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# Homology of balanced complexes via the Fourier transform

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**Abstract** Let $G_0, \dots, G_k$ be finite abelian groups, and let $G_0 * \cdots * G_k$ be the join of the 0-dimensional complexes $G_i$. We give a characterization of the integral $k$-coboundaries of subcomplexes of $G_0 * \cdots * G_k$ in terms of the Fourier transform on the group $G_0 \times \cdots \times G_k$. This provides a short proof of an extension of a recent result of Musiker and Reiner on a topological interpretation of the cyclotomic polynomial.

**Keywords** Simplicial homology · Fourier transform

## 1 Introduction

Let $G_0, \dots, G_k$ be finite abelian groups with the discrete topology, and let $N = \prod_{i=0}^{k}(|G_i| - 1)$. The simplicial join $Y = G_0 * \cdots * G_k$ is homotopy equivalent to a wedge of $N$ $k$-dimensional spheres (see e.g. Theorem 1.3 in [1]). Subcomplexes of $Y$ are called *balanced complexes* (see [5]). Denote the $(k-1)$-dimensional skeleton of $Y$ by $Y^{(k-1)}$. Let $A$ be a subset of $G_0 \times \cdots \times G_k$. Regarding each $a \in A$ as an oriented $k$-simplex of $Y$, we consider the balanced complex

$$X(A) = X_{G_0,\dots,G_k}(A) = Y^{(k-1)} \cup A.$$

In this note we characterize the integral $k$-coboundaries of $X(A)$ in terms of the Fourier transform on the group $G_0 \times \cdots \times G_k$. As an application, we give a short proof of an extension of a recent result of Musiker and Reiner [4] on a topological interpretation of the cyclotomic polynomial.

We recall some terminology. Let $R[G]$ denote the group algebra of a finite abelian group $G$ with coefficients in a ring $R$. By writing $f = \sum_{x \in G} f(x)x \in R[G]$ we

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identify elements of $R[G]$ with $R$-valued functions on $G$. For a subset $A \subset G$, let $R[A] = \{f \in R[G] : \mathrm{supp}(f) \subset A\}$. A character of $G$ is a homomorphism of $G$ into the multiplicative group $\mathbb{C} - \{0\}$. Let $\widehat{G}$ be the character group of $G$, and let $\mathbf{1}$ be the trivial character of $G$. The orthogonality relation asserts that for $\chi \in \widehat{G}$,

$$\sum_{g \in G} \chi(g) = |G| \cdot \delta(\chi, \mathbf{1}), \tag{1}$$

where $\delta(\chi, \mathbf{1}) = 1$ if $\chi = \mathbf{1}$ and is zero otherwise. The Fourier transform is the linear bijection $\mathcal{F} : \mathbb{C}[G] \to \mathbb{C}[\widehat{G}]$ given on $f \in \mathbb{C}[G]$ and $\chi \in \widehat{G}$ by

$$\mathcal{F}(f)(\chi) = \widehat{f}(\chi) = \sum_{x \in G} f(x)\chi(x).$$

Let $G = G_0 \times \cdots \times G_k$. Then $\widehat{G} = \widehat{G}_0 \times \cdots \times \widehat{G}_k$. For $0 \le i \le k$, let

$$L_i = G_0 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_k.$$

We identify the group of integral $k$-cochains $C^k(X(A); \mathbb{Z})$ with $\mathbb{Z}[A]$ and the group of integral $(k-1)$-cochains $C^{k-1}(X(A); \mathbb{Z}) = C^{k-1}(X(G); \mathbb{Z})$ with the $(k+1)$-tuples $\psi = (\psi_0, \ldots, \psi_k)$ where $\psi_i \in \mathbb{Z}[L_i]$. The coboundary map

$$d_{k-1} : C^{k-1}\bigl(X(G); \mathbb{Z}\bigr) \to C^k\bigl(X(G); \mathbb{Z}\bigr)$$

is given by

$$d_{k-1}\psi(g_0, \ldots, g_k) = \sum_{i=0}^{k} (-1)^i \psi_i(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k).$$

For $0 \le i \le k$, let $\mathbf{1}_i$ denote the trivial character of $G_i$, and let

$$\widehat{G}^+ = \bigl(\widehat{G}_0 - \{\mathbf{1}_0\}\bigr) \times \cdots \times \bigl(\widehat{G}_k - \{\mathbf{1}_k\}\bigr).$$

