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Isomorphisms of groups related to flocks

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Abstract A truly fruitful way to construct finite generalized quadrangles is through the detection of Kantor families in the general 5-dimensional Heisenberg group $\mathscr{H}_2(q)$ over some finite field $\mathbb{F}_q$. All these examples are so-called "flock quadrangles". Payne (Geom. Dedic. 32:93–118, 1989) constructed from the Ganley flock quadrangles the new Roman quadrangles, which appeared not to arise from flocks, but still via a Kantor family construction (in some group $\mathscr{G}$ of the same order as $\mathscr{H}_2(q)$). The fundamental question then arose as to whether $\mathscr{H}_2(q) \cong \mathscr{G}$ (Payne in Geom. Dedic. 32:93–118, 1989). Eventually the question was solved in Havas et al. (Finite geometries, groups, and computation, pp. 95–102, de Gruyter, Berlin, 2006; Adv. Geom. 26:389–396, 2006). Payne's Roman construction appears to be a special case of a far more general one: each flock quadrangle for which the dual is a translation generalized quadrangle gives rise to another generalized quadrangle which is in general not isomorphic, and which also arises from a Kantor family. Denote the class of such flock quadrangles by $\mathscr{C}$.

In this paper, we resolve the question of Payne for the complete class $\mathscr{C}$. In fact we do more—we show that flock quadrangles are characterized by their groups.

Several (sometimes surprising) by-products are described in both odd and even characteristic.

Keywords Flock quadrangle · Elation quadrangle · Automorphism group · Heisenberg group · Characterization

1 Introduction

Generalized quadrangles were introduced by Tits in [28] as a subclass of the generalized polygons, the natural geometries of the groups of Lie type of (relative) rank 2.

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Let us formally define them in the finite case. (For definitions not given in the introduction, see the next section, the monographs [19, 23] or the recent paper [18].)

A (finite) *generalized quadrangle* (GQ) of *order* (s,t) is a point-line incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathtt{I}$ is a symmetric point-line incidence relation satisfying the following axioms:

- two distinct points are incident with at most one line;
- each point is incident with $t+1$ lines ($t \geq 1$);
- each line is incident with $s+1$ points ($s \geq 1$);
- if p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $p \mathtt{I} M \mathtt{I} q \mathtt{I} L$.

There is a map D which sends a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ (of order (s,t)) to $\mathcal{S}^D = (\mathcal{B}, \mathcal{P}, \mathtt{I})$, a GQ of order (t,s) which is called the *point-line dual* of $\mathcal{S}$. The presence of this map is called "point-line duality"; for GQs of order (s,t), in any definition or theorem the words "point" and "line" can be interchanged and also the parameters s and t, to obtain the dualized version.

In the last few decades, one of the most fruitful ways to construct finite generalized quadrangles was through the detection of certain Kantor families (arising from a so-called "q-clan") in the 5-dimensional general Heisenberg group $\mathcal{H}_2(q)$ over some finite field $\mathbb{F}_q$. All these examples are so-called "flock quadrangles", and they have order (q^2, q). In [17], Payne constructed from the Ganley flock quadrangles new quadrangles ("Roman quadrangles") which do not come from flocks, but still arise via a Kantor family construction (in some group $\mathcal{G}$ of the same order as $\mathcal{H}_2(q)$). The fundamental question (first asked by Payne in *op. cit.*, see his excellent account [18]) then arose as to whether $\mathcal{H}_2(q) \cong \mathcal{G}$. In [10], Havas et al. showed that for the Roman quadrangles with parameters $(729, 27)$, and using a computer program, the corresponding groups are not isomorphic. In [11], they obtained the result for all Roman quadrangles. The proof consists of showing that non-central elements in $\mathcal{H}_2(q)$ and $\mathcal{G}$ have nonisomorphic centralizers.

