ is naturally defined for $(n, m) \in \mathbb{N}^2$, $(n, m) \ne (0, 0)$.

It follows from (2.2) that $a_i(\mathcal{C}_n, m) = a_i(\mathcal{C}_m, n)$. In [4–6], this equality is named the "Hermite reciprocity". As it has no relation to Hermite and was first proved by Fredman, the term *Fredman's reciprocity* seems to be more appropriate.

From (2.2), one can derive the equality

$$\sum_{(n,m)\in\mathbb{N}^2,\,(n,m)\ne(0,0)} a_i(\mathcal{C}_n, m) x^n y^m = -\sum_{d=1}^{\infty} \frac{c_d(i)}{d} \log\bigl(1 - x^d - y^d\bigr). \tag{2.3}$$

(Cf. [4, Remark 2], [6, Sect. 4].)

3 Symmetric tensor exterior algebra and Poincaré series

As above, let V be a G-module. We consider the Poincaré series of the G-isotypic components in $\mathcal{S}^{\cdot}V \otimes \wedge^{\cdot}V$. Let $(\mathcal{S}^{\cdot}V \otimes \wedge^{\cdot}V)_{G,\chi}$ denote the isotypic component corresponding to an irreducible representation χ. It is a bi-graded vector space and its Poincaré series is the formal power series

$$\mathcal{F}\bigl((\mathcal{S}^{\cdot}V \otimes \wedge^{\cdot}V)_{G,\chi}; s, t\bigr) = \sum_{p,m\ge 0} \dim\bigl(\mathcal{S}^p V \otimes \wedge^m V\bigr)_{G,\chi} s^p t^m.$$

(Clearly, it is a polynomial with respect to t.) It is known that

$$\mathcal{F}\bigl((\mathcal{S}^{\cdot}V \otimes \wedge^{\cdot}V)^G; s, t\bigr) = \frac{1}{\#G} \sum_{\gamma\in G} \frac{\det_V(\mathbb{1} + t\gamma)}{\det_V(\mathbb{1} - s\gamma)},$$

see [1, Theorem 1.33]. A similar argument provides the formula for an arbitrary G-isotypic component:

$$\mathcal{F}\bigl((\mathcal{S}^{\cdot}V \otimes \wedge^{\cdot}V)_{G,\chi}; s, t\bigr) = \frac{\deg(\chi)}{\#G} \sum_{\gamma\in G} \mathrm{tr}\bigl(\chi(\gamma^{-1})\bigr) \frac{\det_V(\mathbb{1} + t\gamma)}{\det_V(\mathbb{1} - s\gamma)}. \tag{3.1}$$

For, in place of the Reynolds operator $\frac{1}{\#(G)}\sum_{\gamma\in G}\gamma$ (the projection to the subspace of G-invariants), one should merely exploit the operator $\frac{\deg(\chi)}{\#(G)}\sum_{\gamma\in G}\operatorname{tr}(\chi(\gamma^{-1}))\gamma$ (the projection to the isotypic component of type χ).

Lemma 3.1 *For the regular representation $\mathcal{R}$ of G, the right-hand side of* (3.1) *can be written as*

$$\frac{\deg(\chi)}{\#G}\sum_{d\geq 1}\Bigg(\sum_{\gamma:\,\mathrm{ord}(\gamma)=d}\operatorname{tr}\big(\chi(\gamma^{-1})\big)\cdot\bigg(\frac{1-(-t)^d}{1-s^d}\bigg)^{(\#G)/d}\Bigg).$$

Proof If $\gamma\in G$ is of order d, then $\langle\gamma\rangle\simeq\mathcal{C}_d$ and each coset of $\langle\gamma\rangle$ in G is a cycle of length d with respect to the multiplication by γ. Hence, in a suitable basis of $\mathcal{R}$, the matrix of γ in $GL(\mathcal{R})$ consists of $(\#G)/d$ diagonal blocks

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \ddots & 0\\ 0 & 0 & \ddots & \ddots & 0\\ 0 & 0 & \ddots & \ddots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}$$

of size d. Since

$$\det\begin{bmatrix} 1 & -s & 0 & \dots & 0\\ 0 & 1 & -s & \ddots & 0\\ 0 & 0 & \ddots & \ddots & 0\\ 0 & 0 & \ddots & \ddots & -s\\ -s & 0 & 0 & \dots & 1\end{bmatrix}=1-s^d,$$

we obtain $\frac{\det_{\mathcal{R}}(\mathbb{1}+t\gamma)}{\det_{\mathcal{R}}(\mathbb{1}-s\gamma)}=(\frac{1-(-t)^d}{1-s^d})^{(\#G)/d}$, which proves the lemma. □

Now, we apply this lemma to the regular representation of $\mathcal{C}_n$. Recall that the number $\binom{a+b+c}{a,\,b,\,c}$ is defined to be $\frac{(a+b+c)!}{a!\,b!\,c!}$.

