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An inductive approach to Coxeter arrangements and Solomon's descent algebra

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Abstract In our recent paper (Douglass et al. arXiv:1101.2075 (2011)), we claimed that both the group algebra of a finite Coxeter group W as well as the Orlik–Solomon algebra of W can be decomposed into a sum of induced one-dimensional representations of centralizers, one for each conjugacy class of elements of W, and gave a uniform proof of this claim for symmetric groups. In this note, we outline an inductive approach to our conjecture. As an application of this method, we prove the inductive version of the conjecture for finite Coxeter groups of rank up to 2.

Keywords Coxeter groups · Reflection arrangements · Descent algebra · Dihedral groups

1 Introduction

Let W be a finite Coxeter group, generated by a set S of simple reflections. If $|S| = r$, then W acts as a reflection group on Euclidean r-space V. The reflection arrangement of W is the hyperplane arrangement consisting of the reflecting hyperplanes in V of all the reflections in W. The Orlik–Solomon algebra $A(W)$ of W is the cohomology

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ring of the complement of the complexified reflection arrangement. It follows from a result of Brieskorn [3] that the algebra $A(W)$ is a W-module of dimension $|W|$. For some history of the computation of $A(W)$ as a W-module, see the introduction of our recent paper [4].

In [4], we claimed that both the group algebra $\mathbb{C}W$ of W (affording the regular character ρ_W) as well as the Orlik–Solomon algebra $A(W)$ (affording the Orlik–Solomon character ω_W) can be decomposed into a sum of induced one-dimensional representations of centralizers, one for each conjugacy class of elements of W, in the following interlaced way.

Conjecture A *Let $\mathcal{R}$ be a set of representatives of the conjugacy classes of W. Then, for each $w \in \mathcal{R}$, there are linear characters $\widetilde{\varphi}_w$ and $\widetilde{\psi}_w$ of $C_W(w)$ such that*

$$\rho_W = \sum_{w\in\mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \widetilde{\varphi}_w, \qquad \omega_W = \sum_{w\in\mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \widetilde{\psi}_w$$

are sums of induced linear characters. Moreover, for each $w \in \mathcal{R}$, the characters $\widetilde{\varphi}_w$ and $\widetilde{\psi}_w$ can be chosen so that

$$\widetilde{\psi}_w = \widetilde{\varphi}_w \epsilon \alpha_w,$$

where ϵ is the sign character of W, and α_w is the determinant on the 1-eigenspace of w.

When W is a symmetric group, the formula for ρ_W has been proved independently by Bergeron, Bergeron, and Garsia [1], Hanlon [6], and Schocker [14]. The formula for ω_W follows from work of Lehrer and Solomon [9], who also checked the identity for ω_W in the case of a dihedral group W. Conjecture 2.1 in [4] is a graded refinement of Conjecture A and the main result in [4] is a uniform proof of this refined conjecture for symmetric groups.

The details of the proof of Conjecture 2.1 in [4] for symmetric groups rely on properties of these groups not shared by other finite Coxeter groups. However, the underlying strategy of the proof using induced characters both generalizes and admits a "relative" version, for pairs (W, W_L), where W_L is a parabolic subgroup of W. In Sect. 4, we formalize this notion in Conjecture C, show how it leads to a proof of Conjecture A, and describe a two-step procedure that can be used to prove this relative conjecture. Prior to that, in Sects. 2 and 3 we review some notation and basic facts about the descent algebra $\Sigma(W)$ and the Orlik–Solomon algebra $A(W)$. In the final section, we apply the methods from Sect. 4 and prove Conjecture C for all pairs (W, W_L) where W is arbitrary and W_L has rank at most 2. As a consequence, we deduce that Conjecture A holds for Coxeter groups of rank 2 or less.

2 Minimal length transversals of parabolic subgroups

The descent algebra of a finite Coxeter group W encodes many aspects of the combinatorics of the minimal length coset representatives of the standard parabolic subgroups of W. In this section, we provide notation and summarize useful properties of these distinguished coset representatives following Pfeiffer [12].

For $J \subseteq S$, let

$$X_J = \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in J\}.$$

Then X_J is a right transversal of the parabolic subgroup $W_J = \langle J \rangle$ of W, consisting of the unique elements of minimal length in their cosets. If we set

$$x_J = \sum_{x \in X_J} x^{-1} \in \mathbb{C}W,$$

then, by Solomon's Theorem [15], the subspace

$$\Sigma(W) = \langle x_J : J \subseteq S \rangle_{\mathbb{C}}$$

is a 2^r-dimensional subalgebra of the group algebra $\mathbb{C}W$, called the descent algebra of W.

For $J \subseteq S$, denote

$$X_J^\sharp = \{x \in X_J : J^x \subseteq S\}.$$

The action of W on itself by conjugation partitions the power set of S into equivalence classes of W-conjugate subsets. We call the class

$$[J] = \{J^x : x \in X_J^\sharp\}$$

of a subset $J \subseteq S$ the *shape* of J, and denote by

$$\Lambda = \{[J] : J \subseteq S\}$$

the set of shapes of W. The shapes parametrize the conjugacy classes of parabolic subgroups of W, since two subsets $J, K \subseteq S$ are conjugate if and only if the corresponding parabolic subgroups W_J and W_K are conjugate. We say that a parabolic subgroup of W has shape $[J]$ if it is conjugate to W_J in W.

Furthermore, for $J \subseteq S$, we define

$$N_J = \{x \in X_J : J^x = J\}.$$

Then N_J is a subgroup of W and by results of Howlett [7], the normalizer of W_J in W is a semi-direct product $N_W(W_J) = W_J \rtimes N_J$.

An element $w \in W$ is called *cuspidal* in case w has no fixed points in the reflection representation of W. Thus, for $J \subseteq S$, an element $w \in W_J$ is cuspidal in the parabolic subgroup W_J if w has no fixed points in the orthogonal complement of $\mathrm{Fix}(W_J)$ in V, where $\mathrm{Fix}(W_J)$ is the fixed point subspace of W_J in V. If w is a cuspidal element in W_J, then the quotient $C_W(w)/C_{W_J}(w)$ is isomorphic to N_J (see [8]).

We consider the character α_J of $N_W(W_J)$, defined, for $w \in N_W(W_J)$, as

$$\alpha_J(w) = \det(w|_{\mathrm{Fix}(W_J)}).$$

Note that W_J is contained in the kernel of α_J and so $\alpha_J(un) = \alpha_J(n)$ for $u \in W_J$, $n \in N_J$.

Lemma 2.1 *Let $J \subseteq S$. For $n \in N_J$ denote by $\sigma_J(n)$ the sign of the permutation induced on J by conjugation with n. Then*

$$\sigma_J(n) = \epsilon(n)\alpha_J(n),$$

for all $n \in N_J$.

Proof Denote by V_J the orthogonal complement of $\mathrm{Fix}(W_J)$ in V. Then V_J affords the reflection representation of the parabolic subgroup W_J, and the decomposition $V = V_J \oplus \mathrm{Fix}(W_J)$ is $N_W(W_J)$-stable. For $n \in N_J$, the matrix of n on V_J is equivalent to the permutation matrix of the conjugation action of n on J and thus has determinant $\sigma_J(n)$. The matrix of n on $\mathrm{Fix}(W_J)$ has determinant $\alpha_J(n)$, by definition. Consequently, the determinant of n on V is $\epsilon(n) = \sigma_J(n)\alpha_J(n)$. □

Pfeiffer and Röhrle [13] call W_J a *bulky* parabolic subgroup of W if $N_W(W_J)$ is isomorphic to the direct product $W_J \times N_J$, or equivalently, if N_J centralizes W_J. Notice that W_J is bulky whenever W_J is a self-normalizing subgroup of W. Suppose W_J is bulky in W. Then $\sigma_J(n) = 1$ for all $n \in N_J$. Consequently, for $u \in W_J$ and $n \in N_J$, we have

$$\epsilon(un)\alpha_J(un) = \epsilon(u). \tag{2.2}$$

Thus, the character $\epsilon\alpha_J = \epsilon_J \times 1_{N_J}$ of $N_W(W_J) = W_J \times N_J$ is the trivial extension of the sign character of W_J.