The construction of Payne appears to be a special case of a more general one: each flock quadrangle $\mathcal{S} = \mathcal{S}(\mathcal{F})$ for which the dual $\mathcal{S}^D$ is a translation generalized quadrangle gives rise to another generalized quadrangle (which is the dual of the translation dual $(\mathcal{S}^D)^*$) which is in general not isomorphic to $\mathcal{S}^D$, and which, by [25], also arises from a Kantor family. (Note that the notion "translation dual" is explained more formally in Sect. 2.2.) Denote the class of such flock quadrangles by $\mathcal{C}$. By reasons to be explained later, see for instance Sect. 2.4, we mainly consider odd characteristic, although some structural results concerning the even case will also be obtained. In odd characteristic, the known members of $\mathcal{C}$ are the Ganley and Kantor–Knuth flock quadrangles (both infinite classes) and the "sporadic" Penttila–Williams flock quadrangle.

$$
\begin{array}{ccccccc}
\mathcal{S} \in \mathcal{C} & \xrightarrow{D} & \mathcal{S}^D & \xrightarrow{*} & (\mathcal{S}^D)^* & \xrightarrow{D} & ((\mathcal{S}^D)^*)^D \\
\downarrow & & & & & & \downarrow [25] \\
\mathcal{H}_2 & & & \xrightarrow{?} & & & \mathcal{G}
\end{array}
$$

In this paper, we resolve the question of Payne for the complete class $\mathscr{C}$. In fact, we do more—we show that the flock quadrangles are characterized by their groups (in any characteristic), a question which was open for quite some time. As an application of the main result, the special case of prime q yields an alternative proof of the main result of [2].

Theorem 1.1 *Let $\mathscr{S}$ be an EGQ of order (q^2, q), q any prime power, with elation group $\mathscr{H}_2(q)$. Then $\mathscr{S}$ is a flock quadrangle.*

In terms of Kantor families, this result reads as follows:

Corollary 1.2 *Each Kantor family of type (q^2, q) in $\mathscr{H}_2(q)$ arises from a q-clan.*

Passing from the latter theorem to the solution of Payne's question goes as follows.

Theorem 1.3 *Let $\mathscr{S} = \mathscr{S}(\mathscr{F})$ be a flock GQ of order (q^2, q) for which the dual $\mathscr{S}^D$ is a TGQ, q odd. Let $\mathscr{G}$ be a group as in the diagram above. Then $\mathscr{G} \cong \mathscr{H}_2(q)$ if and only if $\mathscr{F}$ is a Kantor–Knuth flock if and only if $\mathscr{S} \cong ((\mathscr{S}^D)^*)^D$.*

Proof If $\mathscr{S}(\mathscr{F})$ is a Kantor–Knuth GQ, then $\mathscr{S} \cong ((\mathscr{S}^D)^*)^D$ [16], and the result follows from the fact that flock GQs in odd characteristic admit unique elation groups [24]. If $\mathscr{S}(\mathscr{F})$ is not a Kantor–Knuth GQ, it is known that the dual of $(\mathscr{S}^D)^*$ is not a flock quadrangle [23, Chap. 4]. Our main theorem now implies that its elation group cannot be $\mathscr{H}_2(q)$. □

If $\mathscr{S}(\mathscr{F})$ is the Ganley flock quadrangle, the dual of $(\mathscr{S}^D)^*$ is the Roman quadrangle (which is not isomorphic to a Kantor–Knuth quadrangle), yielding thus the result of Havas et al. [10, 11]. If $\mathscr{S} = \mathscr{S}(\mathscr{F})$ is a flock GQ of order (q^2, q) for which the dual $\mathscr{S}^D$ is a TGQ, and q is even, $\mathscr{S} \cong \mathscr{H}(3, q^2)$ (cf. Johnson [6, 12] or [23, Theorem 5.1.11]). In that case, the analogous question is reduced to the main results of [20, 24]. Other implications will be obtained further in the paper.

This paper arose while writing up [27], in an attempt to understand the importance of symplectic forms coming from commutation in elation groups.

2 Quadrangles, groups and flocks

Let $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathtt{I})$ be a generalized quadrangle of order (s, t).

Suppose $p \not\mathtt{I}\, L$, $(p, L) \in \mathscr{P} \times \mathscr{B}$. Then by $\mathrm{proj}_L p$, we denote the unique point on L collinear with p. Dually, $\mathrm{proj}_p L$ is the unique line incident with p concurrent with L. Let $A \subseteq \mathscr{P}$; then by $A^\perp$ we mean $\cap_{a \in A} a^\perp$ (here $a^\perp$ is the set of points of $\mathscr{P}$ collinear with a, including a). We write $A^{\perp\perp}$ for $(A^\perp)^\perp$. Clearly, if x and y are non-collinear points, $|\{x, y\}^{\perp\perp}| \le t + 1$; if equality holds, we say that $\{x, y\}$ is *regular*. The point x is *regular* provided $\{x, y\}$ is regular for every $y \in \mathscr{P} \setminus x^\perp$.