Theorem 3.2 *The Poincaré series of the $(\mathcal{C}_n,\chi_i)$-isotypic component equals*

$$\begin{aligned}&\mathcal{F}\big((\mathcal{S}^{\cdot}\mathcal{R}\otimes\wedge^{\cdot}\mathcal{R})_{\mathcal{C}_n,\chi_i};s,t\big)\\ &\quad=\frac{1}{n}\sum_{d|n}c_d(i)\frac{(1-(-t)^d)^{n/d}}{(1-s^d)^{n/d}}\\ &\quad=\frac{1}{n}\sum_{d|n}c_d(i)\Bigg(\sum_{a=0}^{n/d}(-1)^{(d+1)a}\binom{n/d}{a}t^{ad}\Bigg)\Bigg(\sum_{b\geq 0}\binom{(n/d)+b-1}{(n/d)-1}s^{bd}\Bigg).\end{aligned}\tag{3.2}$$

Consequently,

$$\dim\big(\mathcal{S}^p\mathcal{R}\otimes\wedge^m\mathcal{R}\big)_{\mathcal{C}_n,\chi_i}=\frac{(-1)^m}{p+n}\sum_{d|n,p,m}(-1)^{m/d}c_d(i)\binom{(n+p)/d}{m/d,\,p/d,\,(n-m)/d}. \tag{3.3}$$

Proof This is a straightforward consequence of Lemma 3.1. If $G=\mathcal{C}_n$, then $\deg(\chi_i)=1$ and $\sum_{\gamma:\,\mathrm{ord}(\gamma)=d}\chi_i(\gamma^{-1})=c_d(n-i)=c_d(i)$, which proves (3.2).

We leave it to the reader to extract the coefficient of t^ms^p in (3.2) and obtain (3.3). □

Letting $n=q+m$ yields a more symmetric form of (3.3):

$$\dim\big(\mathcal{S}^p\mathcal{R}\otimes\wedge^m\mathcal{R}\big)_{\mathcal{C}_{q+m},\chi_i}=\frac{(-1)^m}{p+q+m}\sum_{d|p,q,m}(-1)^{m/d}c_d(i)\binom{(m+p+q)/d}{m/d,\,p/d,\,q/d}. \tag{3.4}$$

As the right-hand side is symmetric with respect to p and q, we get an equality for dimensions of isotypic components related to the regular representations of two cyclic groups, $(\mathcal{C}_{q+m},\mathcal{R})$ and $(\mathcal{C}_{p+m},\tilde{\mathcal{R}})$:

$$\dim\big(\mathcal{S}^p\mathcal{R}\otimes\wedge^m\mathcal{R}\big)_{\mathcal{C}_{q+m},\chi_i}=\dim\big(\mathcal{S}^q\tilde{\mathcal{R}}\otimes\wedge^m\tilde{\mathcal{R}}\big)_{\mathcal{C}_{p+m},\chi_i}. \tag{3.5}$$

For $m=0$, this simplifies to Fredman's reciprocity [7, (4)]. It would be interesting to have a combinatorial interpretation of this symmetry in the spirit of Fredman's approach.

Remark 3.3

(1) Letting $t=0$ in (3.2) or $m=0$ in (3.3), we get known formulae for the isotypic components in the symmetric algebra of $\mathcal{R}$, see [4, 5]. Letting $s=0$ in (3.2) or $n=0$ in (3.3), we get interesting formulae for the isotypic components in the exterior algebra of $\mathcal{R}$, see the next section.
(2) If d is always odd (e.g. at least one of m,p,q is odd), then $(-1)^{m+\frac{m}{d}}=1$ and the right-hand side of (3.4) becomes totally symmetric with respect to p,q,m.

The following is a generalisation of (2.3):

Proposition 3.4

$$\sum_{(p,q,m)\in\mathbb{N}^3,\ p+q+m\ge 1}\dim\big(\mathcal{S}^p\mathcal{R}\otimes\wedge^m\mathcal{R}\big)_{\mathcal{C}_{q+m},\chi_i}\cdot x^py^qz^m$$
$$=-\sum_{d=1}^{\infty}\frac{c_d(i)}{d}\log\big(1-x^d-y^d+(-z)^d\big).$$

Proof By (3.4), the left-hand side equals

$$\sum_{p+q+m\ge 1}\frac{(-1)^m}{p+q+m}\sum_{d|p,q,m}(-1)^{m/d}c_d(i)\binom{(p+q+m)/d}{p/d,q/d,m/d}x^py^qz^m.$$

Letting $p/d=\alpha, q/d=\beta, m/d=\gamma$, we rewrite it as

$$\begin{aligned}
&\sum_{d=1}^{\infty}\frac{c_d(i)}{d}\sum_{\alpha+\beta+\gamma\geq 1}\frac{(-1)^{\gamma}}{\alpha+\beta+\gamma}\binom{\alpha+\beta+\gamma}{\alpha,\beta,\gamma}x^{\alpha d}y^{\beta d}(-z)^{\gamma d}\\
&\quad=\sum_{d=1}^{\infty}\frac{c_d(i)}{d}\sum_{k\geq 1}\biggl(\sum_{\alpha+\beta+\gamma=k}\frac{1}{k}\binom{k}{\alpha,\beta,\gamma}(x^d)^{\alpha}(y^d)^{\beta}\bigl(-(-z)^d\bigr)^{\gamma}\biggr)\\
&\quad=\sum_{d=1}^{\infty}\frac{c_d(i)}{d}\sum_{k\geq 1}\frac{(x^d+y^d-(-z)^d)^k}{k}=-\sum_{d=1}^{\infty}\frac{c_d(i)}{d}\log\bigl(1-x^d-y^d+(-z)^d\bigr). \qquad \square
\end{aligned}$$

Specialising the equality of Proposition 3.4, we get some interesting identities.