For $A \subset G$ and $f \in \mathbb{Z}[G]$, let $f_{|A} \in \mathbb{Z}[A]$ be the restriction of $f$ to $A$. The group

$$\mathrm{B}^k\bigl(X(A); \mathbb{Z}\bigr) = \bigl\{d_{k-1}\psi_{|A} : \psi \in C^{k-1}\bigl(X(G); \mathbb{Z}\bigr)\bigr\}$$

of integral $k$-coboundaries of $X(A)$ is characterized by the following:

**Proposition 1.1** *For any $A \subset G$,*

$$\mathrm{B}^k\bigl(X(A); \mathbb{Z}\bigr) = \bigl\{f_{|A} : f \in \mathbb{Z}[G] \ \textit{such that} \ \mathrm{supp}(\widehat{f}) \subset \widehat{G} - \widehat{G}^+\bigr\}.$$

As an application of Proposition 1.1, we study the homology of a family of balanced complexes introduced by Musiker and Reiner [4]. Let $p_0, \ldots, p_k$ be distinct primes and for $0 \le i \le k$, let $G_i = \mathbb{Z}/p_i\mathbb{Z} = \mathbb{Z}_{p_i}$. Writing $n = \prod_{i=0}^{k} p_i$, let

$$\theta : \mathbb{Z}_n \to G = G_0 \times \cdots \times G_k$$

be the standard isomorphism given by

$$\theta(x) = \big(x(\bmod p_0), \dots, x(\bmod p_k)\big).$$

For any $\ell$, let $\mathbb{Z}_\ell^\times = \{m \in \mathbb{Z}_\ell : \gcd(m, \ell) = 1\}$. Let $\varphi(n) = |\mathbb{Z}_n^\times| = \prod_{i=0}^k (p_i - 1)$ be the Euler function of $n$, and let $A_0 = \{\varphi(n) + 1, \varphi(n) + 2, \dots, n - 2, n - 1\}$. For $A \subset \{0, \dots, \varphi(n)\}$, consider the complex

$$K_A = X\big(\theta(A \cup A_0)\big) \subset \mathbb{Z}_{p_0} * \cdots * \mathbb{Z}_{p_k}.$$

Let $\omega = \exp(\frac{2\pi i}{n})$ be a fixed primitive $n$th root of unity. The $n$th cyclotomic polynomial (see e.g. [2]) is given by

$$\Phi_n(z) = \prod_{j \in \mathbb{Z}_n^\times} \big(z - \omega^j\big) = \sum_{j=0}^{\varphi(n)} c_j z^j \in \mathbb{Z}[z].$$

Musiker and Reiner [4] discovered the following remarkable connection between the coefficients of $\Phi_n(z)$ and the homology of the complexes $K_{\{j\}}$.

**Theorem 1.2** (Musiker and Reiner) *For any* $j \in \{0, \dots, \varphi(n)\}$,

$$\tilde{\mathrm{H}}_i(K_{\{j\}}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/c_j\mathbb{Z}, & i = k - 1, \\ \mathbb{Z}, & i = k \text{ and } c_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The next result extends Theorem 1.2 to general $K_A$. Let

$$c_A = (c_j : j \in A) \in \mathbb{Z}^A$$

and

$$d_A = \begin{cases} \gcd(c_A), & c_A \neq 0, \\ 0, & c_A = 0. \end{cases}$$

**Theorem 1.3** *For any* $A \subset \{0, \dots, \varphi(n)\}$,

$$\tilde{\mathrm{H}}^i(K_A; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = k - 1 \text{ and } d_A = 0, \\ \mathbb{Z}^{|A|-1} \oplus \mathbb{Z}/d_A\mathbb{Z}, & i = k, \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$\tilde{\mathrm{H}}_i(K_A; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/d_A\mathbb{Z}, & i = k - 1, \\ \mathbb{Z}^{|A|}, & i = k \text{ and } d_A = 0, \\ \mathbb{Z}^{|A|-1}, & i = k \text{ and } d_A \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.1 is proved in Sect. 2. It is then used in Sect. 3 to obtain an explicit form of the $k$-coboundaries of $K_A$ (Proposition 3.1) that directly implies Theorem 1.3.