If the order of $\mathscr{S}$ is (t^2, t), and $\{U, V, W\}$ is a line *triad*, that is, three distinct lines mutually non-intersecting, then $|\{U, V, W\}^\perp| = t + 1$ [19, 1.2.4].

2.1 EGQs and STGQs

For a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, we call a point x an *elation point*, if there is an automorphism group H that fixes x linewise and acts sharply transitively on $\mathcal{P} \setminus x^{\perp}$ (the group is called "elation group"). If a GQ has an elation point, it is called an *elation generalized quadrangle* or, shortly, "EGQ". We will frequently use the notation $(\mathcal{S}^x, H)$ to indicate that x is an elation point with associated elation group H. Sometimes we also write $\mathcal{S}^x$ if we do not want to specify the elation group. More details on EGQs can be found in the recent work [26].

Suppose $(\mathcal{S}^x, H) = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is an EGQ of order (s,t), $s \neq 1 \neq t$, and let z be a point of $\mathcal{P} \setminus x^{\perp}$. Let $L_0, L_1, \dots, L_t$ be the lines incident with x, and define $r_i = \mathrm{proj}_{L_i} z$ and $M_i = \mathrm{proj}_z L_i$ for $0 \leq i \leq t$. Put $H_i = H_{M_i}$ and $H_i^* = H_{r_i}$ and $\mathbf{F} = \{H_i | 0 \leq i \leq t\}$. Then $|H| = s^2 t$ and $\mathbf{F}$ is a set of $t+1$ subgroups of H, each of order s. Also, for each i, H_i^* is a subgroup of H of order st containing H_i as a subgroup. Note that if one chooses an other point for z, $\mathbf{F}$ and $\mathbf{F}^*$ rest unchanged up to conjugation by an element of H. The following two conditions are satisfied:

- $H_i H_j \cap H_k = \{\mathbf{1}\}$ for distinct i, j and k;
- $H_i^* \cap H_j = \{\mathbf{1}\}$ for distinct i and j.

If H is a group of order $s^2 t$ and $\mathbf{F}$ (respectively, $\mathbf{F}^*$) is a set of $t+1$ subgroups H_i (respectively, H_i^*) of H of order s (respectively, of order st), and if the aforementioned conditions are satisfied, then the H_i^* are uniquely defined by the H_i, and $(\mathbf{F}, \mathbf{F}^*)$, or just $\mathbf{F}$, is said to be a *Kantor family* or 4*-gonal family* of type (s,t) in H. Using a (now) standard group coset geometry construction, one can then construct a GQ $\mathcal{S}(\mathbf{F}, \mathbf{F}^*)$ which is an EGQ with elation group H; moreover, if we start from $\mathcal{S}$ as above, then $\mathcal{S} \cong \mathcal{S}(\mathbf{F}, \mathbf{F}^*)$ [13, 15].

Remark 2.1 The point z will not be specified below, due to the fact that H acts transitively on $\mathcal{P} \setminus x^{\perp}$. Further, if $A \in \mathbf{F}$, or $A^* \in \mathbf{F}^*$, L_A will denote the line on z stabilized by A. Also, $[A]$ denotes $\mathrm{proj}_x L_A$.

If x is a regular point, it can be shown that H contains a subgroup $\mathbb{S}$ of order t consisting of automorphisms which fix each point of $x^{\perp}$ [27]. Such automorphisms are called *symmetries* with center x, and x is a *center of symmetry* since t is the maximal number of symmetries with center x. We then say that $\mathcal{S}^x$ is a *skew translation quadrangle* (STGQ). We call the STGQ *central* if $\mathbb{S}$ is contained in the center $Z(H)$ of H. As [27] points out, centrality is the key to classifying STGQs.