(A) Taking $x=y=0$ forces $p=q=0$ on the left-hand side, which leads to the equality

$$\sum_{m\geq 1}\dim\bigl(\wedge^m\mathcal{R}\bigr)_{\mathcal{C}_m,\chi_i}z^m=-\sum_{d=1}^{\infty}\frac{c_d(i)}{d}\log\bigl(1+(-z)^d\bigr).$$

For $i=0$, we have

$$c_d(0)=\varphi(d)\quad\text{and}\quad\dim\bigl(\wedge^m\mathcal{R}\bigr)^{\mathcal{C}_m}=\begin{cases}1 & m\text{ odd;}\\ 0 & m\text{ even.}\end{cases}$$

That is, $\frac{z}{1-z^2}=-\sum_{d=1}^{\infty}\frac{\varphi(d)}{d}\log(1+(-z)^d)$. Replacing z with $-z$ and exponentiating, we finally obtain:

$$\exp\biggl(\frac{z}{1-z^2}\biggr)=\prod_{d\geq 1}\bigl(1+z^d\bigr)^{\varphi(d)/d}.$$

(B) Likewise, for $x=z=0$ (or just $x=0$ in (2.3)), we get

$$-\sum_{d=1}^{\infty}\frac{c_d(i)}{d}\log\bigl(1-y^d\bigr)=\begin{cases}y/(1-y) & i=0;\\ 0 & i\neq 0.\end{cases}$$

In particular,

$$\exp\biggl(\frac{-y}{1-y}\biggr)=\prod_{d\geq 1}\bigl(1-y^d\bigr)^{\varphi(d)/d}.$$

4 On the exterior algebra of the regular representation

In case of the exterior algebra of a G-module, the Poincaré series of an isotypic component is actually a polynomial in t, which can be evaluated for any t. Here we gather some practical formulae for the regular representations and for cyclic groups.

First, using (3.1) and Lemma 3.1 with trivial χ and $s=0$, we obtain

$$\mathcal{F}\big((\wedge^\bullet\mathcal{R})^G;t\big)=\frac{1}{\#(G)}\sum_{d\ge 1}\varphi_G(d)\big(1-(-t)^d\big)^{\#(G)/d}.$$

It follows that $\mathcal{F}((\wedge^\bullet\mathcal{R})^G;t)$ always has the factor $1+t$ and

$$\dim(\wedge^\bullet\mathcal{R})^G=\frac{1}{\#(G)}\sum_{d\text{ odd}}\varphi_G(d)2^{\#(G)/d}. \tag{4.1}$$

Note that here G is not necessarily abelian!

Example 4.1 For $G=\mathcal{S}_3$, we have $\varphi_G(1)=1$, $\varphi_G(2)=3$, and $\varphi_G(3)=2$. Therefore,

$$\mathcal{F}\big((\wedge^\bullet\mathcal{R})^{\mathcal{S}_3};t\big)=\frac{1}{6}\big((1+t)^6+3\big(1-t^2\big)^3+2\big(1+t^3\big)^2\big)=1+t+t^2+4t^3+4t^4+t^5.$$

For $G=\mathcal{C}_n$, there are precise assertions for all G-isotypic components in $\wedge^\bullet\mathcal{R}$. Using Theorem 3.2 with $s=0$ and $p=0$, we obtain

$$\mathcal{F}\big((\wedge^\bullet\mathcal{R})_{\mathcal{C}_n,\chi_i};t\big)=\frac{1}{n}\sum_{d|n}c_d(i)\big(1-(-t)^d\big)^{n/d}, \tag{4.2}$$

$$\dim\big((\wedge^m\mathcal{R})_{\mathcal{C}_n,\chi_i}\big)=\frac{(-1)^m}{n}\sum_{d|n,m}(-1)^{m/d}c_d(i)\binom{n/d}{m/d}=:b_i(\mathcal{C}_n,m). \tag{4.3}$$

Again, it is convenient to replace n with $q+m$ in (4.3). Then

$$b_i(\mathcal{C}_{q+m},m)=\dim\big((\wedge^m\mathcal{R})_{\mathcal{C}_{q+m},\chi_i}\big)=\frac{(-1)^m}{q+m}\sum_{d|q,m}(-1)^{m/d}c_d(i)\binom{q/d+m/d}{m/d}.$$

From this we derive the following observation:

Proposition 4.2 *If q or m is odd, then $b_i(\mathcal{C}_{q+m},m)=a_i(\mathcal{C}_q,m)$ and also $b_i(\mathcal{C}_{q+m},m)=b_i(\mathcal{C}_{q+m},q)$.*

Example 4.3 $b_i(\mathcal{C}_{2n-1},n-1)=a_i(\mathcal{C}_{n-1},n)$, and it is the $(n-1)$th Catalan number regardless of i.

Remark If n is odd, then $\wedge^n\mathcal{R}$ is the trivial $\mathcal{C}_n$-module, and therefore $\wedge^m\mathcal{R}\simeq\wedge^{n-m}\mathcal{R}$ as $\mathcal{C}_n$-modules. This “explains” the equality $b_i(\mathcal{C}_n,m)=b_i(\mathcal{C}_n,n-m)$ for n odd.

Substituting $t=1$ in (4.2) yields a nice formula for dimension of the whole isotypic component:

$$\dim\big((\wedge^\bullet\mathcal{R})_{\mathcal{C}_n,\chi_i}\big)=\frac{1}{n}\sum_{d|n,\ d\text{ odd}}c_d(i)\,2^{n/d}. \tag{4.4}$$

For $i=0$, this becomes a special case of (4.1). There is a down-to-earth interpretation of (4.4) that does not invoke Invariant Theory. As in the introduction, choose a basis $\{v_0, v_1, \dots, v_{n-1}\}$ for $\mathcal{R}$ such that v_i has weight χ_i. Then

$$v_{j_1} \wedge \cdots \wedge v_{j_m} \in \left(\wedge^m \mathcal{R}\right)_{\mathcal{C}_n, \chi_i} \quad \Longleftrightarrow \quad j_1 + \cdots + j_m \equiv i \bmod n.$$

Consequently, $\dim(\wedge^{\bullet}\mathcal{R})_{\mathcal{C}_n,\chi_i}$ equals the number of subsets $J \subset \{0, 1, \dots, n-1\}$ such that $|J| \equiv i \bmod n$. (Here $|J|$ stands for the sum of elements of J.) Hence our invariant-theoretic approach proves the following purely combinatorial fact:

$$\#\bigl\{J \subset \{0, 1, \dots, n-1\} \mid |J| \equiv i \bmod n\bigr\} = \frac{1}{n} \sum_{d|n,\ d \text{ odd}} c_d(i) 2^{n/d}.$$

For $i=0$, this is nothing but the number of subsets of $\mathcal{C}_n$ summing to the neutral element (in the additive notation). We provide a similar interpretation for any abelian group.