## 2 *k*-Coboundaries and Fourier transform

*Proof of Proposition 1.1* It suffices to consider the case $A = G$. Let $\psi = (\psi_0, \dots, \psi_k) \in C^{k-1}(X(G); \mathbb{Z})$. Using (1), it follows that for any $\chi = (\chi_0, \dots, \chi_k) \in \widehat{G}$,

$$\begin{aligned}
\widehat{d_{k-1}\psi}(\chi) &= \sum_{g=(g_0,\dots,g_k)\in G} d_{k-1}\psi(g)\chi(g) \\
&= \sum_{(g_0,\dots,g_k)} \sum_{i=0}^{k} (-1)^i \psi_i(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_k) \prod_{j=0}^{k} \chi_j(g_j) \\
&= \sum_{i=0}^{k} (-1)^i \sum_{(g_0,\dots,g_{i-1},g_{i+1},\dots,g_k)} \psi_i(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_k) \\
&\quad \times \prod_{j\neq i} \chi_j(g_j) \sum_{g_i} \chi_i(g_i) \\
&= \sum_{i=0}^{k} (-1)^i \widehat{\psi_i}(\chi_0, \dots, \chi_{i-1}, \chi_{i+1}, \dots, \chi_k) |G_i| \delta(\chi_i, \mathbf{1}_i).
\end{aligned}$$

Therefore $\mathrm{supp}(\widehat{d_{k-1}\psi}) \subset \widehat{G} - \widehat{G}^+$, and so

$$U_1 \stackrel{\text{def}}{=} \mathrm{B}^k\big(X(G); \mathbb{Z}\big) \subset \big\{ f \in \mathbb{Z}[G] : \mathrm{supp}(\widehat{f}) \subset \widehat{G} - \widehat{G}^+ \big\} \stackrel{\text{def}}{=} U_2.$$

Since $X(G)$ is homotopy equivalent to a wedge of $\prod_{i=0}^{k}(|G_i| - 1) = |\widehat{G}^+|$ $k$-dimensional spheres, it follows that $\mathrm{H}^k(X(G); \mathbb{Z}) = \mathbb{Z}[G]/U_1$ is free of rank $|\widehat{G}^+|$ and hence $\operatorname{rank} U_1 = |\widehat{G}| - |\widehat{G}^+|$. On the other hand, the injectivity of the Fourier transform implies that

$$\operatorname{rank} U_2 \le \dim_{\mathbb{C}} \big\{ f \in \mathbb{C}[G] : \mathrm{supp}(\widehat{f}) \subset \widehat{G} - \widehat{G}^+ \big\} = |\widehat{G}| - \big|\widehat{G}^+\big|$$

and therefore $\operatorname{rank} U_2/U_1 = 0$. Since $U_2/U_1 \subset \mathrm{H}^k(X(G); \mathbb{Z})$ is free, it follows that $U_1 = U_2$. □

## 3 The homology of $K_A$

Recall that, in the context of Theorems 1.2 and 1.3, one chooses $G = \mathbb{Z}_{p_0} \times \cdots \times \mathbb{Z}_{p_k}$ and $n = \prod_{j=0}^{k} p_j$. For $h \in \mathbb{Z}[G]$, let $\theta^* h \in \mathbb{Z}[\mathbb{Z}_n]$ be the pullback of $h$ given by $\theta^* h(x) = h(\theta(x))$. For any $\ell$, we identify the character group $\widehat{\mathbb{Z}_\ell}$ with $\mathbb{Z}_\ell$ via the isomorphism $\eta_\ell : \mathbb{Z}_\ell \to \widehat{\mathbb{Z}_\ell}$ given by $\eta_\ell(y)(x) = \exp(2\pi i x y/\ell)$. The Fourier transform on $\mathbb{Z}_\ell$ is then regarded as the automorphism of $\mathbb{C}[\mathbb{Z}_\ell]$ given by

$$\widehat{f}(y) = \sum_{x\in\mathbb{Z}_\ell} f(x) \exp\left(\frac{2\pi i x y}{\ell}\right).$$

Proposition 1.1 implies the following characterization of the integral $k$-coboundaries of $K_A$. For $A \subset \{0, \dots, \varphi(n)\}$, let $\theta_A$ denote the restriction of $\theta$ to $A \cup A_0$, and let $\theta_A^*$ be the induced isomorphism from $\mathbb{Z}[\theta(A \cup A_0)]$ to $\mathbb{Z}[A \cup A_0]$. Let

$$\mathcal{B}(A) = \left\{ f_{|A\cup A_0} : f \in \mathbb{Z}[\mathbb{Z}_n] \text{ such that } \widehat{f}(1) = 0 \right\}.$$

**Proposition 3.1**

$$\theta_A^*\big(\mathrm{B}^k(K_A;\mathbb{Z})\big) = \mathcal{B}(A).$$

*Proof* We first examine the relation between the Fourier transforms on $\mathbb{Z}_n$ and on $G$. Let

$$\lambda = \sum_{j=0}^{k} \prod_{t \neq j} p_t \in \mathbb{Z}_n^\times.$$

For any $h \in \mathbb{Z}[G]$ and $m \in \mathbb{Z}_n$,

$$\begin{aligned} \widehat{\theta^* h}(\lambda m) &= \sum_{x \in \mathbb{Z}_n} \theta^* h(x) \exp\left(\frac{2\pi i x \lambda m}{n}\right) \\ &= \sum_{x \in \mathbb{Z}_n} h\big(\theta(x)\big) \exp\left(\sum_{j=0}^{k} \frac{2\pi i x m}{p_j}\right) = \widehat{h}\big(\theta(m)\big). \end{aligned} \tag{2}$$