2.2 TGQs

If the elation group H of an EGQ $\mathcal{S}^x$ is abelian, we call $\mathcal{S}^x$ a *translation generalized quadrangle* with *translation group* H (and *translation point* x). In that case it can be shown that it is elementary abelian (cf. [19, Chap. 8], [23, Sect. 3.4]), and that $s \leq t$ [19, Chap. 8], [23, Sect. 3.3], if the order of $\mathcal{S}^x$ is (s,t), $s \neq 1 \neq t$. Note that H is uniquely defined—see [23, Theorem 3.3.10]. Let $(\mathbf{F}, \mathbf{F}^*)$ be the associated Kantor family. By [19, Sect. 8.5], see also [23, Sect. 3.4], the ring of endomorphisms of H preserving each element of $\mathbf{F}$ is a field $\mathbb{F}_q$, over which H can be seen as a vector space. Seeing the elements of $\mathbf{F}$ as subspaces (over $\mathbb{F}_q$) of the corresponding projective space $\mathbf{P}$, it appears that if $s \neq t$ or if $s = t$ is odd, one can construct another TGQ $(\mathcal{S}^x)^*$ by interpreting $(\mathbf{F}, \mathbf{F}^*)$ in the dual space of $\mathbf{P}$, see [19, Chap. 8], or [23, Sect. 3.9, Sect. 3.10]. The TGQ $(\mathcal{S}^x)^*$ is the *translation dual* of $\mathcal{S}^x$, and has the same order.

2.3 The general Heisenberg group

The *general Heisenberg group* $\mathcal{H}_n(q)$ (sometimes also written as $\mathcal{H}_n$ if we do not want to specify q) of dimension $2n+1$ over $\mathbb{F}_q$, with n a natural number, is the group of square $(n+2)\times(n+2)$-matrices with entries in $\mathbb{F}_q$, of the following form (and with the usual matrix multiplication):

$$\begin{pmatrix} 1 & \alpha & c \\ 0 & \mathbb{I}_n & \beta^T \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{F}_q^n$, $c \in \mathbb{F}_q$ and with $\mathbb{I}_n$ being the $n \times n$-identity matrix. The group $\mathcal{H}_n$ is isomorphic to the group $\{(\alpha, c, \beta) | \alpha, \beta \in \mathbb{F}_q^n, c \in \mathbb{F}_q\}$, where the group operation $\circ$ is given by $(\alpha, c, \beta) \circ (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \alpha\beta'^T, \beta + \beta')$. The following properties hold for $\mathcal{H}_n$ (defined over $\mathbb{F}_q$).

- $\mathcal{H}_n$ has exponent p if $q = p^h$ with p an odd prime; it has exponent 4 if q is even.
- The center of $\mathcal{H}_n$ is given by $Z = Z(\mathcal{H}_n) = \{(0, c, 0) | c \in \mathbb{F}_q\}$.
- $[\mathcal{H}_n, \mathcal{H}_n] = Z = \Phi(\mathcal{H}_n)$ and $\mathcal{H}_n$ is nilpotent of class 2 ($\Phi(\mathcal{H}_n)$ is the *Frattini subgroup* of H, that is, the intersection of all its maximal subgroups).

Finally recall that the following holds [13, 14].

- Let V be the elementary abelian p-group $\mathcal{H}_2(q)/Z$. The map χ

$$\chi : V \times V \mapsto \mathbb{F}_q : (aZ, bZ) \mapsto [a, b]$$

naturally defines a non-singular bilinear alternating form over $\mathbb{F}_q \equiv Z$. So V can be seen as a 4-dimensional space over $\mathbb{F}_q$, and in the corresponding projective 3-space over $\mathbb{F}_q$, χ defines a symplectic polar space $\mathbf{W}(q)$ of rank 2 (projective index 1).

2.4 Flock quadrangles and q-clans

Let $\mathcal{F}$ be a flock of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$ with equation $X_0X_1 = X_2^2$; so $\mathcal{F}$ is a partition of $\mathcal{K}$ without the vertex consisting of irreducible conics. Then it was noticed in [21] that the equations of the planes generated by the conics (w.r.t. a suitable reference system) define a Kantor family of type (q^2, q) in $\mathcal{H}_2(q)$; that is, to $\mathcal{F}$ corresponds an EGQ $\mathcal{S}^x = \mathcal{S}(\mathcal{F})$, called *flock quadrangle*, of order (q^2, q) with elation group $\mathcal{H}_2(q)$. Also, it happens to be an STGQ w.r.t. x, and Kantor families in $\mathcal{H}_2(q)$ that give rise to flock quadrangles are precisely those related to "q-clans", see [6, 23].

One of the corollaries of the main result of the present note is that Kantor families in $\mathcal{H}_2(q)$ *always* are of this type.