Theorem 4.4 *For an abelian group G, let $\mathcal{N}_G$ denote the number of subsets S of G such that $|S| := \sum_{\gamma\in S}\gamma = 0 \in G$. Then $\mathcal{N}_G = \dim(\wedge^{\bullet}\mathcal{R})^G = \frac{1}{\#(G)} \times \sum_{d \text{ odd}} \varphi_G(d) 2^{\#(G)/d}$.*

Proof In view of (4.1), only the first equality requires a proof. Let $(z_0, \dots, z_{n-1})$ be a basis for $\mathcal{R}$ consisting of G-eigenvectors, $n = \#(G)$. Here the weight of z_i is a linear character χ_i and $\hat{G} = \{\chi_0, \chi_1, \dots, \chi_{n-1}\}$ is the dual group of G. One of the χ_i's is the neutral element of $\hat{G}$, denoted by $\hat{0}$ in the additive notation. Then

$$z_{j_1} \wedge \cdots \wedge z_{j_m} \in \left(\wedge^m \mathcal{R}\right)^G \quad \Longleftrightarrow \quad \chi_{j_1} + \cdots + \chi_{j_m} = \hat{0} \in \hat{G}.$$

Thus, $\dim(\wedge^{\bullet}\mathcal{R})^G$ equals the number of subsets of $\hat{G}$ summing to $\hat{0}$. However, the groups $\hat{G}$ and G are (non-canonically) isomorphic, hence $\mathcal{N}_G = \mathcal{N}_{\hat{G}}$ and we are done. □

5 On the permanent of the Cayley table of an abelian group

In this section, G is an abelian group, $G = \{x_0, x_1, \dots, x_{n-1}\}$. The *Cayley table* of G, denoted $\mathcal{M}_G = (m_{i,j})$, can be regarded as n by n matrix with entries in the polynomial ring $\mathbb{C}[x_0, x_1, \dots, x_{n-1}] \simeq \mathcal{S}^{\bullet}\mathcal{R}$. To distinguish the addition in $\mathbb{C}[x_0, x_1, \dots, x_{n-1}]$ and the group operation in G, the latter is denoted by '$\dotplus$'. By definition, $m_{i,j} = x_i \dotplus x_j$, $i, j = 0, \dots, n-1$. Hence $\mathcal{M}_G$ is a symmetric matrix. The permanent of $\mathcal{M}_G$, $\mathrm{per}(\mathcal{M}_G)$, is a homogeneous polynomial of degree n in x_i's, and it does not depend on the ordering of elements of G. Let $p(G)$ denote the number of formally different monomials occurring in $\mathrm{per}(\mathcal{M}_G)$.

Remark 5.1 In place of the Cayley table, one can consider the matrix $\hat{\mathcal{M}}_G$ with entries $\hat{m}_{i,j} = x_i \ominus x_j$ (the difference in G). Clearly, $\hat{\mathcal{M}}_G$ is obtained from $\mathcal{M}_G$ by rearranging the columns only (or, the rows only), using the permutation on G that takes each element to its inverse. Therefore, $\operatorname{per}(\hat{\mathcal{M}}_G)=\operatorname{per}(\mathcal{M}_G)$ and $\det(\hat{\mathcal{M}}_G)=\pm\det(\mathcal{M}_G)$. Although $\hat{\mathcal{M}}_G$ is not symmetric in general, an advantage is that every entry on the main diagonal is the neutral elements of G.

Example 5.2 For $G=\mathcal{C}_n$ and the natural ordering of its elements (i.e., when x_i corresponds to i), one obtains a generic *circulant matrix* (the latter means that the rows are successive cyclic permutations of the first row). More precisely, $\mathcal{M}_{\mathcal{C}_n}$ (resp., $\hat{\mathcal{M}}_{\mathcal{C}_n}$) is a circulant matrix in Hankel (resp., Toeplitz) form. For instance,

$$\mathcal{M}_{\mathcal{C}_3}=\begin{pmatrix} x_0 & x_1 & x_2\\ x_1 & x_2 & x_0\\ x_2 & x_0 & x_1\end{pmatrix}.$$

Here $\operatorname{per}(\mathcal{M}_{\mathcal{C}_3})=x_0^3+x_1^3+x_2^3+3x_0x_1x_2$. Therefore, $p(\mathcal{C}_3)=4$.

The function $n\mapsto p(\mathcal{C}_n)$ was studied in [3] where it was pointed out that the main result of Hall [8] shows that $p(\mathcal{C}_n)$ equals the number of solutions to

$$\begin{cases}\lambda_0+\cdots+\lambda_{n-1}=n,\\ \sum_{j=0}^{n-1} j\lambda_j\equiv 0 \bmod n.\end{cases}$$

That is, $p(\mathcal{C}_n)=a_0(\mathcal{C}_n,n)$ in our notation. Because results of [8] apply to arbitrary finite abelian groups, one can be interested in $p(G)$ in this more general setting. Below, we give an invariant-theoretic answer using that result of Hall.