Here and in the remainder of the paper, we denote the restrictions of the trivial and the sign character of W to a subgroup U of W by 1_U and ϵ_U, respectively, or by 1_J and ϵ_J, if $U = W_J$ for some $J \subseteq S$. If no confusion can arise, we denote the restrictions of the characters 1_S and ϵ_S of W to any of its subgroups simply by 1 and ϵ, respectively.

Following Bergeron et al. [2], we decompose $\Sigma(W)$ into projective indecomposable modules, using a basis of quasi-idempotents, that naturally arise as follows. For $L, K \subseteq S$, we define

$$m_{KL} = \begin{cases} |X_K \cap X_L^\sharp|, & \text{if } L \subseteq K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(m_{KL})_{K,L\subseteq S}$ is an invertible matrix, and consequently, there is a basis $(e_L)_{L\subseteq S}$ of $\Sigma(W)$ such that

$$x_K = \sum_{L\subseteq S} m_{KL} e_L$$

for $K \subseteq S$. Define, for $\lambda \in \Lambda$, elements

$$e_\lambda = \sum_{L\in\lambda} e_L.$$

Then $\{e_\lambda : \lambda \in \Lambda\}$ is a set of primitive, pairwise orthogonal idempotents in $\Sigma(W)$. In particular,

$$\sum_{\lambda\in\Lambda} e_\lambda = 1 \in \mathbb{C}W.$$

Thus, if we set

$$E_\lambda = e_\lambda \mathbb{C}W,$$

then

$$\mathbb{C}W = \bigoplus_{\lambda \in \Lambda} E_\lambda \tag{2.3}$$

is a decomposition of the group algebra into right ideals. We call the right ideal $E_{[S]}$ the *top component* of $\mathbb{C}W$.

For $\lambda \in \Lambda$, denote by Φ_λ the character of the W-module E_λ. Furthermore, for $L \subseteq S$, denote by Φ_L the character of the top component of the group algebra $\mathbb{C}W_L$. Notice that for $\lambda = [L]$, $\Phi_{[L]}$ is a character of W whereas Φ_L is a character of W_L. If $L = S$, then $W_L = W$ and $\Phi_{[S]} = \Phi_S$. In general, the characters $\Phi_{[L]}$ and Φ_L are related in the following way.

Proposition 2.4 [4, Corollary 3.13] *Let $L \subseteq S$. Then the character Φ_L of W_L extends to a character $\widetilde{\Phi}_L$ of the normalizer $N_W(W_L) = W_L \rtimes N_L$ such that*

$$\Phi_{[L]} = \operatorname{Ind}_{N_W(W_L)}^W \widetilde{\Phi}_L.$$

Remark 2.5 *If W_L is a bulky parabolic subgroup of W, then, by* [4, *Lemma* 3.7] *$\widetilde{\Phi}_L$ is the character $\Phi_L \times 1_{N_L}$ of $N_W(W_L) = W_L \times N_L$ and so $\Phi_{[L]} = \operatorname{Ind}_{W_L \times N_L}^W(\Phi_L \times 1_{N_L})$.*

3 The reflection arrangement and the Orlik–Solomon algebra $A(W)$

A finite Coxeter group of rank r acts as a reflection group on Euclidean space $\mathbb{R}^r$. Here it is convenient to regard this as an action on the complex space $V_\mathbb{C} = \mathbb{C}^r$. Let

$$T = \{s^w : s \in S,\ w \in W\}$$

be the set of reflections of W. For $t \in T$, denote by H_t the reflecting hyperplane of t, i.e., the 1-eigenspace of t. The set of hyperplanes $\mathcal{A} = \{H_t : t \in T\}$ is called the reflection arrangement of W; for details see [11, Chap. 6]. Examples of (the real part of) reflection arrangements in dimension 2 are shown in Figs. 1 and 2.

The *lattice* of $\mathcal{A}$ is the set of all possible intersections of hyperplanes

$$L(\mathcal{A}) = \{H_{t_1} \cap \cdots \cap H_{t_p} : t_1, \dots, t_p \in T\}.$$

For $X \in L(\mathcal{A})$, the pointwise stabilizer

$$W_X = \{w \in W : x.w = x \text{ for all } x \in X\}$$

is a parabolic subgroup of W. We define the *shape* $\operatorname{sh}(X)$ of X to be the shape of W_X, i.e., $\operatorname{sh}(X) = [L] \in \Lambda$ if W_X is conjugate to W_L in W for some $L \subseteq S$. The group W acts on T by conjugation and the W-action on T induces actions of W on $\mathcal{A}$ and

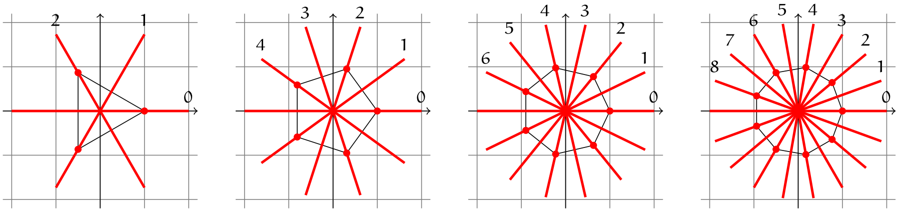

Fig. 1 Hyperplane arrangements of type $I_2(m)$, $m = 3, 5, 7, 9$

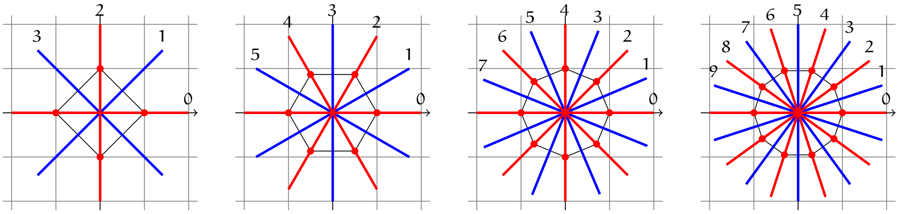

Fig. 2 Hyperplane arrangements of type $I_2(m)$, $m = 4, 6, 8, 10$

$L(\mathcal{A})$. Orlik and Solomon [10] have shown that the normalizer of W_X in W is the setwise stabilizer of X in W, that is,

$$N_W(W_X) = \{w \in W : X.w = X\}.$$

Consequently, the orbits of W on the lattice $L(\mathcal{A})$ are parametrized by the shapes of W. We denote by $\alpha_X \colon N_W(W_X) \to \mathbb{C}$ the linear character of $N_W(W_X)$ defined by

$$\alpha_X(w) = \det(w|_X)$$

for $w \in N_W(W_X)$. Then, for $w \in W$, we have $\alpha_w = \alpha_X$, where $X = \mathrm{Fix}(w)$, the fixed point subspace of w in V. Moreover, for $L \subseteq S$, we have $\alpha_L = \alpha_X$, where $X = \mathrm{Fix}(W_L)$.

The Orlik–Solomon algebra of W is the associative $\mathbb{C}$-algebra $A(W)$, generated as an algebra by elements a_t, $t \in T$, subject to the relations

$$a_t a_{t'} = -a_{t'} a_t$$

for all $t, t' \in T$, and

$$\sum_{i=1}^{p} (-1)^i a_{t_1} \cdots a_{t_{i-1}} \widehat{a_{t_i}} a_{t_{i+1}} \cdots a_{t_p} = 0,$$

where the hat denotes omission, whenever $\{H_{t_1}, \dots, H_{t_p}\}$ is linearly dependent. The action of W on the hyperplanes extends to an action on $A(W)$ via

$$a_t.w = a_{t^w}$$

for $t \in T$, $w \in W$. The algebra $A(W)$ is a skew-commutative, graded algebra

$$A(W) = \bigoplus_{p \geq 0} A^p,$$

where the degree p subspace A^p is spanned by those monomials $a_{t_1} \cdots a_{t_p}$ in $A(W)$ with $\dim H_{t_1} \cap \ldots \cap H_{t_p} = r - p$. Clearly, $A^p = 0$ for $p > r$. We call A^r the *top component* of $A(W)$. We need a refinement of this decomposition, due to Brieskorn [3]. For a subspace $X \in L(\mathcal{A})$ of codimension p, define a subspace

$$A_X = \langle a_{t_1} \cdots a_{t_p} : H_{t_1} \cap \ldots \cap H_{t_p} = X \rangle$$

of $A(W)$. Then $A_{\{0\}} = A^r$ is the top component of $A(W)$. Note that A_X is an embedding of the top component of $A(W_X)$ into $A(W)$. For $w \in W$, we have $A_X.w = A_{X.w}$ and so A_X is an $N_W(W_X)$-stable subspace.