We finally mention that if $\mathcal{S}(\mathcal{F})$ is a flock quadrangle of order (q^2, q), and its dual is a TGQ, then $\mathcal{S}(\mathcal{F}) \cong \mathcal{H}(3, q^2)$ if q is even. If q is odd, the TGQ $\mathcal{S}(\mathcal{F})^D$ is isomorphic to its translation dual if and only if $\mathcal{F}$ is a Kantor–Knuth semifield flock. We refer to [6, 12, 23] for further details.

2.5 Property (G) and flock GQs

Let x be a point of a GQ $\mathcal{S}$ of order (t^2, t), $t \neq 1$, and let U, V be distinct lines incident with x. Then $\mathcal{S}$ satisfies *Property* (*G*) at the pair $\{U, V\}$ if any triad of lines $\{V, W, Z\}$ in $U^\perp$ is 3-regular (which means that $|\{V, W, Z\}^\perp| = |\{V, W, Z\}^{\perp\perp}| = t + 1$). (Note that the definition is symmetric in U and V.) The flag (x, L) has *Property* (*G*) if all pairs $\{L, M\}$ of distinct lines incident with x have Property (G). One says that x has *Property* (*G*) if all pairs $\{U, V\}$ incident with x have Property (G).

It can be shown that if $\mathcal{S}^x$ is a flock quadrangle, the point x satisfies Property (G) [17].

The following theorem was first obtained in odd characteristic in [22], answering a fundamental conjecture of Payne's essay [17]. In the case of even characteristic, relying on [22], it was obtained only much later by Brown [5]. (In odd characteristic, only one flag was required in [22]; later it was shown that one pair of intersecting lines was sufficient [3].)

Theorem 2.2 [5, 22] *A GQ of order* (t^2, t), $t \neq 1$, *satisfying Property* (*G*) *at two distinct flags* (u, L) *and* (u, M) *for some point* u *is isomorphic to a flock GQ.*

3 Special groups and alternating forms

Let H be a special group [8, p. 183] of order q^m, $m \in \mathbb{N} \setminus \{0, 1\}$, with $q = p^h$ a power of the prime p, for which

$$Z(H) = \Phi(H) = [H, H]$$

is (elementary abelian) of order q.

The exponent of H is easily seen to be p or p^2 if p is odd, and 4 if $p = 2$. Note that for p odd both cases occur; think of $\mathcal{H}_2(q)$ and $\langle a, b | a^p = \mathbf{1}, b^p = [a, b], a[a, b] = [a, b]a, b[a, b] = [a, b]b\rangle$. If the exponent of H is p, then $Z(H) = [H, H]$ implies $Z(H) = \Phi(H)$.

3.1 Setting

Define a bi-additive map χ by

$$\chi : V \times V \mapsto \mathbb{F}_q : \big(aZ(H), bZ(H)\big) \mapsto [a, b],$$

where we see $H/Z(H)$ as a vector space V over $\mathbb{F}_p$. We assume that χ "defines" a non-singular bilinear alternating form over $\mathbb{F}_q \equiv Z(H)$; by this we mean that the commuting structure of χ is a (non-singular) symplectic polar space over $\mathbb{F}_q$. The assumption implies that the dimension m of V over $\mathbb{F}_q$ is even, and χ defines a symplectic polar space $\mathbf{W}(2n - 1, q)$ of rank $n - 1$ in the projective space $\mathbf{PG}(2n - 1, q)$ associated to V, with $m = 2n$.

Observation 3.1 *If H admits a Kantor family of type (s,q), then $s \in \{q, q^2\}$, and so $|H| \in \{q^3, q^5\}$.*

Proof This follows from the fact that if H admits such a Kantor family, $|H| = s^2 q$, so $s = q^n$ ($n \in \mathbb{N} \setminus \{0\}$). Higman's inequality yields $n \in \{1, 2\}$. □

Now let H admit a Kantor family $(\mathbf{F}, \mathbf{F}^*)$, with $s = q^2$, and let $\mathcal{S}$ be the GQ of order (q^2, q) arising from $(\mathbf{F}, \mathbf{F}^*)$. Since $\mathcal{S}$ has order (q^2, q), all line spans (of non-concurrent lines) have size 2 [19, 1.4.1]. By a result of [27] (which can also be deduced from the proof of [19, 9.5.1]), it follows that $Z(H)$ can only consist of symmetries about the elation point x, so that x is a center of symmetry with associated group $Z(H) = \mathbb{S}$, and $(\mathcal{S}^x, H)$ is an STGQ. Now define χ as before (still assuming the condition on the alternating form over $\mathbb{F}_q$), and let $\mathbf{W}(q)$ be the corresponding polar space in $\mathbf{PG}(3,q)$—it is a symplectic generalized quadrangle of order q.