Noting that

$$\theta^{-1}\big(\widehat{G}^+\big) = \theta^{-1}\big(\mathbb{Z}_{p_0}^\times \times \cdots \times \mathbb{Z}_{p_k}^\times\big) = \mathbb{Z}_n^\times = \lambda \mathbb{Z}_n^\times,$$

it follows from Proposition 1.1 and (2) that

$$\begin{aligned} \mathrm{B}^k(K_A;\mathbb{Z}) &= \left\{ h_{|\theta(A\cup A_0)} : h \in \mathbb{Z}[G] \text{ such that } \operatorname{supp}(\widehat{h}) \subset \widehat{G} - \widehat{G}^+ \right\} \\ &= (\theta_A^*)^{-1}\left\{ f_{|A\cup A_0} : f \in \mathbb{Z}[\mathbb{Z}_n] \text{ such that } \operatorname{supp}(\widehat{f}) \subset \mathbb{Z}_n - \mathbb{Z}_n^\times \right\}. \end{aligned} \tag{3}$$

Let $\mathcal{P}_n = \{\omega^m : m \in \mathbb{Z}_n^\times\}$ be the set of primitive $n$th roots of 1. The Galois group $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ acts transitively on $\mathcal{P}_n$. Hence, by (3):

$$\begin{aligned} &\theta_A^*\big(\mathrm{B}^k(K_A;\mathbb{Z})\big) \\ &\quad = \left\{ f_{|A\cup A_0} : f \in \mathbb{Z}[\mathbb{Z}_n] \text{ such that } \operatorname{supp}(\widehat{f}) \subset \mathbb{Z}_n - \mathbb{Z}_n^\times \right\} \\ &\quad = \left\{ f_{|A\cup A_0} : f \in \mathbb{Z}[\mathbb{Z}_n] \text{ such that } \widehat{f}(m) = \sum_{x \in \mathbb{Z}_n} f(x)\omega^{mx} = 0 \text{ for all } m \in \mathbb{Z}_n^\times \right\} \\ &\quad = \left\{ f_{|A\cup A_0} : f \in \mathbb{Z}[\mathbb{Z}_n] \text{ such that } \widehat{f}(1) = 0 \right\} = \mathcal{B}(A). \end{aligned}$$

□

**Corollary 3.2** *$\theta_A^*$ induces an isomorphism between $\mathrm{H}^k(K_A;\mathbb{Z})$ and*

$$\mathcal{H}(A) \stackrel{\text{def}}{=} \mathbb{Z}[A \cup A_0]/\mathcal{B}(A).$$

For $j \in A \cup A_0$, let $g_j \in \mathbb{Z}[A \cup A_0]$ be given by $g_j(i) = 1$ if $i = j$ and $g_j(i) = 0$ otherwise. Let $[g_j]$ be the image of $g_j$ in $\mathcal{H}(A)$. The computation of $\mathcal{H}(A)$ depends on the following:

**Claim 3.3**

(i) *$\mathcal{H}(A)$ is generated by* $\{[g_j] : j \in A\}$.
(ii) *The minimal relation between* $\{[g_j]\}_{j \in A}$ *is* $\sum_{j \in A} c_j[g_j] = 0$.

*Proof of* (i) Let $t \in A_0$. There exist $u_0, \ldots, u_{\varphi(n)-1} \in \mathbb{Z}$ such that

$$\sum_{\ell=0}^{\varphi(n)-1} u_\ell \omega^\ell + \omega^t = 0.$$

Let $f \in \mathbb{Z}[\mathbb{Z}_n]$ be given by

$$f(\ell) = \begin{cases} u_\ell, & 0 \le \ell \le \varphi(n) - 1, \\ 1, & \ell = t, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\widehat{f}(1) = \sum_{\ell=0}^{\varphi(n)-1} u_\ell \omega^\ell + \omega^t = 0,$$

it follows that

$$\sum_{j \in A} u_j g_j + g_t = f_{|A \cup A_0} \in \mathcal{B}(A).$$

Hence $[g_t] = -\sum_{j \in A} u_j [g_j]$. □

*Proof of* (ii) Let $f \in \mathbb{Z}[\mathbb{Z}_n]$ be given by $f(\ell) = c_\ell$ if $0 \le \ell \le \varphi(n)$ and zero otherwise. Since $\widehat{f}(1) = \Phi_n(\omega) = 0$, it follows that