We have

$$A(W) = \bigoplus_{X \in L(\mathcal{A})} A_X$$

and if we set

$$A_\lambda = \bigoplus_{\mathrm{sh}(X) = \lambda} A_X,$$

for $\lambda \in \Lambda$, then

$$A(W) = \bigoplus_{\lambda \in \Lambda} A_\lambda$$

is a decomposition of $A(W)$ into W-modules A_λ. Note that $A_{[S]} = A_{\{0\}}$ is the top component of $A(W)$.

For $\lambda \in \Lambda$, denote by Ψ_λ the character of the component A_λ of the Orlik–Solomon algebra $A(W)$. Furthermore, for $L \subseteq S$, denote by Ψ_L the character of the top component of the Orlik–Solomon algebra $A(W_L)$ of the parabolic subgroup W_L of W. Notice that for $\lambda = [L]$, $\Psi_{[L]}$ is a character of W whereas Ψ_L is a character of W_L. If $L = S$, then $\Psi_{[S]} = \Psi_S$. In general, the characters $\Psi_{[L]}$ and Ψ_L are related in the following way, analogous to Proposition 2.4.

Proposition 3.1 [9, §2] *Let $L \subseteq S$. Then the character Ψ_L of W_L extends to a character $\widetilde{\Psi}_L$ of the normalizer $N_W(W_L) = W_L \rtimes N_L$ such that*

$$\Psi_{[L]} = \mathrm{Ind}_{N_W(W_L)}^W \widetilde{\Psi}_L.$$

Remark 3.2 *Suppose that W_L is a bulky parabolic subgroup of W and set $X =$* Fix(W_L). *If* codim $X = p$ *and $t_1, \ldots, t_p$ are in T with $X = H_{t_1} \cap \cdots \cap H_{t_p}$, then $t_1, \ldots, t_p$ are in W_L and so, since N_L centralizes W_L, we have $a_{t_1} \cdots a_{t_p}.n = a_{t_1^n} \cdots a_{t_p^n} = a_{t_1} \cdots a_{t_p}$, for $n \in N_L$. Thus, $\widetilde{\Psi}_L$ is the character $\Psi_L \times 1_{N_L}$ of $N_W(W_L) = W_L \times N_L$ and so $\Psi_{[L]} = \mathrm{Ind}_{W_L \times N_L}^W (\Psi_L \times 1_{N_L})$.*

4 The inductive strategy

Before stating our relative Conjecture C, we briefly review the proof of Conjecture 2.1 in [4] and describe how it leads to a proof of Conjecture A. We first showed that the characters of the top components of $\mathbb{C}W$ and $A(W)$ are related as described in the following conjecture which makes sense for any finite Coxeter group. To this end, let $\mathcal{C}$ be the set of cuspidal conjugacy classes of W and, for $L \subseteq S$, let $\mathcal{C}_L$ denote the set of cuspidal conjugacy classes in W_L. For a class C in $\mathcal{C}$ or $\mathcal{C}_L$, we denote by $w_C \in C$ a fixed representative.

Conjecture B *For each class $C \in \mathcal{C}$, there exist linear characters φ_{w_C} and ψ_{w_C} of the centralizer $C_W(w_C)$ such that the following hold*:

(i) $\Phi_S = \sum_{C\in\mathcal{C}} \operatorname{Ind}_{C_W(w_C)}^W \varphi_{w_C}$;
(ii) $\Psi_S = \sum_{C\in\mathcal{C}} \operatorname{Ind}_{C_W(w_C)}^W \psi_{w_C}$;
(iii) $\psi_{w_C} = \varphi_{w_C}\epsilon$ *for all* $C \in \mathcal{C}$.

Remark 4.1 *If it is known that $\Psi_S = \Phi_S\epsilon_S$, then choosing ψ_{w_C} or φ_{w_C} in such a way that $\psi_{w_C} = \varphi_{w_C}\epsilon$, we have that part* (*iii*) *in the above Conjecture B holds and that* (*i*) *and* (*ii*) *are equivalent statements.*

When W is a symmetric group, every parabolic subgroup W_L of W is a product of symmetric groups and so Conjecture B holds for the group W_L. Thus, for $w_C \in C \in \mathcal{C}_L$, we obtained linear characters φ_{w_C} and ψ_{w_C} of $C_{W_L}(w_C)$ such that the characters Φ_L and Ψ_L of W_L decompose as

$$\Phi_L = \sum_{C\in\mathcal{C}_L} \operatorname{Ind}_{C_{W_L}(w_C)}^{W_L} \varphi_{w_C} \quad \text{and} \quad \Psi_L = \sum_{C\in\mathcal{C}_L} \operatorname{Ind}_{C_{W_L}(w_C)}^{W_L} \psi_{w_C}.$$

We know from Propositions 2.4 and 3.1 that Φ_L and Ψ_L extend to characters $\widetilde{\Phi}_L$ and $\widetilde{\Psi}_L$ of $N_W(W_L)$. The next step in [4] was to show that each φ_{w_C} and ψ_{w_C} extend to characters $\widetilde{\varphi}_{w_C}$ and $\widetilde{\psi}_{w_C}$ of $C_W(w_C)$ in such a way that

$$\widetilde{\Phi}_L = \sum_{C\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w_C)}^{N_W(W_L)} \widetilde{\varphi}_{w_C} \quad \text{and} \quad \widetilde{\Psi}_L = \sum_{C\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w_C)}^{N_W(W_L)} \widetilde{\psi}_{w_C}, \tag{4.2}$$

and moreover that $\widetilde{\psi}_{w_C} = \widetilde{\varphi}_{w_C}\epsilon_S\alpha_L$ for all $C \in \mathcal{C}_L$. Finally, we applied $\operatorname{Ind}_{N_W(W_L)}^W$ to (4.2) and summed over the set of shapes $[L] \in \Lambda$. Conjecture A then followed immediately by transitivity of induction.

Motivated by (4.2), we make the following general conjecture.

Conjecture C *Let $L \subseteq S$. Then, for each $C \in \mathcal{C}_L$, there exist linear characters $\widetilde{\varphi}_{w_C}$ and $\widetilde{\psi}_{w_C}$ of $C_W(w_C)$ such that the following hold*:

(i) $\widetilde{\Phi}_L = \sum_{C\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w_C)}^{N_W(W_L)} \widetilde{\varphi}_{w_C}$;
(ii) $\widetilde{\Psi}_L = \sum_{C\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w_C)}^{N_W(W_L)} \widetilde{\psi}_{w_C}$;
(iii) $\widetilde{\psi}_{w_C} = \widetilde{\varphi}_{w_C}\epsilon_S\alpha_L$ *for all* $C \in \mathcal{C}_L$.

Remark 4.3 *If it is known that $\widetilde{\Psi}_L = \widetilde{\Phi}_L \epsilon_S \alpha_L$, then choosing $\widetilde{\psi}_{w_C}$ or $\widetilde{\varphi}_{w_C}$ in such a way that $\widetilde{\psi}_{w_C} = \widetilde{\varphi}_{w_C} \epsilon_S \alpha_L$, we have that part (iii) in the above Conjecture C holds and that (i) and (ii) are equivalent statements.*

Conjecture B is known to hold in the following cases:

1. W of type A (see [4, Theorem 4.1]);
2. W has rank 2 or less (see Lemmas 5.1 and 5.2, Theorem 5.11).

Conjecture C is known to hold in the following cases:

1. W of type A; all L (see [4, Theorem 5.2]);
2. W arbitrary; W_L is bulky and satisfies Conjecture B (by Theorem 4.7);
3. W arbitrary; $|L| \leq 2$ (see Corollary 5.3, Theorem 5.18).

If Conjecture C holds for all $L \subseteq S$, then Conjecture A is true for W.