Observation 3.2 *The elements of $\mathbf{F}^*$ are elementary abelian.*

Proof For $A \in \mathbf{F}$, $[A, A] \le A \cap [H, H] = A \cap Z(H) = \{\mathbf{1}\}$, so A is abelian. Also, since H is a p-group, we have $\Phi(H) = H^p[H, H]$, implying that $A^p \subseteq Z(H)$. So A has exponent p, and A is elementary abelian. Since $A^* = A\mathbb{S} = AZ(H)$, the observation follows. □

The following observation follows from the fact that $H/\mathbb{S}$ is abelian, and will be used without further notice.

Observation 3.3 *For any $A^* \in \mathbf{F}^*$, A^* fixes $[A]$ pointwise.*

We are ready to obtain the main theorem. Note that it also characterizes 5-dimensional Heisenberg groups, and that it is independent of the characteristic.

Theorem 3.4 *Suppose H is a special p-group of order q^5 for which $Z(H) = \Phi(H) = [H, H]$ is elementary abelian of order q. Suppose H admits a Kantor family of type (q^2, q), and suppose χ defines a non-singular bilinear alternating form over $\mathbb{F}_q$. Then $H \cong \mathcal{H}_2(q)$, and the corresponding generalized quadrangle $\mathcal{S}$ of order (q^2, q) is a flock quadrangle.*

Proof Interpret $\mathbf{F}^*$ in $\mathbf{W}(q)$; it is a set R of $q+1$ mutually disjoint lines (note that in V, $a'Z(H) \sim b'Z(H)$ if and only if $[a', b'] = \mathbf{1}$). Let $A^*, B^* \in \mathbf{F}^*$ be different, and denote the corresponding lines in $\mathbf{W}(q)$ by A', B'. Consider (in $\mathbf{W}(q)$) any point a incident with A', and define $b = \mathrm{proj}_{B'} a$. The line $\mathrm{proj}_b A' = C'$ arises from an abelian subgroup of H of order q^3, denoted by C (it is not an element of $\mathbf{F}^*$). Clearly (in H) $|A^* \cap C| = |B^* \cap C| = q^2$. Let $A \le A^*$ and $B \le B^*$ be the corresponding elements in $\mathbf{F}$; then $|A \cap C| = |B \cap C| = q$, and

$$[A \cap C, B \cap C] = \{\mathbf{1}\}.$$

Interpreted in $\mathcal{S}$, this commutator relation gives a 3-regular triad: $L_B^{A\cap C}$ and $L_A^{B\cap C}$ define the reguli (if $\alpha \in (A \cap C)^\times$, then $[\alpha, B \cap C] = \{\mathbf{1}\}$, so $\{[B]\} \cup L_A^{B\cap C} =$

$\{L_B, L_B^\alpha, [A]\}^\perp$). Letting a vary on A', and A^* and B^* vary in $\mathbf{F}^*$, an easy exercise yields that each flag (x, L) incident with x has Property (G).

By Theorem 2.2, it follows that $\mathcal{S}$ is a flock quadrangle. By [20, 24], a flock quadrangle can only admit more than one elation group (w.r.t. a point) if $\mathcal{S} \cong \mathcal{H}(3, q^2)$ with q even. So in the other cases, we are done. When $\mathcal{S} \cong \mathcal{H}(3, q^2)$, there are exactly two isomorphism classes of elation groups (w.r.t. points), one of nilpotency class 2 (Heisenberg case), and the other of class 3 ("exotic case", see [20]). Since H is of class 2, we are done. □

The generalization of the theorem of Havas et al. [10, 11] was already treated in the introduction.

Remark 3.5 (Local versus global)

(i) Note that only local properties of the symplectic quadrangle are used in order to conclude our main result; in fact, what is only needed (in both characteristics) is that in the "commutation geometry", lines corresponding to distinct elements of $\mathbf{F}^*$ have the property that they are both intersected by $q + 1$ lines of the right size without violating GQ axioms. So instead of demanding that a symplectic quadrangle arise, it is already sufficient to ask that only some part of an abstract quadrangle be present.