Let $\mathbb{S}_n$ denote the symmetric group acting by permutations on $\{0,1,\dots,n-1\}$. Accordingly, $\mathbb{S}_n$ permutes the elements of G by the rule $\pi(x_i):=x_{\pi(i)}$. Recall that

$$\operatorname{per}(m_{i,j})=\sum_{\pi\in\mathbb{S}_n}\prod_{i=0}^{n-1} m_{i,\pi(i)}.$$

For the matrix $\mathcal{M}_G$,

$$\prod_{i=0}^{n-1} m_{i,\pi(i)}=\prod_{i=0}^{n-1}(x_i\dotplus x_{\pi(i)})=\prod_{i=0}^{n-1} x_i^{k_i(\pi)}=:\boldsymbol{x}(\pi)$$

is a monomial in x_i's of degree n. Note that different permutations may result in the same monomial. The following is essentially proved by M. Hall.

Theorem 5.3 [8, n. 3] *A monomial* $\mathfrak{m}=\prod_{i=0}^{n-1}x_i^{k_i}$ *is of the form* $\boldsymbol{x}(\pi)$ *for some* $\pi\in\mathbb{S}_n$ *(i.e., occurs in* $\operatorname{per}(\mathcal{M}_G)$*) if and only if* $\sum_i k_i=n$ *and* $k_0x_0\dotplus\cdots\dotplus k_{n-1}x_{n-1}=0\in G$. [Of course, here k_ix_i stands for $x_i\dotplus\cdots\dotplus x_i$ (k_i times).]

The necessity of the conditions is easy; a non-trivial argument is required for the sufficiency, i.e., for the existence of π.

Theorem 5.4 $p(G)=\dim\mathcal{S}^n(\mathcal{R})^G$.

Proof Let $(z_0, \dots, z_{n-1})$ be a basis for $\mathcal{R}$ consisting of G-eigenvectors. Recall that the weight of z_i is χ_i and $\hat{G} = \{\chi_0, \dots, \chi_{n-1}\}$ is the dual group. The monomial $z_0^{k_0} \cdots z_{n-1}^{k_{n-1}} \in \mathcal{S}^{\cdot}\mathcal{R}$ is a semi-invariant of G of weight $k_0\chi_0 \dotplus \cdots \dotplus k_{n-1}\chi_{n-1} \in \hat{G}$. It follows that

$$\dim \mathcal{S}^n(\mathcal{R})^G = \left\{(k_0, \dots, k_{n-1}) \,\middle|\, \sum_i k_i = n \text{ and } k_0\chi_0 \dotplus \cdots \dotplus k_{n-1}\chi_{n-1} = \hat{0}\right\}.$$

Modulo the passage from G to $\hat{G}$, these conditions coincide with those of Theorem 5.3. Since $G \simeq \hat{G}$, we are done. □

Our next goal is to extend these results to a certain matrix of order $n+1$. We begin with two assertions on $\text{per}(\mathcal{M}_G)$ which are of independent interest.

Proposition 5.5 *There is a natural action* $* : G \times \mathcal{S}_n \to \mathcal{S}_n$ *such that, for* $\gamma \in G$ *and* $\pi \in \mathcal{S}_n$, $\text{sign}(\gamma * \pi) = \text{sign}(\pi)$ *and* $\boldsymbol{x}(\gamma * \pi) = \boldsymbol{x}(\pi)$.

Proof Every $\gamma \in G$ determines a permutation σ_γ on G and thereby an element of $\mathcal{S}_n$. Namely,

$$(x_0, \dots, x_{n-1}) \overset{\sigma_\gamma}{\mapsto} (\gamma \dotplus x_0, \dots, \gamma \dotplus x_{n-1}).$$

Equivalently, $x_{\sigma_\gamma(i)} = x_i \dotplus \gamma$. Define the G-action on $\mathcal{S}_n$ by $\gamma * \pi = \sigma_\gamma \pi \sigma_\gamma$. Hence $\text{sign}(\gamma * \pi) = \text{sign}(\pi)$. Recall that $\boldsymbol{x}(\pi) = \prod_{i=0}^{n-1}(x_i \dotplus x_{\pi(i)})$. Then

$$\boldsymbol{x}(\gamma * \pi) = \prod_{i=0}^{n-1}(x_i \dotplus x_{\sigma_\gamma \pi \sigma_\gamma(i)}) = \prod_{j=0}^{n-1}(x_{\sigma_\gamma^{-1}(j)} \dotplus x_{\sigma_\gamma \pi(j)}),$$

where $j = \sigma_\gamma(i)$. By definition, $x_{\sigma_\gamma \pi(j)} = x_{\pi(j)} \dotplus \gamma$ and $x_j = x_{\sigma_\gamma^{-1}(j)} \dotplus \gamma$. Thus, the linear factors of $\boldsymbol{x}(\gamma * \pi)$ remain the same. □

Remark 5.6 Our action '$*$' can be regarded as a generalisation of Lehmer's "operator S" for circulant matrices [11, p. 45], i.e., essentially, for $G = \mathcal{C}_n$. Using that operator Lehmer proved that, for $n = p$ odd prime,

$$\det(\mathcal{M}_{\mathcal{C}_p}) = x_0^p + \cdots + x_{p-1}^p + pF(x_0, \dots, x_{p-1}),$$

where $F \in \mathbb{Z}[x_0, \dots, x_{p-1}]$. We note that Lehmer's argument applies to $\text{per}(\mathcal{M}_{\mathcal{C}_p})$ as well.