Theorem 4.4 *Suppose that Conjecture C holds for all subsets $L \subseteq S$. Then for each w in a set $\mathcal{R}$ of representatives of the conjugacy classes of W, there are linear characters $\widetilde{\varphi}_w$ and $\widetilde{\psi}_w$ of $C_W(w)$ such that*

(i) *the regular character of W is given by $\rho_W = \sum_{w \in \mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \widetilde{\varphi}_w$,*
(ii) *the Orlik–Solomon character of W is given by $\omega_W = \sum_{w \in \mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \widetilde{\psi}_w$, and*
(iii) *$\widetilde{\psi}_w = \widetilde{\varphi}_w \epsilon \alpha_w$ for all $w \in \mathcal{R}$.*

Proof For $L \subseteq S$, let $\mathcal{R}_L$ be a set of representatives of the classes $\mathcal{C}_L$. For a class $C \in \mathcal{C}_L$, denote by $w_C \in \mathcal{R}_L$ its representative. Let $\mathcal{L}$ be a set of representatives of shapes, so $\Lambda = \{ [L] \mid L \in \mathcal{L} \}$. Then, by [5, Theorem 3.2.12], we may assume without loss that

$$\mathcal{R} = \coprod_{L \in \mathcal{L}} \mathcal{R}_L = \{w_C : C \in \mathcal{C}_L,\ L \in \mathcal{L}\}.$$

Then, by Conjecture C the equality in (iii) holds. By (2.3) and Proposition 2.4, we have

$$\rho_W = \sum_{\lambda \in \Lambda} \Phi_\lambda = \sum_{L \in \mathcal{L}} \operatorname{Ind}_{N_W(W_L)}^W \widetilde{\Phi}_L = \sum_{L \in \mathcal{L}} \sum_{C \in \mathcal{C}_L} \operatorname{Ind}_{C_W(w_C)}^W \widetilde{\varphi}_{w_C},$$

as desired. The formula for ω_W follows in the same way. □

Notice that in the case when $L = S$, Conjecture C is simply a restatement of Conjecture B. In general, Conjecture C for $L \subseteq S$ implies the validity of Conjecture B for the group W_L, as follows.

Proposition 4.5 *Suppose that Conjecture C holds for a subset $L \subseteq S$. Then the restrictions*

$$\varphi_{w_C} = \operatorname{Res}_{C_{W_L}(w_C)}^{C_W(w_C)} \widetilde{\varphi}_{w_C} \quad \text{and} \quad \psi_{w_C} = \operatorname{Res}_{C_{W_L}(w_C)}^{C_W(w_C)} \widetilde{\psi}_{w_C}$$

are linear characters that satisfy Conjecture B for W_L.

Proof By Mackey's theorem, we have

$$\operatorname{Res}_{W_L}^{N_W(W_L)} \operatorname{Ind}_{C_W(w_C)}^{N_W(W_L)} \widetilde{\varphi}_{w_C} = \operatorname{Ind}_{C_{W_L}(w_C)}^{W_L} \operatorname{Res}_{C_{W_L}(w_C)}^{C_W(w_C)} \widetilde{\varphi}_{w_C},$$

since $N_W(W_L) = W_L C_W(w_C)$ (see [8]), and therefore,

$$\begin{aligned}
\Phi_L &= \operatorname{Res}_{W_L}^{N_W(W_L)} \widetilde{\Phi}_L \\
&= \sum_{C \in \mathcal{C}_L} \operatorname{Res}_{W_L}^{N_W(W_L)} \operatorname{Ind}_{C_W(w_C)}^{N_W(W_L)} \widetilde{\varphi}_{w_C} \\
&= \sum_{C \in \mathcal{C}_L} \operatorname{Ind}_{C_{W_L}(w_C)}^{W_L} \operatorname{Res}_{C_{W_L}(w_C)}^{C_W(w_C)} \widetilde{\varphi}_{w_C} \\
&= \sum_{C \in \mathcal{C}_L} \operatorname{Ind}_{C_{W_L}(w_C)}^{W_L} \varphi_{w_C}.
\end{aligned}$$

The formula for Ψ_L follows in the same way. The conclusion that $\psi_{w_C} = \varphi_{w_C}\epsilon$ for $C \in \mathcal{C}_L$ is easily seen to hold. □

Remark 4.6 *Although Conjecture B for W_L formally follows from Conjecture C, as in [4], the characters $\widetilde{\varphi}_{w_C}$ and $\widetilde{\psi}_{w_C}$ of $C_W(w_C)$ arise in practice as extensions of characters φ_{w_C} and ψ_{w_C} of $C_{W_L}(w_c)$ that satisfy Conjecture B for W_L. In particular, if Conjecture B is known to hold for W_L, then using Remark 4.3, to prove Conjecture C for $L \subseteq S$, it suffices to prove that each φ_{w_C} extends to $C_W(w_C)$ in such a way that Conjecture C (i) holds and that $\widetilde{\Psi}_L = \widetilde{\Phi}_L \epsilon_S \alpha_L$.*

When $L \subseteq S$ is such that W_L is a self-normalizing subgroup of W (e.g., if $L = S$), then N_L is the trivial group and Conjecture B for the group W_L vacuously implies Conjecture C for the subset L in this case. More generally, whenever the complement N_L centralizes W_L, i.e., when W_L is bulky in W, Conjecture B for W_L implies Conjecture C for $L \subseteq S$, as follows.

Theorem 4.7 *Let $L \subseteq S$. Suppose that Conjecture B holds for the group W_L and that W_L is a bulky parabolic subgroup of W. Then Conjecture C holds with $\widetilde{\varphi}_{w_C} = \varphi_{w_C} \times 1_{N_L}$ and $\widetilde{\psi}_{w_C} = \psi_{w_C} \times 1_{N_L}$ for each cuspidal class C of W_L.*

Proof As observed in the remark above, it suffices to show that each φ_{w_C} extends to $C_W(w_C)$ in such a way that Conjecture C (i) holds and that $\widetilde{\Psi}_L = \widetilde{\Phi}_L \epsilon_S \alpha_L$.

Because N_L centralizes W_L, we have that the centralizer $C_W(w_C)$ is the direct product of $C_{W_L}(w_C)$, and N_L and so $\widetilde{\varphi}_{w_C}$ is indeed a linear character of $C_W(w_C)$ that extends φ_{w_C}. Thanks to Remark 2.5, $\widetilde{\Phi}_L = \Phi_L \times 1_{N_L}$. Thus, by Conjecture B (i), we have

$$\widetilde{\Phi}_L = \Phi_L \times 1_{N_L} = \left(\sum_{C \in \mathcal{C}_L} \operatorname{Ind}_{C_{W_L}(w_C)}^{W_L} \varphi_{w_C} \right) \times 1_{N_L}$$
$$= \sum_{C \in \mathcal{C}_L} \operatorname{Ind}_{C_{W_L}(w_C) \times N_L}^{W_L \times N_L} (\varphi_{w_C} \times 1_{N_L})$$
$$= \sum_{C \in \mathcal{C}_L} \operatorname{Ind}_{C_W(w_C)}^{N_W(W_L)} \widetilde{\varphi}_{w_C}.$$

Hence Conjecture C (i) holds.

By Remark 3.2, Conjecture B (iii), Lemma 2.1, and Remark 2.5, we have

$$\widetilde{\Psi}_L = \Psi_L \times 1_{N_L} = \Phi_L \epsilon_L \times 1_{N_L} \sigma_L = (\Phi_L \times 1_{N_L}) \epsilon \alpha_L = \widetilde{\Phi}_L \epsilon_S \alpha_L,$$

using the fact that $W_L \subseteq \ker \alpha_L$, whence we are done. □

Combining Theorem 4.7 with the results in [4], we see that if W_L is a product of Coxeter groups of type A and is a bulky parabolic subgroup of W, then Conjecture C holds for $L \subseteq S$. For example, if W_L is of type $A_1 \times A_3$ and W is of type E_6, then the characters φ_{w_C} and ψ_{w_C} constructed in [4] satisfy Conjecture B and so, by Theorem 4.7, they extend to $C_W(w_C)$ and Conjecture C holds. Note, however, that the property of being a bulky parabolic subgroup depends in a fundamental way on the embedding of W_L in W. If W_L is of type $A_1 \times A_3$ and W is of type E_7, then W_L is not bulky and Theorem 4.7 cannot be applied.