$$\sum_{j \in A} c_j g_j = f_{|A \cup A_0} \in \mathcal{B}(A).$$

Hence $\sum_{j \in A} c_j [g_j] = 0$. Conversely, suppose that $\sum_{j \in A} \alpha_j [g_j] = 0$ for integers $\{\alpha_j\}_{j \in A}$. Then there exists an $h \in \mathbb{Z}[\mathbb{Z}_n]$ such that $\widehat{h}(1) = 0$ and $h_{|A \cup A_0} = \sum_{j \in A} \alpha_j g_j$. In particular, $h(\ell) = 0$ for $\ell \ge \varphi(n) + 1$. Let $p(z) = \sum_{\ell=0}^{\varphi(n)} h(\ell) z^\ell$. Then $p(\omega) = \widehat{h}(1) = 0$. Hence $p(z) = r\Phi_n(z)$ for some $r \in \mathbb{Z}$. Therefore $\alpha_j = h(j) = rc_j$ for all $j \in A$. □

*Proof of Theorem 1.3* Corollary 3.2 and Claim 3.3 imply that

$$\mathrm{H}^k(K_A; \mathbb{Z}) \cong \mathcal{H}(A) = \mathbb{Z}[A]/\mathbb{Z}c_A \cong \mathbb{Z}^{|A|-1} \oplus \mathbb{Z}/d_A\mathbb{Z}. \tag{4}$$

The remaining parts of Theorem 1.3 are formal consequences of (4) and the universal coefficient theorem (see e.g. [3]):

$$0 \leftarrow \mathrm{Hom}\big(\mathrm{H}_p(K_A;\mathbb{Z}),\mathbb{Z}\big) \leftarrow \mathrm{H}^p(K_A;\mathbb{Z}) \leftarrow \mathrm{Ext}\big(\mathrm{H}_{p-1}(K_A;\mathbb{Z}),\mathbb{Z}\big) \leftarrow 0. \tag{5}$$

First consider the case $c_A = 0$. By (4) and (5),

$$0 \leftarrow \mathrm{Hom}\big(\mathrm{H}_k(K_A;\mathbb{Z}),\mathbb{Z}\big) \leftarrow \mathbb{Z}^{|A|} \leftarrow \mathrm{Ext}\big(\mathrm{H}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}\big) \leftarrow 0.$$

Therefore $\mathrm{H}_k(K_A;\mathbb{Z}) \cong \mathbb{Z}^{|A|}$, and $\mathrm{H}_{k-1}(K_A;\mathbb{Z})$ is torsion free. The Euler–Poincaré relation

$$\mathrm{rank}\,\mathrm{H}_k(K_A;\mathbb{Z}) = \mathrm{rank}\,\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) + |A| - 1 \tag{6}$$

then implies that $\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) \cong \mathbb{Z}$ and

$$\tilde{\mathrm{H}}^{k-1}(K_A;\mathbb{Z}) \cong \mathrm{Hom}\big(\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}\big) \cong \mathbb{Z}.$$

Next assume that $c_A \neq 0$. By (4) and (5),

$$0 \leftarrow \mathrm{Hom}\big(\mathrm{H}_k(K_A;\mathbb{Z}),\mathbb{Z}\big) \leftarrow \mathbb{Z}^{|A|-1} \oplus \mathbb{Z}/d_A\mathbb{Z} \leftarrow \mathrm{Ext}\,\big(\mathrm{H}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}\big) \leftarrow 0.$$

Therefore $\mathrm{H}_k(K_A;\mathbb{Z}) \cong \mathbb{Z}^{|A|-1}$ and $\mathrm{Ext}\,(\mathrm{H}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}) = \mathbb{Z}/d_A\mathbb{Z}$. It follows by (6) that $\mathrm{rank}\,\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) = 0$. Hence $\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) = \mathbb{Z}/d_A\mathbb{Z}$ and $\tilde{\mathrm{H}}^{k-1}(K_A;\mathbb{Z}) = 0$. □

*Remark* In the proof of (ii) it was observed that the function $f \in \mathbb{Z}[\mathbb{Z}_n]$ given by $f(\ell) = c_\ell$ if $0 \le \ell \le \varphi(n)$ and zero otherwise is the image under $\theta^*$ of a $k$-coboundary of $X(G)$. This fact also appears (with a different proof) in Proposition 24 of [4] and is attributed there to D. Fuchs.

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