(ii) In the even case, one could also consider the quadratic form

$$Q : V \mapsto \mathbb{F}_q : uZ(H) \mapsto u^2.$$

It singles out a hyperbolic quadric induced by the elements of $\mathbf{F}^*$ on the $\mathbf{W}(q)$ defined by χ. So starting from the same conditions as before, but now demanding that Q be a quadratic form over $\mathbb{F}_q$, also is enough to show that we are dealing with a flock GQ. This stresses (i). (A grid is sufficient.)

Remark 3.6 (Subquadrangles) Note that in the even case, the opposite regulus of the one defined by $\mathbf{F}^*$ defines $q + 1$ elementary abelian subgroups of order q^3 of H, in which Kantor families of type (q, q) are induced. Each of these groups defines q^2 TGQs of order q containing x. This is a known fact in q-clan geometry in even characteristic. The mere observation that $\mathbf{F}^*$ "has" a grid structure in V is already enough to conclude this.

The reader notices that in the even case, the aforementioned quadratic form Q singles out the involutions of $\mathcal{H}_2(q)$; in fact, there is more.

Observation 3.7 *The set of involutions of $\mathcal{H}_2(q)$ is $(\cup_{\mathbf{F}^*} A^*)^\times$, where $(\mathbf{F}, \mathbf{F}^*)$ is any Kantor family of type (q^2, q) in $\mathcal{H}_2(q)$, q even.*

The maximal elementary abelian 2-groups of $\mathcal{H}_2(q)$ are the $2(q + 1)$ subgroups corresponding to the lines of the quadric related to Q. Let $(\mathbf{F}, \mathbf{F}^*)$ be any Kantor family (of type (q^2, q)) in $\mathcal{H}_2(q)$. As the elements of $\mathbf{F}^*$ are elementary abelian (and maximal w.r.t. this property), cf. Observation 3.2, it follows that $\mathbf{F}^*$ is completely determined; it is one of only two possible sets $\mathbf{S}^i$, $i = 1, 2$, of $q + 1$ maximal elementary abelian subgroups of $\mathscr{H}_2(q)$ corresponding to the reguli of the quadric of Q. Note that $\mathrm{Aut}(\mathscr{H}_2(q)) \cong \mathscr{H}_2(q)/Z(\mathscr{H}_2(q)) \ltimes \mathbf{O}^+(4,q)$, so that $\mathrm{Aut}(\mathscr{H}_2(q))$ acts transitively on $\{\mathbf{S}^1, \mathbf{S}^2\}$. In other words:

Theorem 3.8 (Tangents in even characteristic) *Let q be even. The set $\mathbf{F}^*$ of tangent spaces of any Kantor family $(\mathbf{F}, \mathbf{F}^*)$ of type (q^2, q) in $\mathscr{H}_2(q)$, or equivalently, of any Kantor family coming from a q-clan, is completely determined, and can always be chosen as the same fixed set of $q+1$ subgroups, independent of the isomorphism class of $\mathbf{F}$.*

It seems that this fact remained unnoticed for a long time.

The following observation is slightly more general than [14, Lemma, p. 154]. It generalizes [2, Lemma 4.6], cf. Remark 3.10(ii).

Observation 3.9 *Let $q = p^h$, p a prime. If p is odd, the elements of $\mathbf{F}^*$ induce a BLT-set in $\mathbf{W}(q)$, while if q is even, we have a regulus of some hyperbolic quadric on $\mathbf{W}(q)$.*