5 Conjectures A, B and C for Coxeter groups of rank up to 2

In this section, we show that Conjecture C holds for $L \subseteq S$ for any S as long as $|L| \leq 2$. Note that because the type of the ambient Coxeter group W is arbitrary, even for types $A_1 \times A_1$ and A_2 Conjecture C is a stronger statement than is proved in [4] for such parabolic subgroups. The strategy we use is to first prove that Conjecture B holds for W when the rank of W is at most 2 and then use the procedure outlined in Remark 4.6. Combining Conjecture C with Theorem 4.7, we conclude that Conjectures A, B, and C all hold in case the rank of W is at most 2.

The top components of Coxeter groups of rank 0 or 1 almost trivially satisfy Conjecture B. For later reference, we record this explicitly in the following lemmas.

Lemma 5.1 *The top component characters of $W_\varnothing$ are $\Phi_\varnothing = 1_\varnothing$ and $\Psi_\varnothing = 1_\varnothing$. Moreover, $W_\varnothing$ satisfies Conjecture B with $\varphi_1 = 1_\varnothing$ and $\psi_1 = 1_\varnothing$.*

Lemma 5.2 *Suppose W is a Coxeter group of rank 1, generated by $S = \{s\}$. Then the top component characters of W are $\Phi_S = \epsilon_S$ and $\Psi_S = 1_S$. Moreover, W satisfies Conjecture B with $\varphi_s = \epsilon_S$ and $\psi_s = 1_S$.*

Proof In this case, the non-trivial conjugacy class $\{s\}$ is the unique cuspidal conjugacy class in W. From the definitions we have $e_{[S]} = e_S = \frac{1}{2}(1 - s)$ and it follows that

W acts on the top component $E_{[S]} = e_{[S]}\mathbb{C}W$ with character $\Phi_{[S]} = \epsilon_S$. Moreover, W acts trivially on the basis $\{a_s\}$ of the top component $A_{[S]}$ of $A(W)$, which therefore affords the trivial character. Thus, $\Psi_{[S]} = 1_S$ and so $\Phi_{[S]} = \Psi_{[S]}\epsilon_S$. Set $\varphi_s = \epsilon_S$ and $\psi_s = 1_S$. Then φ_s and ψ_s obviously satisfy the conclusions of Conjecture B. □

In any finite Coxeter group W, parabolic subgroups of rank 0 and 1 are always bulky. We may thus conclude from Lemmas 5.1 and 5.2 and Theorem 4.7 that Conjecture C holds for $L \subseteq S$ with $|L| \leq 1$.

Corollary 5.3 *Suppose that $L \subseteq S$ has size $|L| \leq 1$. Then Conjecture C holds.*

As a consequence of the corollary, W acts trivially on both the component $E_{[\varnothing]}$ of the group algebra $\mathbb{C}W$ (with character $\Phi_{[\varnothing]} = \widetilde{\Phi}_\varnothing = 1_S$) and the component $A_{[\varnothing]}$ of the Orlik–Solomon algebra $A(W)$ (with character $\Psi_{[\varnothing]} = \widetilde{\Psi}_\varnothing = 1_S$), as one can easily establish directly.

Moreover, the degree 1 component of $A(W)$ is a direct sum of transitive permutation modules, one for each conjugacy class of reflections of W. This agrees with the description of the degree 1 component of $A(W)$ as the permutation representation of W on its reflections, that can easily be obtained directly.

Next we consider the case when W has rank 2. Until further notice, we assume that

$$W = \langle s, t : s^2 = t^2 = (st)^m = 1\rangle.$$

Then W is a Coxeter group of rank two and is of type $A_1 \times A_1$, or $I_2(m)$ for $m \geq 3$, with Coxeter generators $S = \{s, t\}$. For convenience, we regard type $A_1 \times A_1$ as type $I_2(2)$, noting that the general results of this section remain true for $m = 2$.

To prove Conjecture B for W, we first compute the character Φ_S of the top component $E_{[S]}$ of the group algebra $\mathbb{C}W$, and verify that it is a sum of induced linear characters. Then we compute the character Ψ_S of the top component $A_{[S]}$ of the Orlik–Solomon algebra $A(W)$ and verify that $\Psi_S = \Phi_S\epsilon_S$. Conjecture B then follows as observed in Remark 4.1.

As usual, denote by w_0 the longest element of W. Furthermore, we define

$$\mathrm{Av}(U) = \frac{1}{|U|}\sum_{u\in U} u$$

for a subgroup U of W. Recall that $\mathrm{Av}(U)u = \mathrm{Av}(U)$ for all $u \in U$ and that $\mathrm{Av}(U)\mathbb{C}W$ is the permutation module of W on the cosets of U.

Lemma 5.4 $e_S = \mathrm{Av}(\langle w_0\rangle) - \mathrm{Av}(W)$.

Proof By Solomon's theorem [15], the elements

$$x_\varnothing = 1 + s + t + st + ts + \cdots + w_0, \qquad x_s = 1 + t + st + tst + \cdots + w_0 s,$$
$$x_{st} = 1, \qquad x_t = 1 + s + ts + sts + \cdots + w_0 t$$

form a basis of the descent algebra $\Sigma(W)$. Note that $x_t + x_s = x_\varnothing + 1 - w_0$.

For $L \subseteq K \subseteq S$, the numbers $m_{KL} = |X_K \cap X_L^\sharp|$ are easily determined as

$$(m_{KL})_{K,L\subseteq S} = \begin{bmatrix} 2m & . & . & . \\ m & 2 & . & . \\ m & . & 2 & . \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad (m_{KL})^{-1} = \begin{bmatrix} \frac{1}{2m} & . & . & . \\ -\frac{1}{4} & \frac{1}{2} & . & . \\ -\frac{1}{4} & . & \frac{1}{2} & . \\ \frac{m-1}{2m} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

Hence the idempotents e_L are (cf. [2])

$$\begin{aligned} e_\varnothing &= \frac{1}{2m} x_\varnothing, & e_s &= \frac{1}{2} x_s - \frac{1}{4} x_\varnothing, \\ e_{st} &= 1 - \frac{1}{2} x_s - \frac{1}{2} x_t + \frac{m-1}{2m} x_\varnothing, & e_t &= \frac{1}{2} x_t - \frac{1}{4} x_\varnothing. \end{aligned}$$

From $x_t + x_s = 1 + x_\varnothing - w_0$, it follows that $e_s + e_t = \frac{1}{2}(1 - w_0)$, and hence that $e_S = \frac{1}{2}(1 + w_0) - e_\varnothing = \mathrm{Av}(\langle w_0 \rangle) - \mathrm{Av}(W)$, as required. □

As an immediate consequence we obtain the character of the top component of $\mathbb{C}W$.

Corollary 5.5 *The W-module $E_{[S]}$ affords the character $\Phi_S = \mathrm{Ind}_{\langle w_0 \rangle}^W(1) - 1_S$.*

Next we identify linear characters of centralizers of cuspidal elements. Note that the group W consists of m reflections and m rotations. Denote the rotation subgroup of W by $W^+ = \langle st \rangle$. The centralizer of a rotation w is W^+, unless w is central in W. The cuspidal classes of W are exactly the classes of nontrivial rotations, represented by the set $\{(st)^j : j = 1, \dots, \lfloor \frac{m}{2} \rfloor\}$, containing $w_0 = (st)^{m/2}$ in case m is even. The group W^+ is a cyclic group of order m and it has m linear characters χ_j, $j = 0, \dots, m-1$, defined by

$$\chi_j(st) = \zeta_m^j$$

for a primitive mth root of unity ζ_m. In the following arguments, we make frequent use of the fact that the sum of all the nontrivial characters χ_j of W^+ equals the difference of its regular and its trivial character,

$$\sum_{j=1}^{m-1} \chi_j = \mathrm{Ind}_{\{1\}}^{W^+}(1) - 1_{W^+},$$

which obviously follows from $\sum_{j=0}^{m-1} \chi_j = \mathrm{Ind}_{\{1\}}^{W^+}(1)$ and $\chi_0 = 1_{W^+}$.