Proposition 5.7 *Suppose that* $\mathfrak{m}$ *is a monomial in* $\text{per}(\mathcal{M}_G)$ *such that* x_k *occurs in* $\mathfrak{m}$. *If* $x_k = x_i \dotplus x_j$ *for some* i, j, *then there is* $\sigma \in \mathcal{S}_n$ *such that* $\sigma(i) = j$ *and* $\mathfrak{m} = \boldsymbol{x}(\sigma)$.

Proof By the assumption on $\mathfrak{m}$, there is a $\pi \in \mathcal{S}_n$ such that $\mathfrak{m} = \boldsymbol{x}(\pi)$ and $x_k = x_\alpha \dotplus x_\beta$ for some α, β with $\pi(\alpha) = \beta$. If $\{\alpha, \beta\} \neq \{i, j\}$, then we have to correct π. Take $\gamma \in G$ such that $x_i \dotplus \gamma = x_\alpha$. Then $x_\beta \dotplus \gamma = x_j$ and for $\sigma = \gamma * \pi$ we have

$$\sigma(x_i) = \sigma_\gamma \pi \sigma_\gamma(x_i) = \sigma_\gamma \pi(x_\alpha) = \sigma_\gamma(x_\beta) = x_\beta \dotplus \gamma = x_j.$$

Thus, $\sigma(i) = j$ and also $\boldsymbol{x}(\sigma) = \boldsymbol{x}(\pi)$ in view of Proposition 5.5. □

The Cayley table of G is the "addition table" of all elements of G. Define the *extended Cayley table* as an $n+1$ by $n+1$ matrix that the "addition table" of $n+1$ elements of G, with the neutral element is taken twice. More precisely, we assume that $x_0 = x_n = 0$ is the neutral element of G and consider the matrix $\widetilde{\mathcal{M}}_G = (m_{i,j})$, where $m_{i,j} = x_i \dotplus x_j$, $i, j = 0, 1, \ldots, n$. In this context, $\mathcal{S}_{n+1}$ is regarded as permutation group on $\{0, 1, \ldots, n\}$. Then $\operatorname{per}(\widetilde{\mathcal{M}}_G) = \sum_{\tilde{\pi} \in \mathcal{S}_{n+1}} \boldsymbol{x}(\tilde{\pi})$ is a sum of monomials of degree $n+1$.

Example

$$\widetilde{\mathcal{M}}_{C_3} = \begin{pmatrix} x_0 & x_1 & x_2 & x_0 \\ x_1 & x_2 & x_0 & x_1 \\ x_2 & x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 & x_0 \end{pmatrix},$$

$\operatorname{per}(\widetilde{\mathcal{M}}_{C_3}) = 2x_0^4 + 10x_0^2 x_1 x_2 + 4x_0 x_1^3 + 4x_0 x_2^3 + 4x_1^2 x_2^2$.

Theorem 5.8 *The monomial* $\mathfrak{m} = \prod_{i=0}^{n-1} x_i^{k_i}$ *occurs in* $\operatorname{per}(\widetilde{\mathcal{M}}_G)$ *if and only if*

$$\sum_i k_i = n+1 \quad \textit{and} \quad k_0 x_0 \dotplus \cdots \dotplus k_{n-1} x_{n-1} = 0 \in G.$$

Proof "$\Rightarrow$". Suppose $\mathfrak{m} = \boldsymbol{x}(\tilde{\pi})$ for some $\tilde{\pi} \in \mathcal{S}_{n+1}$. Obviously, $\deg \mathfrak{m} = n+1$. Next,

$$k_0 x_0 \dotplus \cdots \dotplus k_{n-1} x_{n-1} = (x_0 \dotplus x_{\tilde{\pi}(0)}) \dotplus (x_1 \dotplus x_{\tilde{\pi}(1)}) \dotplus \cdots \dotplus (x_n \dotplus x_{\tilde{\pi}(n)}) = 0,$$

since the multiset $\{x_0, x_1, \ldots, x_{n-1}, x_n = x_0\}$ is closed with respect to taking inverses.

"$\Leftarrow$". Suppose $\mathfrak{m}$ satisfies the conditions of the theorem.

- $k_0 > 0$. Take $\mathfrak{m}' = x_0^{k_0 - 1} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$. Then $\mathfrak{m}'$ satisfies the conditions of Theorem 5.3. Therefore, $\mathfrak{m}'$ is a monomial of $\operatorname{per}(\mathcal{M}_G)$ and there is a $\pi \in \mathcal{S}_n$ such that $\mathfrak{m}' = \boldsymbol{x}(\pi)$. Embed $\mathcal{S}_n$ into $\mathcal{S}_{n+1}$ as the subgroup preserving the last element n. Let $\tilde{\pi}$ denote π considered as an element of $\mathcal{S}_{n+1}$. Then $\mathfrak{m} = \boldsymbol{x}(\tilde{\pi})$.
- $k_0 = 0$. Choose any binomial $x_i x_j$ in $\mathfrak{m}$ and replace it with $(x_i \dotplus x_j) x_0 = x_k x_0$ (i.e., $x_i \dotplus x_j = x_k$). That is, $\mathfrak{m} = \mathfrak{m}'' x_i x_j$ is replaced with $\mathfrak{m}'' x_k x_0 =: \mathfrak{m}' x_0$. By the previous argument, we can find $\pi \in \mathcal{S}_n$ such that $\mathfrak{m}' = \boldsymbol{x}(\pi)$ and $\mathfrak{m}' x_0 = \boldsymbol{x}(\tilde{\pi})$. Since $x_k = x_i \dotplus x_j$ occurs in $\boldsymbol{x}(\pi)$, we can apply Proposition 5.7 and assume that $\pi(i) = j$ and hence $\tilde{\pi}(i) = j$. Finally, we replace $\tilde{\pi}$ with $\tilde{\pi}\tau$, where the transposition $\tau \in \mathcal{S}_{n+1}$ permutes i and n. One readily verifies that $\boldsymbol{x}(\tilde{\pi}\tau) = \mathfrak{m}'' x_i x_j = \mathfrak{m}$.