Proof For $p = 2$, this was already observed, so let p be odd. Suppose a BLT-set is not induced; let U, V, W be distinct lines in the line set of $\mathbf{W}(q)$ induced by $\mathbf{F}^*$, and let U^*, V^*, W^* be the corresponding subgroups of $\mathscr{H}_2(q)$ (they are elements of $\mathbf{F}^*$). Also, let U', V', W' be the corresponding elements of $\mathbf{F}$. Suppose there is some line X of $\mathbf{W}(q)$ meeting all three of U, V, W, and let X^* be the corresponding group in $\mathscr{H}_2(q)$; then $|U' \cap X^*| = |V' \cap X^*| = |W' \cap X^*| = q$. Let $\mathscr{U}$ be the $(q+1) \times (q+1)$-grid of $\mathscr{S}^x$ corresponding to the identity $[U' \cap X^*, V' \cap X^*] = \{\mathbf{1}\}$ (noting that X^* is abelian); one of its reguli contains $L_{U'}$, the other contains $L_{V'}$. Let $u \in \mathscr{U}$ be the intersection point of $L_{U'}$, $L_{V'}$ and $L_{W'}$, and consider any $\beta \in (W' \cap X^*)^\times$; then it is easy to see that u^β is incident with three lines that meet the point set of $\mathscr{U}$ (namely $L_{W'}, L_{V'}^\beta, L_{U'}^\beta$), contradicting [19, 2.6.1]. (Note that $\mathbb{S} \le X^*$; so X^* has orbits of size q on $[U'] \setminus \{x\}$ and $[V'] \setminus \{x\}$.) □

Remark 3.10

(i) In [14], the latter observation is obtained by using the associated q-clan. Our proof only uses [19, 2.6.1], while the setting is *a priori* more general—we "only" assume that χ defines a symplectic form (the fact that $\mathscr{S}^x$ eventually is a flock GQ, or arises from a q-clan, is not needed).

(ii) In [2], Observation 3.9 is obtained for the special case of q prime (cf. [2, Lemma 4.6]). However, its proof appears to contain a flaw—the conclusion that (in their notation) $C^* \cap H = (C \cap (H \cap A)(H \cap B))Z(E)$ follows from $H = (H \cap A)(H \cap B)Z(E)$ and $C^* = CZ(E)$ is not correct. (The fact that p is odd is not used at that point, while it should—cf. [19, 2.6.1]. As mentioned, when q is a power of 2, the elements of $\mathbf{F}^*$ induce a regulus, so $C^* \cap H \neq (C \cap (H \cap A))(H \cap B))Z(E)$ while still $H = (H \cap A)(H \cap B)Z(E)$ and $C^* = CZ(E)$.) However, the statement is correct by Observation 3.9.

4 Elation quadrangles of order (s, p), p a prime

In the aforementioned paper [2], the following theorem, which complements the result of Bloemen et al. [4] classifying EGQs of order (p, t), was obtained.

Theorem 4.1 (Bamberg et al. [2]) *An EGQ* $(\mathcal{S}^x, H)$ *of order* (s, p) *with* p *a prime*, $s \neq 1$, *is either isomorphic to* $\mathbf{W}(p)$, *or to a flock quadrangle*, *in which case* $s = p^2$.

In this final section, we apply our main result to obtain an alternative and very short proof of Theorem 4.1. As any GQ of order $(s, 2)$ is classical [19, Chap. 6], we suppose that p is odd.

Proof As in [2], we only have to consider the case $s = p^2$, since the case $s < p^2$ reduces to the case $s = p$ by Frohardt's Theorem [7] (and then [4] applies). If $s = p^2$, the aforementioned fact that the center can only consist of symmetries, immediately leads to the fact that $|Z(H)| = p$, so that $\mathcal{S}^x$ is a central STGQ. Any element of $\mathbf{F}$ is an abelian group of order p^2; any element of $\mathbf{F}^*$ is abelian of order p^3. By [9, Sect. 3], either $H/Z(H)$ is elementary abelian, or H contains a subgroup D for which $\mathbf{F}_D = \{A \cap D | A \in \mathbf{F}\}$, $\mathbf{F}^*_D = \{A^* \cap D | A^* \in \mathbf{F}^*\}$ constitutes a Kantor family of type (p, p) in D. In the latter case, there is a subEGQ $\mathcal{S}'$ of order p, which necessarily is isomorphic to $\mathbf{W}(p)$ by [4]. By [1], we have $\mathcal{S}^x \cong \mathcal{H}(3, p^2)$. We henceforth suppose that $H/Z(H)$ is elementary abelian. Then $[H, H] \leq Z(H) = [H, H] = \Phi(H)$; as before, we now see $H/Z(H)$ as a vector space V over $\mathbb{F}_p \equiv Z(H)$. The bi-additive map

$$\chi : V \times V \mapsto \mathbb{F}_p : \big(aZ(H), bZ(H)\big) \mapsto [a, b]$$

then becomes a non-singular bilinear alternating form over $\mathbb{F}_p$, and our main result applies. $\square$

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