We distinguish two cases, depending on the parity of m.

Proposition 5.6 *Suppose that $m = 2k$ with $k > 0$. Let*

$$\varphi_{(st)^j} = \begin{cases} \chi_{2j}, & 0 < j < k, \\ \epsilon_S, & j = k. \end{cases}$$

Then $\varphi_{(st)^j}$ *is a linear character of* $C_W((st)^j)$, *for* $j = 1, \dots, k$, *and*

$$\sum_{j=1}^{k} \mathrm{Ind}_{C_W((st)^j)}^{W}(\varphi_{(st)^j}) = \epsilon_S + \sum_{j=1}^{k-1} \mathrm{Ind}_{W^+}^{W}(\chi_{2j}) = \Phi_S.$$

Proof Note that $C_W((st)^j) = W^+$ and w_0 lies in the kernel of the characters $\varphi_{(st)^j} = \chi_{2j}$, for all $j = 1, \dots, k-1$. Hence the χ_{2j} can be regarded as a full set of nontrivial irreducible characters of the quotient group $W^+/\langle w_0\rangle$, whence their sum $\sum_{j=1}^{k-1} \chi_{2j}$ equals the difference of its regular and its trivial characters. Thus, as a character of W^+, we have

$$\sum_{j=1}^{k-1} \chi_{2j} = \mathrm{Ind}_{\langle w_0\rangle}^{W^+}(1) - 1_{W^+}.$$

Therefore

$$\epsilon_S + \mathrm{Ind}_{W^+}^{W}\left(\sum_{j=1}^{k-1} \chi_{2j}\right) = \epsilon_S + \mathrm{Ind}_{\langle w_0\rangle}^{W}(1) - \mathrm{Ind}_{W^+}^{W}(1) = \mathrm{Ind}_{\langle w_0\rangle}^{W}(1) - 1_S = \Phi_S,$$

where the penultimate equality holds because $\mathrm{Ind}_{W^+}^{W}(1) = 1_S + \epsilon_S$. □

Proposition 5.7 *Suppose that* $m = 2k+1$ *for some* $k > 0$. *For* $j = 1, \dots, k$, *let*

$$\varphi_{(st)^j} = \chi_j.$$

Then $\varphi_{(st)^j}$ *is a linear character of* $C_W((st)^j)$, *for* $j = 1, \dots, k$, *and*

$$\sum_{j=1}^{k} \mathrm{Ind}_{C_W((st)^j)}^{W}(\varphi_{(st)^j}) = \sum_{j=1}^{k} \mathrm{Ind}_{W^+}^{W}(\chi_j) = \Phi_S.$$

Proof We have $C_W((st)^j) = W^+$ and $\mathrm{Res}_{W^+}^{W}(\mathrm{Ind}_{W^+}^{W}(\chi_j)) = \chi_j + \chi_{m-j}$ for all $j = 1, \dots, k$. Hence

$$\begin{aligned}\mathrm{Res}_{W^+}^{W}\left(\sum_{j=1}^{k} \mathrm{Ind}_{W^+}^{W}(\chi_j)\right) &= \sum_{j=1}^{m-1} \chi_j = \mathrm{Ind}_{\{1\}}^{W^+}(1) - 1_{W^+} \\ &= \mathrm{Res}_{W^+}^{W}\big(\mathrm{Ind}_{\langle w_0\rangle}^{W}(1) - 1_S\big) = \mathrm{Res}_{W^+}^{W}(\Phi_S).\end{aligned}$$

It follows that

$$\Phi_S = \sum_{j=1}^{k} \mathrm{Ind}_{W^+}^{W}(\chi_j),$$

since the restrictions of both characters to the subgroup $\langle w_0\rangle$ of W also coincide. □

Proposition 5.8 *Let $\pi_{\mathcal{A}}$ be the character of the permutation action of W on the hyperplane arrangement $\mathcal{A}$. Then W acts on the degree 1 component of $A(W)$ with character $\pi_{\mathcal{A}}$, and W acts on the component $A_{[S]}$ of $A(W)$ with character*

$$\Psi_S = \pi_{\mathcal{A}} - 1_S.$$

Consequently, W acts on $A(W)$ with character $2\pi_{\mathcal{A}}$.

Proof The degree 1 component of $A(W)$ has basis $\{a_t : t \in T\}$ and W acts on it by permuting the basis vectors. In order to analyze the top component of $A(W)$, we make this permutation action explicit as follows.

Label the hyperplanes $H_0, \dots, H_{m-1}$, so that the real part of the hyperplane H_j is spanned by ζ_{2m}^j, where $\zeta_{2m} = e^{2\pi i/2m}$ is a primitive $(2m)$th root of unity, as shown in Figs. 1 and 2.

Let s be the reflection about H_0 (the x-axis) and $ts = (st)^{-1}$ the (anti-clockwise) rotation about the angle $2\pi/m$. Then t is the reflection about H_{m-1}.

The reflection s permutes the hyperplanes according to the rule

$$H_j.s = H_{m-j},$$

for $j = 0, \dots, m-1$, fixing H_0. The rotation ts acts as

$$H_j.ts = H_{j+2},$$

for $j = 0, \dots, m-1$, where the indices are reduced mod m if necessary.

The top component $A_{[S]}$ has a basis $\{a_0 a_j : j = 1, \dots, m-1\}$, where W acts on the indices as indicated above, subject to the relation $a_0 a_j - a_0 a_k + a_j a_k = 0$, i.e.,

$$a_j a_k = a_0 a_k - a_0 a_j.$$

The reflection s fixes H_0 and thus maps $a_0 a_j$ to

$$a_0 a_j.s = a_0 a_{m-j},$$

for $j = 1, \dots, m-1$. The rotation ts maps $a_0 a_j$ to

$$a_0 a_j.ts = a_2 a_{j+2} = \begin{cases} a_0 a_{j+2} - a_0 a_2, & j \neq m-2, \\ -a_0 a_2, & j = m-2. \end{cases}$$

Now define vectors

$$b_0 = -\frac{1}{m} \sum_{j=1}^{m-1} a_0 a_j$$

and, for $j = 1, \dots, m-1$,

$$b_j = a_0 a_j + b_0.$$

Then $b_0.s = b_0$ and $b_j.s = b_{m-j}$ for $j = 1, \dots, m-1$. Moreover, $b_j.ts = b_{j+2}$ for $j = 0, \dots, m-1$, with indices reduced mod m if necessary. Hence the map $a_j \mapsto b_j$

is a W-equivariant bijection from the basis $\{a_j : j = 0, \ldots, m-1\}$ of the degree 1 component to a spanning set $\{b_j : j = 0, \ldots, m-1\}$ of $A_{[S]}$. Clearly, $\sum_{j=0}^{m-1} b_j = 0$ and so the character of W on $A_{[S]}$ is $\pi_{\mathcal{A}} - 1_S$. □

Lemma 5.9 *The element $a_0 a_{m-1}$ generates the top component $A_{[S]}$ as $\mathbb{C}W$-module.*

Proof Let $M = a_0 a_{m-1}.\mathbb{C}W$. Then M contains the elements

$$a_0 a_1 = a_0 a_{m-1}.s, \qquad a_1 a_2 = -a_0 a_{m-1}.ts, \qquad a_0 a_2 = a_0 a_1 + a_1 a_2,$$

and, by induction, the elements

$$a_{j-1} a_j = a_{j-3} a_{j-2}.ts, \quad \text{and} \quad a_0 a_j = a_0 a_{j-1} + a_{j-1} a_j,$$

for $j > 2$. Consequently, M contains the basis $\{a_0 a_j : j = 1, \ldots, m-1\}$ of $A_{[S]}$, whence $M = A_{[S]}$. □

Proposition 5.10 $\Psi_S = \Phi_S \epsilon_S$.

Proof We distinguish two cases.

If m is odd, then $\pi_{\mathcal{A}} = \operatorname{Ind}_{\langle s \rangle}^W(1)$, since $C_W(s) = \langle s \rangle$ and all reflections are conjugates of s. Hence

$$\Psi_S = \operatorname{Ind}_{\langle s \rangle}^W(1) - 1_S = \operatorname{Ind}_{\langle w_0 \rangle}^W(1) - 1_S = \Phi_S$$

and $\Phi_S = \Phi_S \epsilon_S$, since $\Phi_S(w) = 0$ for all $w \in W$ with $\epsilon_S(w) = -1$.