□

Corollary 5.9 *The number of different monomials in* $\operatorname{per}(\widetilde{\mathcal{M}}_G)$ *equals* $\dim(\mathcal{S}^{n+1}\mathcal{R})^G$.

The proof is almost identical to that of Theorem 5.4 and left to the reader.

It follows from Frobenius' theory of group determinants (see, e.g. [10, Sect. 2]) that, for abelian groups, $\det(\mathcal{M}_G)$ is the product of linear forms in x_i's. In case of generic circulant matrices, this fact plays an important role in [11, 13]. For future use, we provide a quick derivation. Recall that $G = \{x_0, x_1, \ldots, x_{n-1}\}$ and

$\hat{G} = \{\chi_0, \chi_1, \ldots, \chi_{n-1}\}$. Consider the n by n complex matrix $\mathcal{K}_G$, with $(\mathcal{K}_G)_{i,j} = (\chi_j(x_i))$, and the vectors $v_j = \sum_{i=0}^{n-1} \chi_j(x_i)x_i \in \mathcal{R}$, $j = 0, 1, \ldots, n-1$.

Proposition 5.10 *Under the above notation, we have*:

(1) v_j *is an eigenvector of* G *corresponding to the weight* χ_j^{-1};
(2) $\det(\mathcal{M}_G) \cdot \det(\mathcal{K}_G) = \det(\overline{\mathcal{K}_G}) v_0 v_1 \cdots v_{n-1}$, *where 'bar' stands for the complex conjugation*;
(3) $\det(\overline{\mathcal{K}_G}) / \det(\mathcal{K}_G)$ *equals the sign of the permutation* $\pi_0 \in \mathcal{S}_n$ *that takes each* x_i *to its inverse. Hence* $\det(\mathcal{M}_G) = \operatorname{sign}(\pi_0) v_0 v_1 \cdots v_{n-1}$.

Proof

(1) Obvious.
(2) It is easily seen that $(\mathcal{M}_G \cdot \mathcal{K}_G)_{ij} = \chi_j(x_i)^{-1} v_j = \overline{\chi_j(x_i)} v_j = (\overline{\mathcal{K}_G})_{i,j} v_j$.
(3) Assuming that x_0 is the neutral element, compare the coefficient of x_0^n in both parts of the equality in (2). □

Note that $\widetilde{\mathcal{M}}_G$ has equal columns and hence $\det(\widetilde{\mathcal{M}}_G) = 0$.

Remark 5.11

1. The set of vectors $\{v_j\}$ is closed with respect to complex conjugation, and letting $z_j = \overline{v_j} = \sum_i \overline{\chi_j(x_i)} x_i$ one obtains the eigenvector corresponding to χ_j.
2. The orthogonality relations for the characters imply that $\mathcal{K}_G (\overline{\mathcal{K}_G})^t = n \mathbb{1}_n$; that is, $\frac{1}{\sqrt{n}} \mathcal{K}_G$ is unitary and $|\det(\mathcal{K}_G)|^2 = n^n$.

For the sake of completeness, we mention some other easy properties.

Proposition 5.12 *Suppose* $\gamma \in G$ *and* $\pi \in \mathcal{S}_n$.

(1) $\gamma \cdot \boldsymbol{x}(\pi) = \boldsymbol{x}(\pi \sigma_\gamma^{-1})$, *where '·' stands for the natural* G*-action on* $\mathcal{S}^n \mathcal{R}$;
(2) $\boldsymbol{x}(\pi) = \boldsymbol{x}(\pi^{-1})$;
(3) $\operatorname{per}(\mathcal{M}_G) \in (\mathcal{S}^n \mathcal{R})^G$;
(4) *If* $\hat{G}$ *has a unique element of order* 2, *say* ψ, *then* $\det(\mathcal{M}_G)$ *is a semi-invariant of weight* ψ. *In all other cases*, $\det(\mathcal{M}_G) \in (\mathcal{S}^n \mathcal{R})^G$.

Proof

(1) $\gamma \cdot \boldsymbol{x}(\pi) = \gamma \cdot \prod_{i=0}^{n-1} (x_i \dotplus x_{\pi(i)}) = \prod_{i=0}^{n-1} (x_i \dotplus x_{\pi(i)} \dotplus \gamma) = \prod_{i=0}^{n-1} (x_{\sigma_\gamma(i)} \dotplus x_{\pi(i)}) = \boldsymbol{x}(\pi \sigma_\gamma^{-1})$.
(2) Obvious.
(3) Follows from (1).
(4) Proposition 5.10 shows that $\det(\mathcal{M}_G)$ is a semi-invariant whose weight equals the sum of all elements of $\hat{G}$. The sum of all elements of an abelian group is known to be the neutral element unless the group has a unique element of order 2, in which case the sum is this unique element. □

Note that $\operatorname{per}(\widetilde{\mathcal{M}}_G)$ is an element of $\mathcal{S}^{n+1}\mathcal{R}$, but it does not belong to $(\mathcal{S}^{n+1}\mathcal{R})^G$.

6 Some open problems

Associated with previous results on $\mathrm{per}(\mathcal{M}_G)$, there are some interesting problems. Let $d(G)$ denote the number of different monomials in $\det(\mathcal{M}_G)$. In view of possible cancellations, we have $d(G) \le p(G)$. Using the factorisation of $\det(\mathcal{M}_{\mathcal{C}_n})$ and theory of symmetric functions, Thomas [13] proved that $d(\mathcal{C}_n) = p(\mathcal{C}_n)$ whenever n is a prime power. He also computed these values up to $n = 12$ (e.g. $d(\mathcal{C}_6) = 68 < 80 = p(\mathcal{C}_6)$) and suggested that the converse could be true.