If m is even, then $\operatorname{Ind}_{\langle w_0 \rangle}^W(1)\epsilon_S = \operatorname{Ind}_{\langle w_0 \rangle}^W(1)$ and

$$\Phi_S \epsilon_S = \big(\operatorname{Ind}_{\langle w_0 \rangle}^W(1) - 1_S\big)\epsilon_S = \operatorname{Ind}_{\langle w_0 \rangle}^W(1) - \epsilon_S = \pi_{\mathcal{A}} - 1_S = \Psi_S,$$

since $\pi_{\mathcal{A}} - \operatorname{Ind}_{\langle w_0 \rangle}^W(1) = 1_S - \epsilon_S$, as can be easily verified. □

We can now conclude that Conjecture B holds for W of rank 2.

Theorem 5.11 *Let W be a Coxeter group of rank 2, generated by $S = \{s, t\}$. Then, with notation as above, the top component characters of W are $\Phi_S = \operatorname{Ind}_{\langle w_0 \rangle}^W(1) - 1_S$ and $\Psi_S = \pi_{\mathcal{A}} - 1_S = \Phi_S \epsilon_S$. Moreover, W satisfies Conjecture B with $\varphi_{(st)^j} = \chi_j$ in case m odd, while $\varphi_{w_0} = \epsilon_S$ and $\varphi_{(st)^j} = \chi_{2j}$ in case m even.*

Proof Apply Propositions 5.6, 5.7, and 5.10, and Remark 4.3. □

Corollary 5.12 *Suppose that W is a Coxeter group with rank at most 2. Then Conjecture A holds for W.*

Proof By Lemmas 5.1 and 5.2, and Theorem 5.11, Conjecture B holds for all parabolic subgroups of W. By Theorem 4.4, it suffices to show that Conjecture C holds for all subsets $L \subseteq S$. If $|L| = 0, 1$, this follows from Corollary 5.3. It follows from Theorem 5.11 that Conjecture C holds when the rank of W and $|L|$ are both equal 2. □

It follows in particular from Corollary 5.12 that every Coxeter group of type $I_2(m)$ satisfies Conjecture A. We list the corresponding decomposition of the regular character ρ_W into characters $\Phi_{[L]} = \mathrm{Ind}_{N_W(W_L)}^W \widetilde{\Phi}_L$ and the decomposition of the Orlik–Solomon character ω_W into characters $\Psi_{[L]} = \mathrm{Ind}_{N_W(W_L)}^W \widetilde{\Psi}_L$ in Table 1 below. In Table 1, the left character table covers the case $m = 2k$ and the right character table covers the case $m = 2k + 1$. The columns of the character tables are labeled by representatives of the conjugacy classes of W, where the parameter in $(st)^i$ is $i = 1, \ldots, k-1$ for $m = 2k$, and $i = 1, \ldots, k$ for $m = 2k+1$. An entry '.' in the table stands for the value 0. As observed in Proposition 5.8, the rank 1 component of ω_W is the permutation character of the action of W on the set $\mathcal{A}$ of hyperplanes. In case $m = 2k$, the constituent $\Psi_{[\{s\}]}$ corresponds to the action on the W-orbit of the hyperplane H_s, and whether the element s has 2 or 1 fixed points in this action depends on whether k is even or odd. In such a situation, an entry of the form '$x \mid y$' in the table stands for 'x if k is even and y if k is odd'.

We saw in Theorem 5.11 that Conjecture B holds when W has rank 2 and we saw in Corollary 5.3 that Conjecture C holds when the subset $L \subseteq S$ has size $|L| \leq 1$. In the rest of this section, we prove that if the parabolic subgroup W_L has rank 2, then Conjecture C holds for any overgroup W. A similar result when W_L is a product of symmetric groups would reduce the proof of Conjecture A to considering only a small number of cases.

From now on, W is a finite Coxeter group, generated by S with $|S| \geq 3$ and W_L is a rank 2 parabolic subgroup of W with $L = \{s, t\} \subseteq S$. The elements x_K and e_K are defined relative to the ambient set S. We use a superscript to indicate this ambient set when it is not equal to S. Thus, for $K \subseteq L$, x_K^L denotes a basis element of the descent algebra of W_L.

If W_L is bulky, then W_L satisfies Conjecture C, by Theorem 4.7.

Suppose W_L is not bulky. Then N_L does not centralize W_L and so N_L contains an element inducing the nontrivial graph automorphism γ on W_L, interchanging s and

Table 1 The characters Φ_λ and Ψ_λ for $I_2(m)$; $m = 2k$, $m = 2k + 1$

	1	s	t	w_0	$(st)^i$
$\Phi_{[\varnothing]}$	1	1	1	1	1
$\Phi_{[\{s\}]}$	k	$. \mid 1$	$. \mid -1$	$-k$	.
$\Phi_{[\{t\}]}$	k	$. \mid -1$	$. \mid 1$	$-k$	.
$\Phi_{[S]}$	$m-1$	-1	-1	$m-1$	-1
ρ_W	$2m$	.	.	.	.
$\Psi_{[\varnothing]}$	1	1	1	1	1
$\Psi_{[\{s\}]}$	k	$2 \mid 1$	$. \mid 1$	k	.
$\Psi_{[\{t\}]}$	k	$. \mid 1$	$2 \mid 1$	k	.
$\Psi_{[S]}$	$m-1$	1	1	$m-1$	-1
ω_W	$2m$	4	4	$2m$	.

	1	s	$(st)^i$
$\Phi_{[\varnothing]}$	1	1	1
$\Phi_{[\{s\}]}$	m	-1	.
$\Phi_{[S]}$	$m-1$	.	-1
ρ_W	$2m$	.	.
$\Psi_{[\varnothing]}$	1	1	1
$\Psi_{[\{s\}]}$	m	1	.
$\Psi_{[S]}$	$m-1$	.	-1
ω_W	$2m$	2	.

t. In this case, s and t are conjugate in W and so W_L is either of type $A_1 \times A_1$ or of type $I_2(m)$ for odd $m > 2$. We distinguish two cases accordingly.

First, suppose that W_L is of type $A_1 \times A_1$. Then W_L has exactly one cuspidal element $w = st = ts$, which is central in W_L and invariant under N_L, hence central in $N_W(W_L)$. We have

$$\varphi_w = \Phi_L = \epsilon_L, \quad \text{and} \quad \psi_w = \Psi_L = 1_L,$$

by Corollary 5.5 and Proposition 5.8. Parts (i) and (ii) of Conjecture C are therefore trivially satisfied, with

$$\widetilde{\varphi}_w = \widetilde{\Phi}_L, \quad \text{and} \quad \widetilde{\psi}_w = \widetilde{\Psi}_L,$$

which exist by Propositions 2.4 and 3.1.

For part (iii) of Conjecture C, note that the idempotent

$$f = \frac{1}{4}(1 - s - t + st)$$

spans a subspace of $\mathbb{C}W_L$ affording the character Φ_L. As in the proof of Lemma 5.4,

$$e_L^L = 1 - \frac{1}{2}x_s^L - \frac{1}{2}x_t^L + \frac{1}{4}x_\varnothing^L = \frac{1}{4}(1 + st) - \frac{1}{4}(s + t) = f,$$

and thus $e_L^L f = e_L^L$ is a basis of the top component of W_L which is centralized by N_L. Hence $\widetilde{\varphi}_w(un) = \varphi_w(u)$, for $u \in W_L$ and $n \in N_L$. Moreover, note that $a_L = a_s a_t$ spans the top component of $A(W_L)$, and that $e_L n = e_L$, whereas $a_L.n = \sigma_L(n) a_L$ for $n \in N_L$. It follows that $\widetilde{\psi}_L(un) = \psi_L(u)\sigma_L(n) = \varphi_L(u)\epsilon(u)\epsilon(n)\alpha_L(n) = \widetilde{\varphi}_L(un)\epsilon(un)\alpha_L(un)$, for $u \in W_L$ and $n \in N_L$, as desired. This proves the following proposition.

Proposition 5.13 *Suppose $L = \{s, t\} \subseteq S$ is such that W_L is of type $A_1 \times A_1$. Then Conjecture C holds for $L \subseteq S$.*

Second, suppose that W_L is of type $I_2(m)$ where $m = 2k + 1$. Recall the characters $\chi_j \colon st \mapsto \zeta_m^j$ for $j = 1, \dots, m - 1$. The centrally primitive idempotent in $\mathbb{C}\langle st\rangle$ affording χ_j is

$$f_j = \frac{1}{m} \sum_{k=0}^{m-1} \zeta_m^{jk} (st)^{-k}.$$

The cuspidal conjugacy classes in W_L are represented by $c_j = (st)^j$ for $j = 1, \dots, k$ and $C_{W_L}(c_j)$ is the rotation subgroup $W^+ = \langle st\rangle$ of W. Moreover, for $j = 1, \dots, k$ the characters $\varphi_{(st)^j} = \chi_j$ satisfy the conclusions of Conjecture B for W_L and the line $\mathbb{C}e_L^L f_j$ in $e_L^L \mathbb{C}W_L$ affords the character $\varphi_{(st)^j}$ of W^+. As usual, denote by w_L the longest element of W_L. Note that $f_j^{w_L} = f_{m-j}$, for $j = 1, \dots, k$, since $(st)^{w_L} = (st)^{-1}$, and that $e_L^L f_j = \mathrm{Av}(\langle w_L\rangle) f_j$, by Lemma 5.4, since $\mathrm{Av}(W_L) f_j = \sum_{k=0}^{m-1} \zeta_m^{jk} \mathrm{Av}(W_L) = 0$, for $j = 1, \dots, m - 1$. Obviously, the graph automorphism γ

swaps $e_L^L f_j$ and $e_L^L f_{m-j}$, as does right multiplication by w_L:

$$\begin{aligned} e_L^L f_j w_L &= \mathrm{Av}(\langle w_L \rangle) f_j w_L = \mathrm{Av}(\langle w_L \rangle) w_L\, f_j^{w_L} \\ &= \mathrm{Av}(\langle w_L \rangle) f_j^{w_L} = \mathrm{Av}(\langle w_L \rangle) f_{m-j} = e_L^L f_{m-j}. \end{aligned}$$

Recall from [4, Lemma 3.2] that N_L centralizes e_L^L and that $N_W(W_L)$ acts naturally on $\mathbb{C}W_L$ with $a.nw = n^{-1}anw$ for a in $\mathbb{C}W_L$, n in N_L, and w in W_L. Suppose z is in $C_W(c_j)$. Then $z = nw$, where n is in N_L and w is in W_L. If n centralizes W_L, then

$$e_L^L f_j \cdot z = n^{-1} e_L^L f_j n w = e_L^L f_j w$$

and w is in W^+. If n does not centralize W_L, then nw_L centralizes W_L and $nw = (nw_L)(w_L w)$, so

$$e_L^L f_j \cdot z = \big(w_L n^{-1}\big) e_L^L f_j (n w_l)(w_L w) = e_L^L f_j (w_L w)$$

and $w_L w$ is in W^+. It follows that the characters $\varphi_{(st)^j}$ of W^+ extend to characters $\widetilde{\varphi}_{(st)^j}$ of the centralizers $C_W(c_j)$ with

$$\widetilde{\varphi}_{(st)^j}(z) = \varphi_{(st)^j}(v),$$

where if $z = nw$ with $n \in C_W(W_L)$, then $v = w$, and if $z = nw$ with $n \notin C_W(W_L)$, then $v = w_L w$. For $j = 1, \dots, k$, define $M_j = e_L^L f_j \mathbb{C}W_L$. Then M_j is a $N_W(W_L)$-module with basis $\{e_L^L f_j, e_L^L f_{m-j}\}$ and character $\mathrm{Ind}_{C_W(c_j)}^{N_W(W_L)} \widetilde{\varphi}_{(st)^j}$. Clearly, $e_L^L \mathbb{C}W_L \cong \bigoplus_{j=1}^k M_j$ is a decomposition of $e_L^L \mathbb{C}W_L$ as a direct sum of $N_W(W_L)$-modules. By [4, Corollary 3.13], the character of $e_L^L \mathbb{C}W_L$ is $\widetilde{\Phi}_L$, and so we conclude that

$$\widetilde{\Phi}_L = \sum_{j=1}^k \mathrm{Ind}_{C_W(c_j)}^{N_W(W_L)} \widetilde{\varphi}_{(st)^j}.$$

Thus, part (i) of Conjecture C holds. By Remark 4.3, to show that Conjecture C holds, it suffices to show that $\widetilde{\Psi}_L = \widetilde{\Phi}_L \epsilon_S \alpha_L$. Define $a_L = a_s a_t$ in $A(W)$, and recall from Lemma 5.9 that $a_L \mathbb{C}W_L$ is isomorphic to the top component of $A(W_L)$. Since m is odd, we have $W_L = W^+ \cup w_L W^+$ and thus

$$a_L \mathbb{C}W^+ = a_L \mathbb{C}W_L,$$

since $a_L w_L = a_s a_t . w_L = a_t a_s = -a_s a_t = -a_L$. Define $f_0 = \mathrm{Av}(W^+)$. Then the idempotents f_j for $j = 0, \dots, m-1$ form a Wedderburn basis of the group algebra $\mathbb{C}W^+$ and so the module $a_L \mathbb{C}W^+$ is spanned by the elements $\{a_L f_j : j = 0, \dots, m-1\}$. Since

$$\begin{aligned} a_L f_0 &= \sum_{k=0}^{m-1} a_0 a_{m-1}.(ts)^k = \sum_{k=0}^{m-1} a_{k+1} a_k \\ &= a_0 a_{m-1} - a_0 a_1 + \sum_{k=1}^{m-2} a_0 a_k - a_0 a_{k+1} = 0, \end{aligned}$$

we see that $\{a_L f_j : j = 1, \dots, m-1\}$ is a $\mathbb{C}$-basis of $a_L \mathbb{C} W_L$. By construction $f_j st = \zeta_m^j f_j$ and by definition $\epsilon(st) = \alpha_L(st) = 1$. Therefore,

$$a_L f_j.st = \zeta_m^j a_L f_j \quad \text{and} \quad e_L^L f_j.st = \epsilon(w)\alpha_L(w)\zeta_m^j e_L^L f_j. \tag{5.14}$$

We have seen that $f_j w_L = w_L f_{m-j}$, $a_L.w_L = -a_L$, and in Lemma 5.4 that $e_L^L w_L = e_L^L$. Also, $\epsilon(w_L) = -1$, and $\alpha_L(w_L) = 1$. Therefore,

$$a_L f_j.w_L = a_L f_{mj} \quad \text{and} \quad e_L^L f_j.w_L = \epsilon(w_L)\alpha_L(w_L) e_L^L f_{m-j}. \tag{5.15}$$

For n in N_L, we have $f_j.n = n f_j^n$, $a_L n = \sigma_L(n) a_L$, where by Lemma 2.1 $\sigma_L(n) = \epsilon(n)\alpha_L(n)$, and $e_L^L.n = n^{-1} e_L^L n = e_L^L$. Therefore,

$$a_L f_j.n = \sigma_L(n) a_L f_j^n \quad \text{and} \quad e_L^L f_j.n = \epsilon(n)\alpha_L(n)\sigma_L(n) e_L^L f_j^n. \tag{5.16}$$

Because $N_W(W_L)$ is generated by st, w_L, and N_L, it follows from (5.14), (5.15), and (5.16) that $\widetilde{\Psi}_L(w) = \widetilde{\Phi}_L(w)\epsilon_S(w)\alpha_L(w)$ for w in $N_W(W_L)$. This proves the following proposition.

Proposition 5.17 *Suppose $L = \{s, t\} \subseteq S$ is such that the order m of st is odd. Then Conjecture C holds for $L \subseteq S$.*

We summarize Propositions 5.13, 5.17, and Theorem 4.7 for rank 2 parabolic subgroups as follows.

Theorem 5.18 *Suppose that W_L is a rank 2 parabolic subgroup of W. Then Conjecture C holds for W_L.*

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