Problem 1 *What are necessary/sufficient conditions on a finite abelian group G for the equality $d(G) = p(G)$? Specifically, is it still true that the condition '$\#(G)$ is a prime power' is sufficient?*

The equality $\det(\mathcal{M}_G) = \mathrm{sign}(\pi_0)v_0v_1\cdots v_{n-1}$ might be helpful in resolving Problem 1. The following problem is more general and vague.

Problem 2 *Let $\mathfrak{m}$ be a monomial that satisfies conditions of Theorem 5.3. Is there a group-theoretic (or invariant-theoretic) interpretation of the coefficient of $\mathfrak{m}$ in* $\mathrm{per}(\mathcal{M}_G)$ *or* $\det(\mathcal{M}_G)$*?*

For $G = \mathcal{C}_2 \oplus \mathcal{C}_2$, we have $p(G) = d(G) = 11$. Because $p(\mathcal{C}_4) = d(\mathcal{C}_4) = 10$, one may speculate that $p(G) \ge p(\mathcal{C}_n)$ and $d(G) \ge d(\mathcal{C}_n)$ if $\#(G) = n$. By Theorem 5.4, $p(G)$ is the coefficient of t^n in the Poincaré series $\mathcal{F}((\mathcal{S}^{\cdot}\mathcal{R})^G; t)$, and one can consider a related

Problem 3 *Is it true that* $[t^m]\mathcal{F}((\mathcal{S}^{\cdot}\mathcal{R})^G; t) \ge [t^m]\mathcal{F}((\mathcal{S}^{\cdot}\mathcal{R})^{\mathcal{C}_n}; t)$ *for any $m \in \mathbb{N}$?*

Given $G = \{x_0, \dots, x_{n-1}\}$, a family of matrices $\mathcal{M}_{G,l} \in \mathrm{Mat}_l(\mathbb{C}[x_0, \dots, x_{n-1}])$, $l \ge n$, is said to be *admissible*, if $\mathcal{M}_{G,n} = \mathcal{M}_G$, $\mathcal{M}_{G,l}$ is a principal submatrix of $\mathcal{M}_{G,l+1}$, and the number of different monomials in $\mathrm{per}(\mathcal{M}_{G,l})$ equals $\dim(\mathcal{S}^l\mathcal{R})^G$.

Problem 4 *For what G, does an admissible family exist?*

So far, we only have matrices $\mathcal{M}_{G,l}$ for $l = n, n+1$. It is possible to jump up to $l = 2n$ by letting $\mathcal{M}_{G,2n} = \left(\begin{smallmatrix}\mathcal{M}_G & \mathcal{M}_G \\ \mathcal{M}_G & \mathcal{M}_G\end{smallmatrix}\right)$. It is the addition table for two consecutive sets of group elements, and it can be proved that $\mathrm{per}(\mathcal{M}_{G,2n})$ has the required property. Then, similarly to the construction of the extended Cayley table, one defines a larger matrix $\mathcal{M}_{G,2n+1}$. This procedure can be iterated, so one obtains a suitable collection of matrices of orders $kn, kn+1$, $k \in \mathbb{N}$. However, it is not clear whether it is possible to define matrices $\mathcal{M}_{G,l}$ for all other l. Maybe the reason is that, for arbitrary abelian G, there is no natural ordering of its elements. But, for a cyclic group, one does have a natural ordering, and we provide a conjectural definition of an admissible family of matrices.

For $G = \mathcal{C}_n$, it will be convenient to begin with the circulant matrix in the Toeplitz form, see Example 5.2. That is to say, our initial matrix is $\hat{\mathcal{M}}_{\mathcal{C}_n} = (\hat{m}_{i,j})$, where

$\hat{m}_{i,j} = x_{i-j}$, $i, j = 0, 1, \ldots, n-1$, and the subscripts of x's are interpreted $\pmod n$. For any $l \geq n$, we then define the entries of $\hat{\mathcal{M}}_{\mathcal{C}_n,l}$ by the same formula, only the range of i, j is extended. In particular, $\hat{\mathcal{M}}_{\mathcal{C}_n,l}$ is a Toeplitz matrix for any l.

Example

$$\hat{\mathcal{M}}_{\mathcal{C}_{3,5}} = \begin{pmatrix} x_0 & x_1 & x_2 & x_0 & x_1 \\ x_2 & x_0 & x_1 & x_2 & x_0 \\ x_1 & x_2 & x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 & x_0 & x_1 \\ x_2 & x_0 & x_1 & x_2 & x_0 \end{pmatrix}.$$

Conjecture 6.1 *For $l \geq n$, the monomial $x_0^{\lambda_0} x_1^{\lambda_1} \cdots x_{n-1}^{\lambda_{n-1}}$ occurs in* $\operatorname{per}(\hat{\mathcal{M}}_{\mathcal{C}_n,l})$ *if and only if*

$$(*) \qquad \lambda_0 + \cdots + \lambda_{n-1} = l \quad \textit{and} \quad \sum_{j=1}^{n-1} j\lambda_j \equiv 0 \bmod n.$$

In particular, the number of different monomials in $\operatorname{per}(\hat{\mathcal{M}}_{\mathcal{C}_n,l})$ *equals* $a_0(\mathcal{C}_n, l)$.

It is not hard to verify the necessity of $(*)$ and that the conjecture is true for $n = 2$.

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